

CARTESIAN PRODUCT OF AUTOMATA

MRIDUL DUTTA¹, SANJOY KALITA, AND HELEN K. SAIKIA

ABSTRACT. Automaton is a system that spontaneously gives output from input. Here input may be energy, information, materials etc. The system works without intervention of man. Simply automaton (plural: automata or automations) is a self-operating machine. This article discusses and analyzes two automata to construct a product machine from two given machines. The cross-product operation has been used as a common framework for the design and study of new interconnection networks with various properties. Here we defined the Cartesian product of two automata $\Sigma_1 = (Q_1, A_1, B_1, F_1, G_1)$ and $\Sigma_2 = (Q_2, A_2, B_2, F_2, G_2)$ as another system $\Sigma = \Sigma_1 \times \Sigma_2 = (Q, A, B, F, G)$ where $Q = Q_1 \times Q_2$, $A = A_1 \cup A_2$, $B = B_1 \cup B_2$ and F and G are transition function and output functions. A Separable system is a generalization of automata. With different examples we have seen that the system Σ is also an automata. We also observed that if Σ_1 and Σ_2 are two separable systems then Σ too a separable system.

1. INTRODUCTION

In automata theory different types of automata are discussed. Automata is a five tuples consisting of a set of states, inputs, outputs, one transition function and one output functions in between these three sets. A Semi automaton and automaton is defined by Ginzburg(1968) firstly. Every cyclic automaton is isomorphic in the classical sense to an automaton whose state-space is a quotient of its semi group[10]. W. Dorfler (1978)[2] has also studies on the Cartesian

¹corresponding author

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Product of Automata. A good number of researchers Rabin, Scott, Gecseg, Peak, Fleck, Weeg, Bravel, Yukio etc. (approx. 1972-1980) have done considerable work along this line[9]. It is suggested by Muir and Warner (1981), Warner (1981) that there are advantages in using more general morphisms between automata, namely pseudomorphisms[10]. Most famous near-ringer G. F. Pliz, W. M. I. Holecombe, J. L. Casti, G. Hofer, R. Lidl, K.C. Chowdhury et al and many others have done considerable work (approx.1970-2000), on various aspects of Non-linear dynamical systems, near-rings with chain conditions, automata, reachability, feedback system etc[1]. For every semi automata where state set is a group, G.F. Pilz, T.J. Laffey, M.R.I.A. has associate its works on syntactic near-rings [6]. Authors Corsini, Chvalina, Leoreanu, Masami Ito, Takahiro Ito, Mitsuhiro Fujio, S.C. Hsieh etc. all have done their works on different algebraic structure of automata in between (1993-2012)[3]. Most recently, J Baskar Babujeo, J Julie, N. Subashini, K. Thiagaranjans, M. Novak, S. Krehlik, D. Stanek, S. Krehlik etc. have done their work on Automata resulting from graph operations, n -ary Cartesian composition and cartesian product of automata[4, 5, 6, 7].

The beginning of it saw the first proper automata consideration. Since then the theory of automata has developed much and at present a sophisticated theory with numbers applications in various area, namely Biology, Sociology, Economic, Engineering, Cryptography, Mathematics etc. In recent years its connection with computer science, near-ring, dynamical systems, rooted tree, coding theory etc.

2. PRELIMINARIES

A *Semi automata*[4, 6] is a triple $\Sigma = (Q, A, F)$ where Q and A are set of states and set of inputs respectively. For any semi automata $\Sigma = (Q, A, F)$ we obtain a collection of mappings $F_a : Q \rightarrow Q$, one for each $a \in A$, which are given by $F_a(q) = F(q, a)$. If the input a_1 is followed by the input a_2 , the semi automaton moves from the state $q \in Q$ first into $F_{a_1}(q)$ and then into $F_{a_2}(F_{a_1}(q))$. If we extend (as usual) A to the free word monoid A^* over A (consisting of all finite sequences of elements of A , including the empty sequence Λ), we therefore obtain $F_{a_1 a_2} = F_{a_2} F_{a_1}$.

An *automata*[6, 8] is a system of is a quintuple $\Sigma = (Q, A, B, F, G)$, where Q is a set of states, A is a set of inputs, B is a set of outputs, $F : Q \times A \rightarrow Q$ and

$G : Q \times A \rightarrow B$ are functions usually known as state transition function and output function respectively.

