

## NUMERICAL SOLUTION FOR DUFFING-VAN DER POL OSCILLATOR VIA BLOCK METHOD

AHMAD FADLY NURULLAH RASEDEE<sup>1</sup>, MOHAMAD HASSAN ABDUL SATHAR, TZE JIN WONG,  
LEE FENG KOO, AND NUR AINNA RAMLI

**ABSTRACT.** The Duffing-Van Der Pol Oscillators is an interesting differential problem with many applications in mechanics and fluid dynamics. Among its most known application involves the chaotic motions in periodic-self excited oscillators and occasionally used to model hydrodynamic forces on canonical structures. The current work aims to provide a numerical solution for the Duffing-Van Der Pol Oscillators in second order differential form using a two-point block method. The two-point block algorithm will be coded in C programming to minimize computational cost (calculation time). Results will show that the proposed method is a practical tool for solving nonlinear differential equation.

### 1. INTRODUCTION

The development of multistep method in solving ordinary differential equation (ODEs) throughout the years has been immense. These developments include techniques for solving higher order ODES. Previous methods would propose solving these higher order ODEs by reducing them to a system of ODEs. These reduction methods were considered as robust until authors such as Gear [1], Krogh [2] and Suleiman [3] proposed techniques for solving these problems directly. Gear [1] provided a Nordsiek version of the multistep method for solving

<sup>1</sup>*corresponding author*

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two second order problems. This was then rivalled by Krogh [2], when he introduced a variation of the divided difference approach. In his research, Krogh suggests that the back values were to be interpolated for any point of the derivative. The works in Krogh [2] was then extended by Suleiman [3] using an adaption of the divided difference approach for solving higher order nonstiff ODEs (dth order) which is currently known as the Direct Integration (DI) method. In [4], Rasedee then developed a one-point predict-corrector algorithm in backward difference form for solving higher order ODEs directly. His work was then continued by Md Ijam et al. [5]. Md Ijam converted Rasedee's [4] backward difference method into a two-point sequential block algorithm. Since then, many variation works derived from Suleiman's initial algorithm can be found. These works include Suleiman et al. [6], Rasedee et al. [7] and others.

## 2. METHODOLOGY

Although the Duffing-Van Der Pol oscillator can be used to model of hydrodynamic forces on canonical structures in fluid dynamics, it is frequently related with the study of chaotic motion of nonlinear periodic self-excited oscillators with applications commonly found in electrical circuits. The Duffing-Van Der Pol oscillator has the following general form

$$\frac{d^2y}{dt^2} - \alpha (1 - t^2) \frac{dy}{dt} + \omega_0 t + \delta t^3 = f(t),$$

given then constants  $\alpha$ ,  $\omega_0$  and  $\delta$  where  $\alpha > 0$  and  $f(t)$  function in time.

The current research will establish a two-point block method for solving higher order ODEs in the form of second order Duffing-Van Der Pol oscillator. Consider the second order ODE

$$(2.1) \quad y'' = f(t, y, y'),$$

with  $t \in [a, b]$  and the initial condition

$$y(a) = \eta, \quad y'(a) = \eta'.$$

The proposed two-point block method is constructed in an explicit-implicit mode where the explicit coefficients acts as the predictor and the implicit coefficients acts as the corrector. Construction of the predictor-corrector algorithm begins with the

derivation of the predictor. The derivation of the predictor begins by integrating (2.1) once. Integrating (2.1) once, with the limit of integration  $t_n$  to  $t_{n+b}$ , gives

$$(2.2) \quad y'(t_{n+b}) = y'(t_n) + h \sum_{j=0}^{k-1} p\alpha_{b,1,j} \nabla^j f_n, \quad p\alpha_{b,1,j} = (-1)^j \int_0^b \binom{-s}{j} ds,$$

where  $b = 1, 2$  and  $f(t, y, y')$  is approximated by the Newton Gregory polynomial

$$(2.3) \quad P_n(t) = \sum_{j=0}^{k-1} (-1)^j \binom{-s}{j} \nabla^j f_n, \quad s = \frac{t - t_n}{h},$$

at  $k$  back values. Next, we denote the generating function by  ${}_pG_{b,1}(t)$  where

$$(2.4) \quad {}_pG_{b,1}(t) = \sum_{j=0}^{\infty} p\alpha_{b,1,j} t^j.$$

Then  $\alpha_{b,1,j}$  from (2.2) is replaced with (2.3) thus

$${}_pG_{b,1}(t) = \sum_{j=0}^{\infty} (-t)^j \int_0^b \binom{-s}{j} ds.$$

By mathematical inference, the first and second point explicit generating function can be established as follows,

$$(2.5) \quad {}_pG_{b,1}(t) = - \left[ \frac{(1-t)^{-b}}{\log(1-t)} - \frac{1}{\log(1-t)} \right],$$

