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GENERALISED MATHEMATICAL MODEL FOR PERIODIC POINTS OF PERIOD 2^N AND THEIR ORIENTATION IN ONE DIMENSIONAL DISCRETE DYNAMICAL SYSTEM

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ABSTRACT. In this paper the mathematical model given by Dutta et al. [5] is given in generalised form and graph theoretic properties of the model is discussed using Horizontal visibility graph which throws a light on the orientation of periodic points in time series scenario.

First the set V_n [5] is considered in time series scenario and this set is denoted as I_n and I_n is defined as $I_n = V_{n,0} \cup V_{n,1} \cup V_{n,2} \cup \ldots \cup V_{n,m} \cup \ldots$, $m = 0, 1, 2, \ldots \infty$ where $V_{n,0}$ is the set V_n i.e $V_{n,0} = \{(a_1, 0), (a_2, 1), \ldots, (a_{2^n}, 2^n - 1)\}$ where each $a_i \in \{0, 1, 2, \ldots, 2^n - 1\}$. A mathematical modelling of I_n is given and each $V_{n,m}$ can be obtained from $V_{n-1,m}$ by using mathematical induction. A horizontal visibility graph is constructed in I_n and sufficient results have been given for calculating the degree of various elements of I_n.

1. INTRODUCTION

In recent years many authors are using Horizontal visibility graph to connect time series, non-linear dynamics and graph theory. HVG has been introduced by Lacasa [1] for mapping a time series by geometric criteria to a directed network. Lacasa et al [2]have highlighted a method which has helped to measure

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real valued time series irreversibility by combining two different tools the horizontal visibility algorithm and the Kullback-Leibler divergence. By this method they have mapped a time series to a directed network according to a geometric criterion. This method also has helped them to distinguish between reversible and irreversible stationary time series, including analytical and numerical studies of its performance. Ravetti et al. [3] have used Horizontal visibility graph algorithm to distinguish between deterministic and stochastic components in time series.. Weidong et al. [4] have measured the nonlinear interactions between non-stationary time series on multiple time scales. They have proposed a graph-theoretic method by the node degrees relationship in the context of horizontal visibility graphs (HVGs) which connects the gap among time series analysis, multiscale analysis, and graph theory.

Dutta et al. [5] have considered the set of periodic points of logistic map and obtained a mathematical model $'V_n'$ and defined Horizontal visibility graph (V_n, E_n). They have partitioned the set V_n in n level sets and derived important property [6] that no two elements in the same level set of V_n are adjacent. They have given sufficient results for finding degree of an element by which the orientation of a periodic points of period 2^n can be predicted.

2. Main results

2.1. Mathematical model of $\mathbf{V_n}$ [5] based on time series.

Here we consider the vertex set V_n [5] taken in infinite times i.e the periodic points are considered in time series scenario. Let the new set be I_n.

If $V_n = \{(a_1, 0), (a_2, 1), \ldots, (a_{2^n}, 2^n - 1)\}$ then I_n is defined as: $I_n = V_{n,0} \cup V_{n,1} \cup V_{n,2} \cup \ldots \cup V_{n,m} \cup \ldots$. Here $V_{n,0}$ is the same set as V_n, i.e., $V_{n,0} = \{(a_1, 0), (a_2, 1), \ldots, (a_{2^n}, 2^n - 1)\}$, where each $a_i \in \{0, 1, 2, \ldots, 2^n - 1\}$ and $V_{n,1} = \{(a_1 + 2^n, 0), (a_2 + 2^n, 1), \ldots, ((a_{2^n} + 2^n), 2^n - 1)\}$.

Similarly, $V_{n,2} = \{(a_1 + 2^n.2\, 0), (a_2 + 2^n.2\ \ 1), \ldots, (a_n + 2^n.2\ \ 2^n - 1))\}$.

In general $V_{n,m} = (a_1 + 2^n m\, 0), (a_2 + 2^n m, 1) + \ldots + (a_n + 2^n m\ \ 2^n - 1))$.

Each $V_{n,m}$ is defined with the help of Mathematical induction from $V_{n-1,m}$ in the following way:

Let, $V_{0,m} = \{(\ 2^n m, 0\ ), (1 + 2^n m, 1)\}$, $m = 0, 1, 2, \ldots$. Let $V_{n-1,m}$ contains 2^{n-1} points. Then $V_{n,m}$ is defined as follows:

$V_{n,1,m} = \{(2k, i) \mid (k, i)\ \in\ V_{n-1\ m}\ i = 0, 1, 2, \ldots, 2^{n-2} - 1\}$,

$V_{n,2,m} = \{(2k+4, i)|(k,i) \in \ V_{n-1\,m} \quad i = 2^{n-2}, \ldots, 2^{n-1}-2\}$,
$V_{n,3,m} = \{(2+2^n m \quad 2^{n-1}-1)\}$,
$V_{n,4,m} = \{(2k+1\,2^{n-1}+i)\ |(k,i) \ \in \ V_{n-1\,m} \ i = 0,1,2,\ldots,2^{n-1}-1\}$.
Then $V_{n,m} = \ \bigcup_{i=1}^{4} V_{n,i,m}$ and $I_n = \ \bigcup_m V_{n,m}$.