A system Σ (as defined above) is called *separable*[8] if Q, A, B are groups (written additively, but not necessarily abelian) and if there are maps $\alpha : Q \rightarrow Q$, $\gamma : Q \rightarrow B$ and homomorphisms $\beta : A \rightarrow Q$ and $\delta : A \rightarrow B$ such that $F(q, a) = \alpha(q) + \beta(a)$, $G(q, a) = \gamma(q) + \delta(a)$, $\forall q \in Q$ and $a \in A$. We then denote Σ by $(Q, A, B, \alpha, \beta, \gamma, \delta)$ or simply by $(\alpha, \beta, \gamma, \delta)$. Σ is called zero-symmetric if $\alpha(0) = \gamma(0) = 0$.

If $\Sigma_1 = (Q_1, A_1, F_1)$ and $\Sigma_2 = (Q_2, A_2, F_2)$ are two semi automata with $Q_1 = \{p_1, p_2, p_3, \dots, p_m\}$ and $Q_2 = \{q_1, q_2, q_3, \dots, q_n\}$ then the Cartesian product of Σ_1 and Σ_2 is $\Sigma = \Sigma_1 \times \Sigma_2$ where $\Sigma = (Q, A, F)$ with $Q = Q_1 \times Q_2$, $|Q| = mn$, $A = A_1 \cup A_2$, $F : (Q_1 \times Q_2) \times A \rightarrow Q_1 \times Q_2$ is defined by $F((p_i, q_j), a) = (p_k, q_l)$, $1 \leq i \leq m$, $1 \leq j \leq n$ iff (i) $p_i = p_k$ and $F_2(q_j, a) = q_l$ for $q_j, q_l \in Q_2$ and $a \in A_2$ or (ii) $q_j = q_l$ and $F_1(p_i, a) = p_k$ for $p_i, p_k \in Q_1$ and $a \in A_1$.

3. CARTESIAN PRODUCT OF AUTOMATA

In this section we defined Cartesian product of two automata. We also gave some examples in support of this definition and found that the product of two automata is again an automata. Moreover we observed that the product automata carries some behaviour of the parental automata.

We defined the Cartesian product of two automata as follows:

If $\Sigma_1 = (Q_1, A_1, B_1, F_1, G_1)$ and $\Sigma_2 = (Q_2, A_2, B_2, F_2, G_2)$ are two automata with $Q_1 = \{p_1, p_2, p_3, \dots, p_m\}$ and $Q_2 = \{q_1, q_2, q_3, \dots, q_n\}$ then the Cartesian product of Σ_1 and Σ_2 denoted as $\Sigma = \Sigma_1 \times \Sigma_2$ is given by $\Sigma = (Q, A, B, F, G)$ where $Q = Q_1 \times Q_2$, $A = A_1 \cup A_2$, $B = B_1 \cup B_2$. Transition function $F : (Q_1 \times Q_2) \times A \rightarrow Q_1 \times Q_2$ is defined by $F((p_i, q_j), a) = (p_r, q_s)$, iff any of the following conditions hold

$$F_1(p_i, a) = p_r \text{ and } q_j = q_s \text{ } a \in A_1 \text{ or } p_i = p_r \text{ and } F_2(q_j, a) = q_s \text{ if } a \in A_2.$$

Output function $G : (Q_1 \times Q_2) \times A \rightarrow B$ is defined by $G((p_i, q_j), a) = b_t$, iff any of the following conditions hold

$$G((p_i, q_j), a) = \begin{cases} G_1(p_i, a) & \text{if } a \in A_1 \text{ and } F_1(p_i, a) = p_r \\ & \text{or} \\ G_2(q_j, a) & \text{if } a \in A_2 \text{ and } F_2(q_j, a) = q_s \end{cases}$$

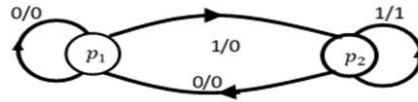
Example 1. Let $\Sigma_1 = (Q_1, A_1, B_1, F_1, G_1)$ with $Q_1 = \{p_1, p_2\}$, $A_1 = \{0, 1\}$, $B_1 = \{0, 1\}$, transition function $F_1 : Q_1 \times A_1 \rightarrow Q_1$ and output function $G_1 : Q_1 \times A_1 \rightarrow B_1$ are defined in Table 1