Subsequent to the first integration, equation (2.1) is then integrated twice in similar manner to prior integration, resulting in

$$y(t_{n+b}) = y(t_n) + hy'(t_n) + h^2 \sum_{j=0}^{k-1} p\alpha_{b,2,j} \nabla^j f_n,$$

where

$$p\alpha_{b,2,j} = \int_0^b \frac{(b-s)}{1!} \binom{-s}{j} ds.$$

Then, the generating function of the second integration is denoted by

$${}_pG_{b,2}(t) = \sum_{j=0}^{\infty} p\alpha_{b,2,j} t^j.$$

Replacing  $\alpha_{b,2,j}$  in  ${}_pG_{2,1}$  and consequently solving the integral

$${}_pG_{b,2}(t) = \sum_{j=0}^{\infty} (-t)^j \int_0^b (b-s) \binom{-s}{j} ds$$

yields the following

$$(2.6) \quad {}_pG_{b,2}(t) = \left[ \frac{1}{\log(1-t)} - \frac{{}_pG_{b,1}(t)}{\log(1-t)} \right].$$

As for the latter, the corrector can be established similar to the predictor with minor differences. This include changing the limit of integration to  $t \in [-b, 0]$  and denoting the Newton -Gregory polynomial as

$$P_n(t) = \sum_{j=0}^k (-1)^j \binom{-s}{j} \nabla^j f_{n+b}, \quad s = \frac{t - t_{n+b}}{h}.$$

This produces the following corrector formula

$$y(t_{n+b}) = y(t_n) + hy'(t_n) + h^2 \sum_{j=0}^k c\alpha_{b,2,j} \nabla^j f_{n+b},$$

with the corresponding implicit generating function

$${}_cG_{b,1}(t) = \sum_{j=0}^{\infty} c\alpha_{b,2,j} t^j,$$

and by mathematical induction can be written as

$$(2.7) \quad {}_cG_{b,1}(t) = - \left[ \frac{1}{\log(1-t)} - \frac{(1-t)^b}{\log(1-t)} \right].$$

By repeating similar process for the second integration establishes the following

$$(2.8) \quad {}_cG_{b,2}(t) = \left[ \frac{2(1-t)^b}{\log(1-t)} - \frac{{}_cG_{b,1}(t)}{\log(1-t)} \right].$$

### 3. THE COEFFICIENTS

The purpose of a two-point block method is to reduce computational cost. In line with reducing the computational cost, establishing a recursive relationship between explicit and implicit integration coefficients allows authors to reduce numerous lines of computational code. Next, we derive the relationship between the explicit and implicit coefficients. Firstly, the first order implicit integration generating functions from (2.7) are rearranged as follows

$$(3.1) \quad \begin{aligned} {}_cG_{1,1}(t) &= -(1-t) \left[ \frac{(1-t)^{-1}}{\log(1-t)} - \frac{1}{\log(1-t)} \right], \\ {}_cG_{2,1}(t) &= -(1-t)^2 \left[ \frac{(1-t)^{-2}}{\log(1-t)} - \frac{1}{\log(1-t)} \right]. \end{aligned}$$

Next, the first order explicit generating functions from (2.4) and (2.5) are substituted in (2.8) respectively, thus providing a recursive relationship as shown below

$${}_cG_{1,1}(t) = (1-t) {}_pG_{1,1}(t), \quad {}_cG_{2,1}(t) = (1-t) {}_pG_{2,1}(t).$$

Then the second order explicit generating functions (equation (2.6)) and the second order implicit generating functions (equations (2.8)) are rewritten in similar nature to its first order counterpart, hence establishing the following relationship

$${}_cG_{1,2}(t) = (1-t) {}_pG_{1,2}(t), \quad {}_cG_{2,2}(t) = (1-t) {}_pG_{2,2}(t).$$

The integration coefficients can then be established with expanding the generating functions(3.1) by way of (2.3) and (2.6) then rewritten as

$$\sum_{j=0}^k c\alpha_{b,2,j} = p\alpha_{b,2,j}.$$

More detailed discussions and example of coefficients can be found in the works by Rasedee et al. [8]. The upcoming section will present numerical results of the proposed method.