2.2. Definition of the graph $\mathbf{G_n(I_n,E_n)}$.

A horizontal visibility graph $G_n(I_n, E_n)$ is formed where the elements of I_n are considered as vertices of the Horizontal Visibility Graph and the edge set E_n[5] is taken as $E_n \ = \ \{((n_1, i),(n_2, j))| \ \mathrm{N}((n_1, i),(n_2, j)) \ = \ 0\}$, where $N((n_1, i),$ $(n_2, j))$ represents the number of elements of the form (k, k_1) such that $k_1 \geq i$ or j and $n_1 < k < n_2$.

2.3. Level q sets $(\mathbf{q \leq n+1})$ of $\mathbf{I_n = \ \cup V_{n,m}}$, $\mathbf{m = 0, 1, 2, \ldots}$.

There exist n level sets [6] of V_n where the elements of each level sets is defined. Since $(i+2^n m, j)$ exists in I_n iff (i, j) exists in $V_{n,0}$ i.e V_n so level sets of I_n can be defined as

$$I_{n,1} = \{(i+2^n m, j)|\,(i,j) \in \mathit{level}\ 1\ \mathit{set\ of}\ V_{\mathrm{n}}\ ,\ m = 0,1,2,\ldots\}$$

$$I_{n,2} = \{(i+2^n m, j)|\,(i,j) \in \mathit{level}\ 2\ \mathit{set\ of}\ V_n\ \ m = 0,1,2,3,\ldots\}$$

$$\vdots$$

$$I_{n,n} = \{(i+2^n m, j)|\,(i,j) \in \mathit{level}\ n\ \mathit{set\ of}\ V_{\mathrm{n}}\ ,\ m = 0,1,2,3,\ldots\}$$

$$I_{n,(n+1)} = \{(i+2^n m, j)|\,(i,j) \in \mathit{level}\ (n+1)\ \mathit{set\ of}\ V_{\mathrm{n}}\ ,\ m = 0,1,2,3,\ldots\}.$$

These $I_{n,1}$, $I_{n,2}$, $\ldots$, $I_{n,n}$, $I_{n,(n+1)}$ sets are called level $1, 2, \ldots, (n+1)$ sets of I_n.

Theorem 2.1. *There exist edge between* $(2^n-1+2^n i \ \ 2^n-1)$ *and* $(2^n-1+2^n j \ \ 2^n-1)$ *if and only if i and* j *are consecutive.*

Proof. The element $(2^n-1, 2^n-1)$ [5] exists in V_n. Therefore $(2^n-1+2^n m, 2^n-1$ also exists in I_n. In $V_{n,0}$ there are 2^n-1 elements. For each m $V_{n,1} \ \ V_{n,2}$ ... $V_{n,m}$ sets also contain 2^n-1 elements. There exist edge between $(2^n-1+2^n(m-1),$ $2^n-1)$ and $(2^n-1+2^n m 2^n-1)$ because there does not exist any (i, j) where [$2^n-1+2^n\,(m-1)] < i < [2^n-1+2^n m]$ by which $(2^n-1+2^n m\, 2^n-1)$ is adjacent as $j < 2^n-1$. So it can be said that $(2^n-1+2^n i, 2^n-1)$ and $(2^n-1+2^n j, 2^n-1)$ are adjacent if i and j are consecutive.

Conversely let there exist edge between $(2^n-1+2^n i \ \ 2^n-1)$ and $(2^n-1+$ $2^n j \ \ 2^n-1)$. To show i and j are consecutive. On the contrary we assume that there exist edge between $(2^n - 1 + 2^n i \;\; 2^n - 1)$ and $(2^n - 1 + 2^n j \, 2^n - 1)$ when i and j are not consecutive. So, there exist one p, where p, j are consecutive and $2^n - 1 + 2^n i \; < 2^n - 1 + 2^n p \; < 2^n - 1 + 2^n j$. Thus, there will be edge between $(2^n - 1 + 2^n p, 2^n - 1)$ and $(2^n - 1 + 2^n j, 2^n - 1)$ and hence the theorem. $\square$

Theorem 2.2. *There can not exist any edge between* (i, j) *and* (p, q) *where* $(i, j) \in V_{n,m}$ *and* $(p, q) \in V_{n,m+1}$ *and* $i < 2^n - 1 + 2^n m$ *and* $p < 2^n - 1 + 2^n(m + 1)$.

