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SOME PROPERTIES OF MULTIPLICATIVE S-METRIC SPACES

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ABSTRACT. The purpose of this paper is to introduce multiplicative S-metric space as a generalization of multiplicative d-metric space and to investigate some properties of it.

1. INTRODUCTION

In 1906, Frechet introduced the notion of metric space. Later, it has been generalized in many ways by several authors. In this direction, A. E. Bashirov, E. M. Kurplnara and A. Ozyapici [1] introduced a new concept called multiplicative metric in the year 2008 and studied some properties of multiplicative derivatives and multiplicative integrals. In 2012, Florack and Assen [2] used this concept in biomedical image analysis. In the same year Özavşar and Çevikel [3] studied topological properties and introduced multiplicative contraction by which they proved some fixed point theorems in multiplicative d-metric spaces. In 2012, Sedghi, Shobe and Aliouche [4] introduced S-metric space as a generalization of G-metric space and metric space and proved several fixed and common fixed point results in the setting of S-metric spaces(For example, see [4-6]).

In this paper, we introduce multiplicative S-metric space and study some of its topological properties. Further, we observed that the set of positive real numbers

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is complete with respect to some multiplicative S-metric but it is not complete in usual sense.

2. PRELIMINARIES

In this section, we present some definitions which will be used later in this paper.

Definition 2.1. *Let X be a non empty set. Then we say that a function S: $X^3 \to [0,\infty)$ is a multiplicative S-metric on X iff it satisfies the following for all α,β,γ and $\theta \in X$:*

P1) S$(\alpha,\beta,\gamma) \geq 1$;

P2) S(α,β,γ)=1 iff $\alpha = \beta = \gamma$

; P3) S(α,β,γ) $\leq$S(α,α,θ) S(β,β,θ) S(γ,γ,θ).

Here (X,S) is called a multiplicative S-metric space.

Example 1. *Let $X = [0,\infty)$ and S(α,β,γ):=$t^{max\{\alpha,\beta,\gamma\}}$ for α,β,$\gamma \in X$ and fixed $t > 1$. For α,β,$\gamma \in X$, we have max$\{\alpha,\beta,\gamma\} \geq 0$ and hence $t^{max\{\alpha,\beta,\gamma\}} \geq 1$. Thus S$(\alpha,\beta,\gamma) \geq 1$. Now S($\alpha$,$\beta$,$\gamma$)=1 iff max$\{\alpha,\beta,\gamma\} = 0$ iff $\alpha = \beta = \gamma$. Now for α,β,γ and $\theta \in X$,*

$$
\begin{aligned}
S(\alpha,\beta,\gamma)=t^{max\{\alpha,\beta,\gamma\}} &\leq t^{max\{max\{\alpha,\alpha,\theta\},max\{\beta,\beta,\theta\},max\{\gamma,\gamma,\theta\}\}} \\
&\leq t^{max\{\alpha,\alpha,\theta\}+max\{\beta,\beta,\theta\}+max\{\gamma,\gamma,\theta\}} \\
&= S(\alpha,\alpha,\theta)\, S(\beta,\beta,\theta)\, S(\gamma,\gamma,\theta).
\end{aligned}
$$

Thus S(α,β,γ) $\leq$S(α,α,θ) S(β,β,θ) S(γ,γ,θ) for all α,β,γ and $\theta \in X$. Therefore, S is a multiplicative S-metric on X.

Definition 2.2. *We say that a sequence (α_n) in multiplicative S-metric space (X,S) multiplicative S-converges to some $\alpha \in X$ iff for each $\epsilon > 1$, there exists a H$\in \mathbb{N}$ such that S$(\alpha_n,\alpha_n,\alpha) < \epsilon$,$\forall$n$\geq$H.*

Definition 2.3. *We say that a sequence (α_n) in multiplicative S-metric space (X,S) is multiplicative S-Cauchy sequence in X iff for each $\epsilon > 1$, there exists a H$\in \mathbb{N}$ such that S$(\alpha_n,\alpha_n,\alpha_m) < \epsilon$, $\forall$n,m$\geq$H.*

