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# $(m,q)$-COMPLEX SYMMETRIC TRANSFORMATIONS ON A COMPLEX-VALUED METRIC SPACE

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ABSTRACT. In this paper, we introduce and studied the concept of $(m,q)$-complex symmetric transformation on a complex valued metric space. A self transformation $\Phi$ on a complex-valued metric space $(\mathbf{E},\mathbf{d}_{\mathbb{C}})$ is said to be an $(m,q)$-complex symmetric transformation if $\Phi$ satisfies for all $u,v\in\mathbf{E}$,

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{m-k}v\big)^q=0,$$

for some positive integer $m$ and a real number $q\in(0,\ \infty)$.

## 1. INTRODUCTION

The authors in [2] has introduced the notion of complex valued metric space. They had defined a partial order $\precsim$ over the set of complex numbers $\mathbb{C}$ as follows: let $u,v\in\mathbb{C}$,

$$u\precsim v \text{ if and only if}\begin{cases}\Re e(u)\leq\Re e(v)\\ \\ \Im m(u)\leq\Im m(v),\end{cases}.$$

It was observed that

$$u\precsim v\Longleftrightarrow\begin{cases}\Re e(u)=\Re e(v);\\ \\ \Im m(u)<\Im m(v),\end{cases}\qquad u\precsim v\Longleftrightarrow\begin{cases}\Re e(u)<\Re e(v);\\ \\ \Im m(u)=\Im m(v)\end{cases}$$

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$$u \precnsim v \Longleftrightarrow \begin{cases} \Re e(u) < \Re e(v); \\ \\ \Im m(u) < \Im m(v) \end{cases} \qquad u \precsim v \Longleftrightarrow \begin{cases} \Re e(u) = \Re e(v); \\ \\ \Im m(u) = \Im m(v). \end{cases}$$

In a similar way as in real metric space, the concept of complex valued metric space has defined as follows:

**Definition 1.1.** *(see [2]). Let* $\mathbf{E}$ *be a nonempty set. A map* $\mathbf{d}_{\mathbb{C}} : \mathbf{E} \times \mathbf{E} \to \mathbb{C}$ *is called a complex-valued metric on* $\mathbf{E}$*, if it satisfies the following conditions:*

$$\begin{cases} \text{(i)} & 0 \precsim \mathbf{d}_{\mathbb{C}}(u,v) \quad \textit{for all} \quad u,v \in \mathbf{E} \\ \\ \text{(ii)} & \mathbf{d}_{\mathbb{C}}(u,v) = 0 \Longleftrightarrow u = v. \\ \\ \text{(iii)} & \mathbf{d}_{\mathbb{C}}(u,v) = d_{\mathbb{C}}(v,u) \quad \textit{for all } u,v \in \mathbf{E}, \\ \\ \text{(iv)} & \mathbf{d}_{\mathbb{C}}(u,v) \precsim \mathbf{d}_{\mathbb{C}}(u,w) + \mathbf{d}_{\mathbb{C}}(w,v) \quad \textit{for all } u,v,w \in \mathbf{E}. \end{cases}$$

The pair $(\mathbf{E}, \mathbf{d}_{\mathbb{C}})$ is called a complex-valued metric space.

Let $(\mathbf{E},\ \mathbf{d_E})$ and $(\mathbf{F},\ \mathbf{d_F})$ be a metric spaces. A map $\Phi : (\mathbf{E},\ \mathbf{d_E}) \to (\mathbf{F},\ \mathbf{d_F})$ is said to be an isometry if $\Phi$ satisfies

$$\mathbf{d_F}\big(\Phi u,\ \Phi v\big) = \mathbf{d_E}\big(u,\ v\big),\ \ \forall\, u,v \in E.$$

Recall that a bounded linear transformation $\Phi : \mathbf{H} \longrightarrow \mathbf{H}$ when $\mathbf{H}$ is a Hilbert space is called

(1) $m$-isometry if

$$\Phi^{*m}\Phi^m - \binom{m}{1}\Phi^{*m-1}\Phi^{m-1} + \cdots + (-1)^{m-1}\binom{m}{1}\Phi^*\Phi + (-1)^m I_{\mathbf{H}} = 0, \tag{1.1}$$

( see [1]).

(2) $m$-symmetric if ( [7])

$$\Phi^m - \binom{m}{m-1}\Phi^*\Phi^{m-1} + \cdots + (-1)^{m-1}\binom{m}{1}\Phi^{*m-1}\Phi + (-1)^m\Phi^{*m} = 0. \tag{1.2}$$

$(3)$ $m$-complex symmetric if

$$C\Phi^m C - \binom{m}{m-1}\Phi^* C\Phi^{m-1}C + \cdots + (-1)^{m-1}\binom{m}{1}\Phi^{*m-1}C\Phi C + (-1)^m C\Phi^{*m}C = 0 \tag{1.3}$$

where $C$ is a conjugation transformation on $\mathbf{H}$ ( see [4]). Recall that a transformation $C$ is said to be a conjugation if $C$ satisfying the following conditions ( [6]):

(i) $C$ is antilinear: $C(\alpha u + \beta v) = \overline{\alpha}C(u) + \overline{\beta}C(v)$,

(ii) $C^2 = I$ and $\langle Cu\ ,\ Cv\rangle_{\mathbf{H}} = \langle v\ ,\ u\rangle_{\mathbf{H}}$. $\big($see [4]$\big)$.