Proof. Since $(2^n - 1 + 2^n m \, 2^n - 1) \in V_{n,m}$ and $(2^n - 1 + 2^n(m + 1)) \, 2^n - 1) \in V_{n,(m+1)}$ so there can not exist any edge between $(i, j) \in V_{n,m}$ and $(p, q) \in V_{n,(m+1)}$ where $i < 2^n - 1 + 2^n m$ and $p < 2^n - 1 + 2^n \, (m + 1)$ as both j, q is less than $2^n - 1$ and hence the theorem. $\square$

Theorem 2.3. *Degree of* $(2^n - 1 + 2^n m \;\; 2^n - 1)$ *is* $2n + 2 \;\; m > 0$ *and* $2n + 1$ *when* $m = 0$.

Proof. First we find those elements (x, y) where x is less than $(2^n - 1 + 2^n m)$ and by which $(2^n - 1 + 2^n m \;\; 2^n - 1)$ is adjacent.

For this we first show that $(2^n - 1 + 2^n m - 2^{p-1} \; , \; j_p \;) \in$ level p set for $p = 1, 2, 3, \ldots, n$.

When $p = 1$, $2^n - 1 + 2^n m - 2^{p-1} = 2^n - 1 - 2^0 + 2^n m = 2^n - 1 - 1 + 2^n m = 1 + 2 + 2^2 + \cdots + 2^{n-1} - 1 + 2^n m = 2 + 2^2 + \cdots + 2^{n-1} + 2^n m = 2[1 + 2 + \cdots + 2^{n-2} + 2^n m]$.

Therefore, $(2^n - 1 + 2^n m - 1, \; j_1 \;) \in$ level 1 set.

When $p = 2$, $2^n - 1 + 2^n m - 2^{p-1} = 2^n - 1 - 2 + 2^n m = 2^n - 1 - 2 + 2^n m = 1 + 2 + 2^2 + \cdots + 2^{n-1} - 2 + 2^n m = 1 + 2^2 + \cdots + 2^{n-1} + 2^n m = 1 + 2^2 \, [1 + 2 + \cdots + 2^{n-3}] + 2^n m$.

Therefore, $(2^n - 1 + 2^n m - 2, \; j_1 \;) \in$ level 2 set.

In a similar manner when $p = n$ we have,

$$
\begin{aligned}
& 2^n - 1 + 2^n m - 2^{n-1} = \; 2^n - 1 - 2^{n-1} + 2^n m \\
&= \left(1 + 2 + 2^2 + \cdots + 2^{n-1} \right) - 2^{n-1} + 2^n m \\
&= 1 + 2 + 2^2 + \cdots + 2^{n-2} + 2^n m \\
&= \left(1 + 2 + 2^2 + \cdots + 2^{n-2} \right) + 2^n m.
\end{aligned}
$$

So, $(2^n - 1 + 2^n m - 2^{n-1} \; , \; j_n) \in$ level n set.

Thus, for $p = 1, 2, 3, \ldots, n$, $(2^n - 1 + 2^n m - 2^{p-1} \; , \; j_p \;) \in$ level p set.

Another element of level p set is either $(2^n - 1 + 2^n m - 2^{p-1} - 2^p \;\; b_{k_1})$ or $(2^n - 1 + 2^n m - 2^{p-1} + 2^p \;\; b_{k2})$ for some $b_{k_1}$and b_{k_2}.

Now,

$$2^n - 1 + 2^n m - 2^{p-1} - 2^p < 2^n - 1 + 2^n m - 2^{p-1}.$$

But

$$2^n - 1 + 2^n m - 2^{p-1} + 2^p = 2^n - 1 + 2^n m - 2^{p-1}(1-2)$$
$$=2^n - 1 + 2^n m + 2^{p-1} > \ 2^n - 1 + 2^n m.$$

So, $(2^n - 1 + 2^n m \ \ 2^n - 1)$ is adjacent with $(2^n - 1 + 2^n m - 2^{p-1}\ ,\ j\ )$, for some j, where $2^n - 1 + 2^n m - 2^{p-1}$ is less than $2^n - 1 + 2^n m$.

Exactly in a similar manner it can be said that $(2^n - 1 + 2^n m \ \ 2^n - 1)$ is adjacent with $(2^n - 1 + 2^n m + 2^{p-1}\ ,\ t_p)$ for $p = 1, 2, \ldots n$ where $2^n - 1 + 2^n m + 2^{p-1}$ is greater than $2^n - 1 + 2^n m$.

So, degree of $(2^n - 1 + 2^n m \ 2^n - 1) \geq 2n$.

By Theorem 2.1, $(2^n - 1 + 2^n m \ 2^n - 1)$ is adjacent with $(2^n - 1 + 2^n(m-1)2^n - 1)$ and $(2^n - 1 + 2^n(m+1) \ 2^n - 1)$which lie in level $(n+1)$ set. So degree of $(2^n - 1 + 2^n m \ 2^n - 1)$ is $2n+2$.

When $m = 0$, $(2^n - 1 + 2^n m \ 2^n - 1)$ is adjacent with $(2^n - 1 + 2^n(m+1) \ 2^n - 1)$ and hence degree $2n+1$ and hence the theorem. $\square$

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