Definition 2.4. *We say that a multiplicative S-metric space (X,S) is multiplicative S-complete iff every multiplicative S-Cauchy sequence in X is multiplicative S-convergent in X.*

Definition 2.5. *Let (X,S) and (Y,S') be two multiplicative S-metric spaces. Then we say that f: $X \to Y$ is multiplicative S-continuous at some point $\alpha \in X$ iff for every $r > 1$,there exists $\eta > 1$ such that f(B(α, η)) $\subset$ B(f(α), r). Here we say that f is multiplicative S-continuous on X iff it is multiplicative S-continuous at every point of X.*

Definition 2.6. *We say that a subset A of multiplicative S-metric space (X,S) is multiplicative S-bounded in X iff there exist $\alpha \in X$ and $\epsilon > 1$ such that A$\subset$B(α, ϵ).*

3. Main results

Theorem 3.1. *In multiplicative S-metric space (X,S), we have S(α,α,β)=S(β,β,α) for all α,$\beta \in X$.*

Proof. For $\alpha, \beta \in X$, S$(\alpha, \alpha, \beta) \leq$S$(\alpha, \alpha, \alpha)S(\alpha, \alpha, \alpha)S(\beta, \beta, \alpha)$=S$(\beta, \beta, \alpha)$. Thus, S$(\alpha, \alpha, \beta) \leq$S$(\beta, \beta, \alpha)$. By interchanging α and β, we get S$(\beta, \beta, \alpha) \leq$S$(\alpha, \alpha, \beta)$ and hence the result proved. ☐

Theorem 3.2. *In multiplicative S-metric space (X,S), $\alpha_n \to \alpha$ iff S(α_n,α_n,α) $\to 1$, as $n \to \infty$.*

Proof.

($\Longrightarrow$)Then for every $\epsilon > 1$, there exists H$\in \mathbb{N}$ such that S$(\alpha_n, \alpha_n, \alpha) < \epsilon$, $\forall n \geq$H. In particular, we have S$(\alpha_n, \alpha_n, \alpha) < 1 + \frac{1}{n}$ for every $n \geq$ H. Now letting $n \to \infty$, we have $\lim\limits_{n\to\infty}$S$(\alpha_n, \alpha_n, \alpha) \leq 1$.Note that S$(\alpha_n, \alpha_n, \alpha) \geq 1$ for every $n \in \mathbb{N}$.It follows that $\lim\limits_{n\to\infty}$S$(\alpha_n, \alpha_n, \alpha) \geq 1$ and the results follows.

($\Longleftarrow$) Let $\epsilon > 1$. As S(α_n,α_n,α) $\to 1$, then there exists H$'$ $\in \mathbb{N}$ such that $|$S$(\alpha_n, \alpha_n, \alpha) - 1| < \epsilon - 1$,$\forall n \geq$ H$'$. This will imply that S$(\alpha_n, \alpha_n, \alpha) < \epsilon$ $\forall n \geq$ H$'$ and hence the result proved. ☐

Theorem 3.3. *In multiplicative S-metric space (X,S), if there exist two sequences (α_n) and (β_n) in X such that $\lim\limits_{n\to\infty} \alpha_n = \alpha$ and $\lim\limits_{n\to\infty} \beta_n = \beta$, then $\lim\limits_{n\to\infty}$S$(\alpha_n, \alpha_n, \beta_n)$=S$(\alpha, \alpha, \beta)$.*

Proof. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{S}(\alpha_n, \alpha_n, \beta_n) &\leq \mathrm{S}(\alpha_n, \alpha_n, \alpha)\mathrm{S}(\alpha_n, \alpha_n, \alpha)\mathrm{S}(\beta_n, \beta_n, \alpha) \\ &\leq \mathrm{S}(\alpha_n, \alpha_n, \alpha)\mathrm{S}(\alpha_n, \alpha_n, \alpha)\mathrm{S}(\beta_n, \beta_n, \beta) \end{aligned}$$

$$S(\beta_n, \beta_n, \beta)S(\alpha, \alpha, \beta),$$

Therefore

$$S(\alpha_n, \alpha_n, \beta_n) \leq S(\alpha_n, \alpha_n, \alpha)S(\alpha_n, \alpha_n, \alpha)\ S(\beta_n, \beta_n, \beta)S(\beta_n, \beta_n, \beta)S(\alpha, \alpha, \beta)$$

for every $n \in \mathbb{N}$.Now letting $n \to \infty$,we have $\lim_{n\to\infty} S(\alpha_n, \alpha_n, \beta_n) \leq S(\alpha, \alpha, \beta)$.

