A PROOF FOR THE VOLUME OF SOLIDS REVOLUTION IN POLAR COORDINATES

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ABSTRACT. The purpose of this note is to give another proof for the volume of solids revolution in polar coordinates.

1. INTRODUCTION

We know that the volume of solids revolution in polar coordinates given as follow (see e.g., [1]):

Theorem 1.1. *The volume (V) of the solid generated by the revolution of the area bounded by the curve $r = r(\theta)$ and radii vectors $\theta = \theta_1$, $\theta = \theta_2$.*

(i) about the initial line $\theta = 0$ (i.e., the x-axis) is

$$V = \int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin\theta d\theta.$$

(ii) about the line $\theta = \dfrac{\pi}{2}$ (i.e., the y-axis) is

$$V = \int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \cos\theta d\theta.$$

(iii) about any line $\theta = \gamma$ is

$$V = \int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin(\theta - \gamma)d\theta.$$

Most of the literature does not provide proof of this theorem. However, one can also prove it by using triple integrals. It may come as a surprise that the almost trivial below prove has not been given before.

2. PROOF OF THEOREM 1.1

For the case (i), consider the volume as the sum of the lateral areas of infinitesimally thin cones (see Figure 1).
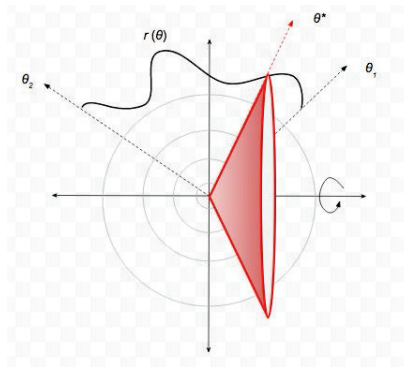

FIGURE 1.

More precisely, the "area" is really the volume of a hollowed-out cone
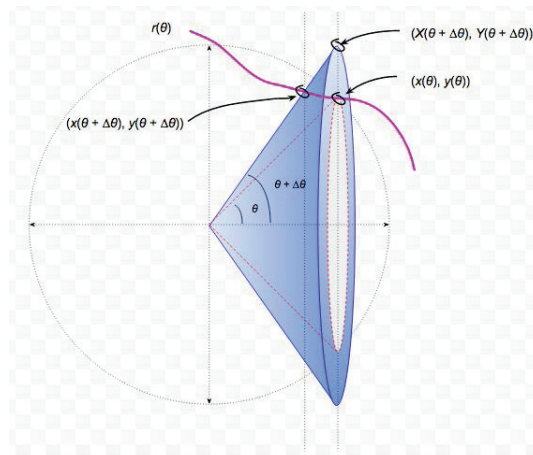

FIGURE 2.

Recall that the volume of the cone with base radius $y(\theta)$ and slant height $h = \sqrt{r(\theta)^2 - y(\theta)^2}$ is

$$V = \frac{\pi}{3} y(\theta)^2 \sqrt{r(\theta)^2 - y(\theta)^2}.$$

Then, we have

$$\Delta V = \frac{\pi}{3} \left(Y(\theta + \Delta\theta)^2 \sqrt{R(\theta + \Delta\theta)^2 - Y(\theta + \Delta\theta)^2} - y(\theta)^2 \sqrt{r(\theta)^2 - y(\theta)^2} \right),$$

where $R(\theta + \Delta\theta)$ is the slant height of the hollowed-out cone, the distance from the origin to the point $(X(\theta + \Delta\theta); Y(\theta + \Delta\theta))$ (see Figure 2 above).

On the other hand, we have

$$\begin{cases} X(\theta + \Delta\theta) & = x(\theta), \\ Y(\theta + \Delta\theta) & = r(\theta)\cos\theta\tan(\theta + \Delta\theta), \\ R(\theta) & = r(\theta)\cos\theta\sec(\theta + \Delta\theta). \end{cases}$$

So, we get

$$\begin{aligned}
\Delta V &= \frac{\pi}{3} \Bigg(r(\theta)^2 \cos^2\theta \tan^2(\theta + \Delta\theta) \\
&\quad \cdot \sqrt{r(\theta)^2 \cos^2\theta \sec^2(\theta + \Delta\theta) - r(\theta)^2 \cos^2\theta \tan^2(\theta + \Delta\theta)} \\
&\quad - r(\theta)^2 \sin^2\theta \sqrt{r(\theta)^2 - r(\theta)^2 \sin^2\theta} \Bigg) \\
&= \frac{\pi}{3} r(\theta)^3 \left(\cos^3\theta \tan^2(\theta + \Delta\theta) - \sin^2\theta \cos\theta \right) \\
&= \frac{\pi}{3} r(\theta)^3 \left(\cos^3\theta \tan^2(\theta + \Delta\theta) - \cos^3\theta \tan^2\theta \right) \\
&= \frac{\pi}{3} r(\theta)^3 \cos^3\theta \left(\tan^2(\theta + \Delta\theta) - \tan^2\theta \right).
\end{aligned}$$

This follows that

$$\begin{aligned}
dV &= \lim_{\Delta\theta \to 0} \frac{\Delta V}{\Delta\theta} = \frac{\frac{\pi}{3} r(\theta)^3 \cos^3\theta \left(\tan^2(\theta + \Delta\theta) - \tan^2\theta \right)}{\Delta\theta} \\
&= \frac{\pi}{3} r(\theta)^3 \cos^3\theta \lim_{\Delta\theta \to 0} \frac{\tan^2(\theta + \Delta\theta) - \tan^2\theta}{\Delta\theta} \\
&= \frac{\pi}{3} r(\theta)^3 \cos^3\theta \, d(\tan^2\theta) \, d\theta = \frac{2\pi}{3} r(\theta)^3 \sin\theta \, d\theta.
\end{aligned}$$

Therefore, we conclude that

$$V = \int_{\theta_1}^{\theta_2} dV = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r(\theta)^3 \sin\theta \, d\theta.$$

Thus, the case (i) is proved. The formula for (ii) can be derived in a similar fashion.