TABLE 1. Transition table and Output table of Σ_1

F_1	0	1
p_1	p_1	p_2
p_2	p_1	p_2

G_1	0	1
p_1	0	0
p_2	0	1

And the state diagram of Σ_1 is given by Figure 1

FIGURE 1. State diagram of Σ_1

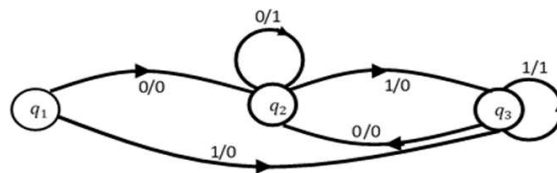
Again let $\Sigma_2 = (Q_2, A_2, B_2, F_2, G_2)$ with $Q_2 = \{q_1, q_2, q_3\}$, $A_2 = \{0, 1\}$, $B_2 = \{0, 1\}$, transition function $F_2 : Q_2 \times A_2 \rightarrow Q_2$ and output function $G_2 : Q_2 \times A_2 \rightarrow B_2$ are defined by Table 2

TABLE 2. Transition table and Output table of Σ_2

F_2	0	1
q_1	q_2	q_3
q_2	q_2	q_3
q_3	q_2	q_3

G_2	0	1
q_1	0	0
q_2	1	0
q_3	0	1

And the state diagram of Σ_2 is given by Figure 2

FIGURE 2. State diagram of Σ_2

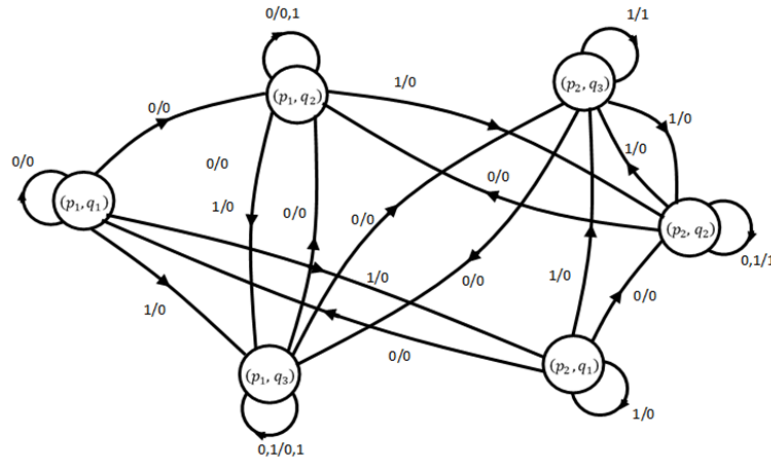
Then $\Sigma = \Sigma_1 \times \Sigma_2$ where $\Sigma = (Q, A, B, F, G)$ with $Q = Q_1 \times Q_2 = \{(p_1, q_1), (p_1, q_2), (p_1, q_3), (p_2, q_1), (p_2, q_2), (p_2, q_3)\}$, $A = A_1 \cup A_2 = \{0, 1\}$, $B = B_1 \cup B_2 = \{0, 1\}$ and transition function and output function $F : Q \times A \rightarrow Q$ and $G : Q \times A \rightarrow B$ are given by Table 3

 TABLE 3. Transition table and Output table of Σ

F	0	1
(p_1, q_1)	$(p_1, q_1), (p_1, q_2)$	$(p_2, q_1), (p_1, q_3)$
(p_1, q_2)	(p_1, q_2)	$(p_2, q_2), (p_1, q_3)$
(p_1, q_3)	$(p_1, q_3), (p_1, q_2)$	$(p_2, q_3), (p_1, q_3)$
(p_2, q_1)	$(p_1, q_1), (p_2, q_2)$	$(p_2, q_1), (p_2, q_3)$
(p_2, q_2)	$(p_1, q_2), (p_2, q_2)$	$(p_2, q_2), (p_2, q_3)$
(p_2, q_3)	$(p_2, q_3), (p_2, q_2)$	(p_2, q_3)