## 4. RESULTS AND DISCUSSIONS

Most real-life applications of engineering system behaviour are in the form of nonlinear processes which nearly impossible to be solve analytically. In this research, examples selected are of coupled vibrations in the form of nonlinear Duffing-Van Der Pol oscillators. The examples selected are of various types of Duffing-Van Der Pol oscillators to highlight the accuracy of the method.

**Example 1.** *Nonlinear oscillator,*

$$\frac{d^2y(t)}{dt^2} = - (1 + y^2(t)) \frac{dy(t)}{dt} - y(t) + 2\cos(t) - \cos^2(t),$$

with the following initial conditions

$$y(0) = 0, \quad \frac{dy(0)}{dt} = 1,$$

and the analytic solution given by

$$y(t) = \cos(t).$$

Example 1 is a non-homogeneous Duffing-Van Der Pol oscillator with an analytical solution. It is selected as a control environment to validate the accuracy of the proposed method. Numerical results for Example 1 are illustrated in Table 1. Results displayed in Table 1 compares the proposed two point backward difference method (2PB) with the Alternative Variation Iteration method (AVI). The AVI which was established in [9] has proven to be successful in approximating solutions for Duffing-Van Der Pol oscillator. Results show that AVI manages to obtain a better approximate for  $t = 0.1$  and  $t = 0.2$  but is outperformed by the 2PB method thereafter.

**Example 2.** *Nonlinear oscillator,*

$$\frac{d^2y(t)}{dt^2} = \frac{1}{10} (1 - y^2(t)) \frac{dy(t)}{dt} - y(t) - \frac{1}{100}y^3(t),$$

with the following initial conditions

$$y(0) = 2, \quad \frac{dy(0)}{dt} = 0.$$

Example 2 is a homogeneous nonlinear oscillator without any known analytical solution which was obtained from [8]. From Table 1, the absolute error provided substantiate the reliability of the 2PB method. To further explore the validity of

TABLE 1. Comparison of numerical results for Problem 1 between AVI and 2BP method

t	AVI		2PB	
	Approximation	Absolute Error	Approximation	Absolute Error
0.1	0.099833503766	8.71190E-8	0.09983108741	2.32923E-6
0.2	0.198672306412	2.97562E-6	0.19866463190	4.69889E-6
0.3	0.295544761414	2.45548E-5	0.29551318542	7.02124E-6
0.4	0.389532415382	1.14073E-4	0.38940906938	9.27293E-6
0.5	0.479813376489	3.87838E-4	0.47941410725	1.14314E-5
0.6	0.565724718495	1.08225E-3	0.56462899848	1.34749E-4
0.7	0.646848217682	2.63053E-3	0.64420230404	1.53832E-4
0.8	0.723119718202	5.76363E-3	0.71733895367	1.71372E-5
0.9	0.794955142884	1.16282E-2	0.78330819004	1.87196E-5
1.0	0.863375606677	2.19046E-2	0.84145087018	2.01146E-5

its accuracy, for Problem 2 the approximated solution attained by the 2PB method is then compared with the Lindstedts method (LSD) and one point backward difference method (1PB). Numerical approximation presented in Table 2 shows that accuracy of the 2PB method is as competitive as its numerical counterpart.

**Example 3.** *Nonlinear oscillator;*

$$\frac{d^2y(t)}{dt^2} = \cos(0.7t) + (1 - y^2(t)) \frac{dy(t)}{dt} + y(t) - y^3(t),$$

with the following initial conditions

$$y(0) = 0.1, \quad \frac{dy(0)}{dt} = -0.2.$$

Example 3 is a nonhomogeneous nonlinear oscillator without any known analytical solution. For this particular problem, the divide difference direct integration method (DD) and 1PB method were selected for comparison. These methods were selected because of their similar nature with the 2PB. Both methods are also variation of the multistep method programmed in predictor-corrector mode. This allows for a more well-matched comparison. Table 3 exhibits numerical result for DD, 1PB and 2PB method for the time step,  $h = 0.5$  in the interval  $0 \leq t \leq 4.5$ . The approximation of the 2PB shows to be comparable at every time step to its numerical peers. Next, Figure 1 displays the plotted graph of the 2PB method for every