We now prove for the case (iii). The ray $\alpha = \gamma$ coincides with the line $y = \tan(\gamma).x$. Let $\ell(\alpha)$ be the polar representation of the line perpendicular to $\alpha = \gamma$ passing through $(x(\theta); y(\theta))$ (see Figure 3).
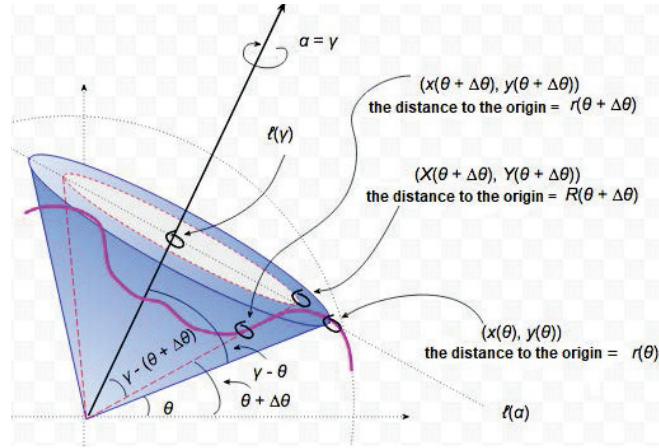

FIGURE 3.

Since the slope of $\ell(\alpha)$ is $-\cot(\gamma)$, it has equation

$$y - y(\theta) = -\cot(\gamma)(x - x(\theta))$$
$$\text{or}\quad \ell(\alpha)\sin\alpha - r(\theta)\sin\theta = -\cot(\gamma)[\ell(\alpha)\cos\alpha - r(\theta)\cos\theta]$$
$$\text{or}\quad \ell(\alpha) = r(\theta)\frac{\cos(\theta - \gamma)}{\cos(\alpha - \gamma)}.$$

So, we deduce that

$$\ell(\gamma) = r(\theta)\cos(\theta - \gamma)$$
$$\ell(\theta + \Delta\theta) = R(\theta + \Delta\theta) = r(\theta)\frac{\cos(\theta - \gamma)}{\cos((\theta + \Delta\theta) - \gamma)}.$$

From Figure 3, we see that the radius R_{out} of the outer blue cone is the distance from $(x(\theta); y(\theta))$ to $\ell(\gamma)$ and that the radius R_{in} of the inner cone that is cut out is the distance from $(X(\theta + \Delta\theta); Y(\theta + \Delta\theta))$ to $\ell(\gamma)$. Then, by using the law of

cosines, we have the following:

$$
\begin{aligned}
R_{\text{out}}^2 &= r(\theta)^2 + \ell(\gamma)^2 - 2r(\theta)\ell(\gamma)\cos(\theta - \gamma) \\
&= r(\theta)^2 \sin^2(\theta - \gamma) \\
&= r(\theta)^2 \cos^2(\theta - \gamma)\tan^2(\theta - \gamma),
\end{aligned}
$$

$$
\begin{aligned}
R_{\text{in}}^2 &= R(\theta + \Delta\theta)^2 + \ell(\gamma)^2 - 2R(\theta + \Delta\theta)\ell(\gamma)\cos((\theta + \Delta\theta) - \gamma) \\
&= R(\theta)^2 \cos^2(\theta - \gamma)\tan^2(\theta - \gamma).
\end{aligned}
$$

Putting everything together, the volume of the hollowed-out cone is

$$
\Delta V = \frac{\pi}{3}\left(R_{\text{out}}^2\sqrt{r(\theta)^2 - R_{\text{out}}^2} - R_{\text{in}}^2\sqrt{R(\theta + \Delta\theta)^2 - R_{\text{in}}^2}\right).
$$

Then, similarly as the case (i), we obtain

$$
dV = \lim_{\Delta\theta \to 0}\frac{\Delta V}{\Delta\theta} = \frac{\pi}{3}r(\theta)^3\cos^3(\theta - \gamma)d(\tan^2(\theta - \gamma))\,d\theta = \frac{2\pi}{3}r(\theta)^3\sin(\theta - \gamma)\,d\theta.
$$

Thus, we get the formula

$$
V = \int_{\theta_1}^{\theta_2} dV = \int_{\theta_1}^{\theta_2}\frac{2\pi}{3}r(\theta)^3\sin(\theta - \gamma)\,d\theta.
$$

This completes the proof of the theorem.

REFERENCES

[1] A. K. SHARMA: *Application Of Integral Calculus*, Discovery Publishing House, India, 2005.

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