G	0	1
(p_1, q_1)	0	0
(p_1, q_2)	$0, 1$	0
(p_1, q_3)	0	$0, 1$
(p_2, q_1)	0	$1, 0$
(p_2, q_2)	$0, 1$	$1, 0$
(p_2, q_3)	0	1

The state diagram of Σ is given by Figure 3


 FIGURE 3. State diagram of Σ

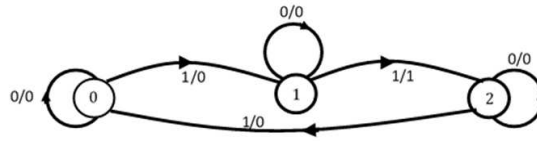
Example 2. Let $\Sigma_1 = (Q_1, A_1, B_1, F_1, G_1)$ is a automata with $Q_1 = (\mathbb{Z}_3, +)$, $A_1 = (\mathbb{Z}_2, +)$, $B_1 = (\mathbb{Z}_2, +)$ the transition function $F_1 : Q_1 \times A_1 \rightarrow Q_1$ i.e. $F_1 : \mathbb{Z}_3 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_3$ defined by $F_1(p_1, a_1) = p_1 + a_1$ and output function $G_1 : Q_1 \times A_1 \rightarrow B_1$ i.e. $G_1 : \mathbb{Z}_3 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ defined by $G_1(p_1, a_1) = p_1 a_1$.

The transition functin and output function are given in Table 4.

TABLE 4. Transition table and Output table of Σ_1

F_1	0	1
0	0	1
1	1	2
2	2	0

G_1	0	1
0	0	0
1	0	1
2	0	0

FIGURE 4. State diagram of Σ_1

The state diagram of Σ_1 is given by Figure 4.

Again let $\Sigma_2 = (Q_2, A_2, B_2, F_2, G_2)$ is an automata with $Q_2 = (\mathbb{Z}_2, +)$, $A_2 = (\mathbb{Z}_3, +)$, $B_2 = (\mathbb{Z}_3, +)$, transition function $F_2 : Q_2 \times A_2 \rightarrow Q_2$ i.e. $F_2 : \mathbb{Z}_2 \times \mathbb{Z}_3 \rightarrow \mathbb{Z}_2$ defined by $F_2(q_1, a_2) = q_1 + a_2$ and output function $G_2 : Q_2 \times A_2 \rightarrow B_2$ i.e. $G_2 : \mathbb{Z}_2 \times \mathbb{Z}_3 \rightarrow \mathbb{Z}_2$ defined by $G_2(q_1, a_2) = q_1 a_2$.

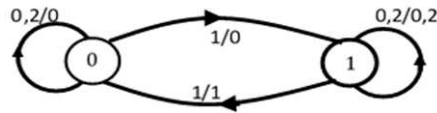
The transition function and output function are given in Table 5.

TABLE 5. Transition table and Output table of Σ_2

F_2	0	1	2
0	0	1	0
1	1	0	1

G_2	0	1	2
0	0	0	0
1	0	1	2

The state diagram of Σ_2 is given by Figure 5

FIGURE 5. State diagram of Σ_2

Then $\Sigma = \Sigma_1 \times \Sigma_2$ where $\Sigma = (Q, A, B, F, G)$ with $Q = Q_1 \times Q_2 = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}$, $A = A_1 \cup A_2 = \{0, 1, 2\}$, $B = B_1 \cup B_2 = \{0, 1, 2\}$, the transition function $F : Q \times A \rightarrow Q$ i.e. $F : (\mathbb{Z}_3 \times \mathbb{Z}_2) \times \mathbb{Z}_3 \rightarrow \mathbb{Z}_3 \times \mathbb{Z}_2$ and output function $G : Q \times A \rightarrow B$ i.e. $G : (\mathbb{Z}_3 \times \mathbb{Z}_2) \times \mathbb{Z}_3 \rightarrow \mathbb{Z}_3$ will be given.