TABLE  
2. Comparison  
of numerical approx-  
imation for Problem  
2

t	LSD	1PB	2PB
0.1	1.98971	1.98971	1.98971
0.2	1.95936	1.95936	1.95936
0.3	1.90980	1.90980	1.90980
0.4	1.84202	1.84202	1.84202
0.5	1.75702	1.75702	1.75702
0.6	1.65586	1.65586	1.65586
0.7	1.53958	1.53958	1.53958
0.8	1.40922	1.40923	1.40923
0.9	1.26581	1.26586	1.26586
1.0	1.11033	1.11054	1.11054

TABLE  
3. Comparison  
of numerical approx-  
imation for Problem  
3

t	LSD	1PB	2PB
0.0	0.1	0.1	0.1
0.5	0.13038	0.13047	0.13050
1.0	0.52399	0.52428	0.52433
1.5	1.33649	1.33689	1.33685
2.0	1.74809	1.74814	1.74806
2.5	1.42639	1.42613	1.42613
3.0	0.88625	0.88594	0.88594
3.5	0.16497	0.16456	1.64555
4.0	-1.12614	-1.12676	-1.12676
4.5	-2.08061	-2.08006	-2.08006

point  $t$  paired with its corresponding approximated solution  $y$  whereas Figure 2 plots approximated points of  $y$  and the corresponding approximated  $y'$ . To provide a more comprehensive understanding of the oscillatory nature of the Duffing Van Der Pol oscillator, the approximation interval is extended to  $0 \leq t \leq 200$ .

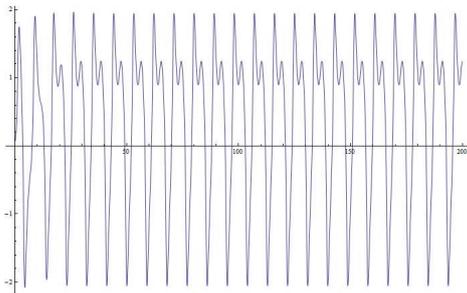


FIGURE 1. Numerical approximation of  $y$  for every time step  $t$

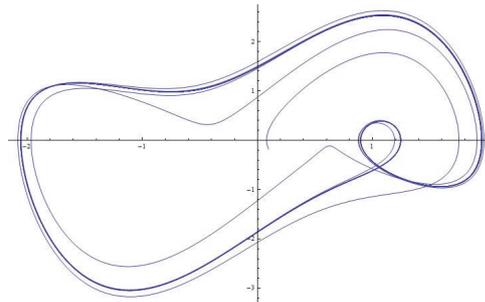


FIGURE 2. Numerical approximation of  $y'$  for against approximation of  $y$ .

## 5. CONCLUSIONS

The 2PB method is feasible method for solving ODEs with period solutions noticeably the Duffing-Van Der Pol oscillator. Numerical results prove the accuracy of the 2PB method, which increases significantly when using finer time steps. Notes on convergence of the 2PB method can be found in [10].

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FAKULTI EKONOMI DAN MUAMALAT, UNIVERSITI SAINS ISLAM MALAYSIA,  
78100 NILAI, NEGERI SEMBILAN, MALAYSIA.  
*Email address:* fadlynurullah@usim.edu.my, nurainna.ramli@usim.edu.my

CENTRE OF FOUNDATION STUDIES FOR AGRICULTURAL SCIENCE, UNIVERSITI PUTRA MALAYSIA,  
43400 UPM SERDANG, SELANGOR, MALAYSIA.  
*Email address:* mohdhasan@upm.edu.my

DEPARTMENT OF BASIC SCIENCES AND ENGINEERING,  
FACULTY OF AGRICULTURE AND FOOD SCIENCE, UNIVERSITI PUTRA MALAYSIA,  
BINTULU CAMPUS, 97008 BINTULU, SARAWAK, MALAYSIA.  
*Email address:* w.tzejin@upm.edu.my

DEPARTMENT OF BASIC SCIENCES AND ENGINEERING,  
FACULTY OF AGRICULTURE AND FOOD SCIENCE, UNIVERSITI PUTRA MALAYSIA,  
BINTULU CAMPUS, 97008 BINTULU, SARAWAK, MALAYSIA.  
*Email address:* leefeng@upm.edu.my

FAKULTI EKONOMI DAN MUAMALAT, UNIVERSITI SAINS ISLAM MALAYSIA,  
78100 NILAI, NEGERI SEMBILAN, MALAYSIA.  
*Email address:* nurainna.ramli@usim.edu.my