Consider

$$\begin{aligned} S(\alpha, \alpha, \beta) &\leq S(\alpha, \alpha, \alpha_n)S(\alpha, \alpha, \alpha_n)\ S(\beta, \beta, \alpha_n) \\ &\leq S(\alpha, \alpha, \alpha_n)S(\alpha, \alpha, \alpha_n) \\ &\qquad S(\beta, \beta, \beta_n)S(\beta, \beta, \beta_n)\ S(\alpha_n, \alpha_n, \beta_n) \\ &= S(\alpha_n, \alpha_n, \alpha)S(\alpha_n, \alpha_n, \alpha) \\ &\qquad S(\beta_n, \beta_n, \beta)S(\beta_n, \beta_n, \beta)\ S(\alpha_n, \alpha_n, \beta_n) \end{aligned}$$

Thus, $S(\alpha, \alpha, \beta) \leq S(\alpha_n, \alpha_n, \alpha)S(\alpha_n, \alpha_n, \alpha)S(\beta_n, \beta_n, \beta)S(\beta_n, \beta_n, \beta)\ S(\alpha_n, \alpha_n, \beta_n)$ for every $n \in \mathbb{N}$. Now letting $n \to \infty$, we have $S(\alpha, \alpha, \beta) \leq \lim_{n\to\infty} S(\alpha_n, \alpha_n, \beta_n)$ and hence the result follows. □

Theorem 3.4. *In multiplicative S-metric space (X,S), (α_n) is a multiplicative S-Cauchy sequence in X iff S(α_n,α_n,α_m) $\to 1$,as $n, m \to \infty$.*

Proof. The proof is similar to that of the Lemma 3.2. and hence omitted. □

Theorem 3.5. *Let (X,S) and (Y,S') be two multiplicative S-metric spaces. Then f: $X \to Y$ is continuous at point $\alpha \in X$ iff f$\alpha_n \to$ fα for every sequence (α_n) in X with $\alpha_n \to \alpha$ in X.*

Proof.

($\Longrightarrow$) Let $\alpha_n \to \alpha$ in X. Now we show that f$\alpha_n \to$ fα in Y. For this, let $\epsilon > 1$. Since f is continuous at point $\alpha \in X$, there exists $\delta > 1$ such that f(B(α, δ)) $\subset$ B(fα, ϵ). Now $\alpha_n \to \alpha$ implies that there exists H$\in \mathbb{N}$ such that $S(\alpha_n, \alpha_n, \alpha) < \delta\ \forall n \geq$H. For $n \geq$H, we have $\alpha_n \in$ B(α, δ)). It follows that f$\alpha_n \in$ B(fα, ϵ)) for $n \geq$H and hence f$\alpha_n \to$ fα in Y.

($\Longleftarrow$)Suppose that f is not continuous at $\alpha \in X$. Then for any $\delta > 1$, there exists a $\epsilon > 1$ such that f(B(α, δ)) $\nsubseteq$ B(fα, ϵ). In particular, for every $n \in \mathbb{N}$ there exists $\alpha_n \in$ B$(\alpha, 1 + \frac{1}{n})$ such that f$\alpha_n \notin$ B(fα, ϵ). Thus, $S(\alpha_n, \alpha_n, \alpha) < 1 + \frac{1}{n}$, but S(f$\alpha_n$, f$\alpha_n$, f$\alpha$) $\geq \epsilon$ for every $n \in \mathbb{N}$. By letting $n \to \infty$, we have $\lim_{n\to\infty} S(\alpha_n, \alpha_n, \alpha) = 1$, but (f$\alpha_n$) does not converge to f$\alpha$ in Y-contradiction and hence the result follows. □

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