The transition function and output function are given in Table 6

TABLE 6. Transition table and Output table of Σ

F	0	1	2
$(0, 0)$	$(0, 0)$	$(1, 0), (0, 1)$	$(0, 0)$
$(0, 1)$	$(0, 1)$	$(1, 1), (0, 0)$	$(0, 1)$
$(1, 0)$	$(1, 0)$	$(2, 0), (1, 1)$	$(1, 0)$
$(1, 1)$	$(1, 1)$	$(2, 1), (1, 0)$	$(1, 1)$
$(2, 0)$	$(2, 0)$	$(0, 0), (2, 1)$	$(2, 0)$
$(2, 1)$	$(2, 1)$	$(0, 1), (2, 0)$	$(2, 1)$

G	0	1	2
$(0, 0)$	0	0	0
$(0, 1)$	0	0, 1	2
$(1, 0)$	0	1, 0	0
$(1, 1)$	0	1	2
$(2, 0)$	0	0	0
$(2, 1)$	0	0, 1	2

And the state diagram of Σ is given by Figure 6

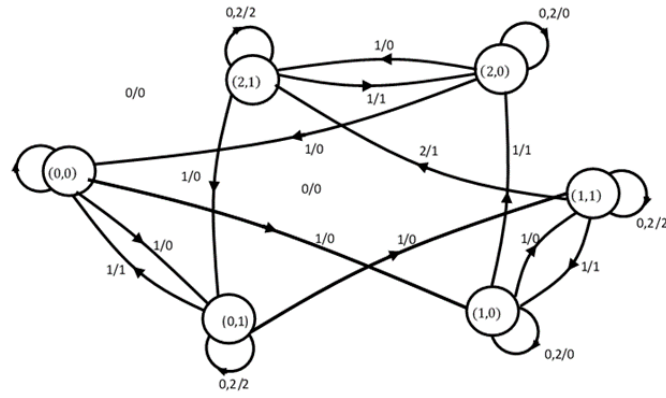


FIGURE 6. State diagram of Σ

Example 3. We consider following two automata $\Sigma_1 = (Q_1, A_1, B_1, F_1, G_1) = (\mathbb{R}, \mathbb{R}, \mathbb{R}, F_1, G_1)$ and $\Sigma_2 = (Q_2, A_2, B_2, F_2, G_2) = (\mathbb{R}, \mathbb{Z}, \mathbb{Z}, F_2, G_2)$ where $(\mathbb{R}, +)$ and $(\mathbb{Z}, +)$ are groups. The functions $F_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $G_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are defined by

$$F_1(p_1, a_1) = p_1^2 + \sin p_1 + 3a_1, \forall p_1, a_1 \in \mathbb{R}$$

$$G_1(p_1, a_1) = e^{p_1} - 1 + \pi a_1, \forall p_1, a_1 \in \mathbb{R}$$

and $F_2 : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{R}$ and $G_2 : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{Z}$ are defined by

$$F_2(q_1, a_2) = q_1^3 + 3a_2, \forall q_1 \in \mathbb{R}, a_2 \in \mathbb{Z}$$

$$G_2(q_1, a_2) = [q_1] + a_2, \forall q_1 \in \mathbb{R}, a_2 \in \mathbb{Z}$$

Then their product will be given by $\Sigma = \Sigma_1 \times \Sigma_2 = (Q, A, B, F, G) = (\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mathbb{R}, F, G)$ where $F : (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ defined by

$$F((p_i, q_j), a) = (p_i^2 + \sin p_i + 3a, q_j) \text{ if } a \in \mathbb{R} \text{ or } (p_i, q_j^3 + 3a), a \in \mathbb{Z}$$

and $G : (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$G((p_i, q_j), a) = \begin{cases} e^{p_i} - 1 + \pi a \text{ if } a \in \mathbb{R} \text{ and } F_1(p_i, a) = p_r \\ \text{or} \\ [q_j] + a \text{ if } a \in \mathbb{Z} \text{ and } F_2(q_j, a) = q_s \end{cases}$$

$\Sigma = \Sigma_1 \times \Sigma_2 = (Q, A, B, F, G)$ also forms an automata.

Example 4. We consider the automata $\Sigma_1 = (\mathbb{Z}_2, \mathbb{Z}, \mathbb{Z}_3, F_1, G_1)$ where $(\mathbb{Z}_2, +)$, $(\mathbb{Z}_3, +)$ and $(\mathbb{Z}, +)$ are groups and $F_1 : \mathbb{Z}_2 \times \mathbb{Z} \rightarrow \mathbb{Z}_2$ and $G_1 : \mathbb{Z}_2 \times \mathbb{Z} \rightarrow \mathbb{Z}_3$ are defined by

$$F_1(p_1, a_1) = 1 + \begin{cases} [0] \text{ if } a_1 \text{ is even} \\ [1] \text{ if } a_1 \text{ is odd} \end{cases}, p_1 \in \mathbb{Z}_2, a_1 \in \mathbb{Z}$$

$$G_1(p_1, a_1) = 2 + a_1(\text{mod}3) \forall p_1 \in \mathbb{Z}_2, a_1 \in \mathbb{Z}.$$

Let us consider another automata $\Sigma_2 = (\mathbb{C}, \mathbb{C}, \mathbb{C}, F_2, G_2)$ where $(\mathbb{C}, +)$ is a group and $F_2 : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ and $G_2 : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ are defined by

$$F_2(q_1, a_2) = q_1 + 1 + \overline{a_2} \forall q_1, a_2 \in \mathbb{C}$$

$$G_2(q_1, a_2) = q_1^2 + 3a_2, \forall q_1, a_2 \in \mathbb{C}.$$

Then their product $\Sigma = \Sigma_1 \times \Sigma_2 = (Q, A, B, F, G)$ where $Q = \mathbb{Z}_2 \times \mathbb{C}$, $A = \mathbb{Z} \cup \mathbb{C} = \mathbb{C}$, and $B = \mathbb{Z}_3 \cup \mathbb{C} = \mathbb{C}$, the functions $F : (\mathbb{Z}_2 \times \mathbb{C}) \times \mathbb{C} \rightarrow \mathbb{Z}_2 \times \mathbb{C}$ and $G : (\mathbb{Z}_2 \times \mathbb{C}) \times \mathbb{C} \rightarrow \mathbb{C}$ are defined by

$$F((p_i, q_j), a) = \begin{cases} \left(1 + \begin{cases} [0] \text{ if } a \text{ is even} \\ [1] \text{ if } a \text{ is odd} \end{cases}, q_j \right) \text{ if } a \in \mathbb{Z}_2 \\ \text{or} \\ (p_i, q_1 + 1 + \overline{a}) \text{ if } a \in \mathbb{C} \end{cases}$$

$$G((p_i, q_j), a) = \begin{cases} 2 + a(\text{mod} 3) \text{ if } a \in \mathbb{Z}_2 \text{ and } F_1(p_i, a) = p_r \\ \text{or} \\ q_j^2 + 3a \text{ if } a \in \mathbb{C} \text{ and } F_2(q_j, a) = q_s \\ \text{for some } p_r \in \mathbb{Z}_2 \text{ and } q_s \in \mathbb{C} \end{cases},$$

Therefore $\Sigma = \Sigma_1 \times \Sigma_2 = (Q, A, B, F, G)$ forms again an automata.

4. OBSERVATIONS

From above examples it is clear that the definition of Cartesian product given in this paper is well defined. The definition also preserve various properties of the given automata in their product automata. As we have seen in examples 3 and example 4 that the given automata are separable and that property is also displayed by their product automata.

5. FUTURE SCOPE

We observed that the product of automata is separable if the given automata are separable. We also know that with a separable automata we can define an algebraic structure[8] by defining new operations on it, so we can surely extend our study of automata linking it to algebra for more fruitful outcomes.

6. CONCLUSION

In our study we used the usual notation of automata and defined the product automata with respect to both transition function and output function. Many author have defined product automata earlier but our approach is little different from those. This approach surely give us a way to construct product machine as well as provides opportunity to study then in terms of graph theory and algebra.

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DEPARTMENT OF MATHEMATICS
DUDHNOI COLLEGE
DUDHNOI-783124, GOALPARA, ASSAM, INDIA
Email address: mridulduttamc@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GAUHATI
GUWAHATI-781014, ASSAM INDIA
Email address: sanjoykalita1@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GAUHATI
GUWAHATI-781014, ASSAM INDIA
Email address: hsaikia@yahoo.com