In [3] the authors extended (1.1) to general real metric space as follows: A map $\Phi\,:\,(\mathbf{E},\mathbf{d}_{\mathbb{R}}) \to (\mathbf{E},\mathbf{d}_{\mathbb{R}})$ is said to be $(m,q)$-isometric mapping for some integer $m \in \mathbb{N}$ an $q \in (0,\infty)$, if

$$\mathbf{d}_{\mathbb{R}}\big(\Phi^m u,\ \Phi^m v\big)^q - \binom{m}{1}\mathbf{d}_{\mathbb{R}}\big(\Phi^{m-1}u,\ \Phi^{m-1}v\big)^q + \cdots$$
$$+\ (-1)^{m-1}\binom{m}{m-1}\mathbf{d}_{\mathbb{R}}\big(\Phi u,\ \Phi v\big)^q + (-1)^m\mathbf{d}_{\mathbb{R}}\big(u,\ v\big)^q = 0;$$

for all $(u,v) \in \mathbf{E}^2$ $\big($ see [3]$\big)$.

In this work, our goal is to extend (1.2) and (1.3) to general complex-valued metric space $(\mathbf{E},\mathbf{d}_{\mathbb{C}})$.

The main results of the paper are described in Section two, We prove that if $\Phi$ is an $(m,q)$-complex symmetric, then it is $(m+1,q)$-complex symmetric (Proposition 2.1). We show that is $\Phi$ is an $(m,q)$-complex symmetric, then $\Phi^2$ is an $(m,q)$-complex symmetric. In particular, if $\Phi$ is an $(2,q)$-complex symmetric so is its power $\Phi^n$ for all positive integer $n$ (Theorem 2.2). Moreover we show that a transformation $\Phi\,:\,(\mathbf{E},\ \mathbf{d}_{\mathbb{C}}) \to (\mathbf{E},\ \mathbf{d}_{\mathbb{C}})$ is an $(m,q)$-complex symmetric if and only if $\Phi\,:\,(\mathbf{E},\ \widetilde{\mathbf{d}_{\mathbb{K}}}) \to (\mathbf{E},\ \widetilde{\widetilde{\mathbf{d}_{\mathbb{K}}}})$ is an complex symmetric for some complex valued metric $\widetilde{\mathbf{d}_{\mathbb{K}}}$ and $\widetilde{\widetilde{\mathbf{d}_{\mathbb{K}}}}$ associated to $\Phi$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ ( Theorem 2.3 and Proposition 2.6).

## 2. $(m,q)$-COMPLEX SYMMETIC TRANSFORMATIONS

In this section, we define the concept of $(m,q)$-complex symmetric of transformation on a complex valued metric space. Several properties of this family of transformations are examined.

**Definition 2.1.** *Let* $(\mathbf{E},\ \mathbf{d}_{\mathbb{C}})$ *and* $(\mathbf{E},\ \mathbf{d}'_{\mathbb{C}})$ *be complex valued metric spaces. A transformation* $\Phi : (\mathbf{E},\ \mathbf{d}_{\mathbb{C}}) \to (\mathbf{E},\ \mathbf{d}'_{\mathbb{C}})$ *is said to be a complex symmetric if* $\Phi$ *satisfies*

$$\mathbf{d}_{\mathbb{C}}\big(\Phi u,\ v\big) = \mathbf{d}'_{\mathbb{C}}\big(u,\ \Phi v\big),\ \ \forall\, u, v \in E.$$

**Definition 2.2.** *A self transformation* $\Phi$ *on a complex valued metric space* $(\mathbf{E}, \mathrm{d}_{\mathbb{C}})$ *is said to be an* $(m,q)$*-complex symmetric mapping if* $S$ *satisfies for all* $u, v \in \mathbf{E}$,

$$\sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^k x,\ \Phi^{m-k} y\big)^q = 0, \tag{2.1}$$

*for some positive integer* $m$ *and a real number* $q \in (0,\ \infty)$. *We said that* $\Phi$ *is an* $(m,q)$*-symmetric transformation if*

$$\sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{R}}\big(\Phi^k x,\ \Phi^{m-k} y\big)^q = 0.$$

**Remark 2.1.**
*(1) If* $m = 1$, *(2.1) is equivalent to* $\mathbf{d}_{\mathbb{C}}\big(\Phi u,\ v\big)^q = \mathbf{d}_{\mathbb{C}}\big(u, \Phi v\big)^q$.

*(2)* (i) If $m = 2$, (2.1) is equivalent to

$$\mathbf{d}_{\mathbb{C}}\big(u,\ \Phi^2 v\big)^q - 2\mathbf{d}_{\mathbb{C}}\big(\Phi u,\ \Phi v\big)^q + \mathbf{d}_{\mathbb{C}}\big(u, \Phi^2 v\big)^q = 0,\ \ \forall\, u, v \in \mathbf{E}.$$

**Example 1.** *Let* $\mathbf{E} = \mathbb{C}^2$ *and* $\mathbf{d}_{\mathbb{C}}$ *be a complex valued metric on* $\mathbf{E}$ *define by*

$$\mathbf{d}_{\mathbb{C}}\big((a,b),(u.v)\big) = |a-u| + |b-v|,\ (a,b),(u,v) \in \mathbb{C}^2.$$

*Consider the map* $\Phi : \mathbf{E} \to \mathbf{E}$ *defined by* $\Phi(u,v) = (v,\ u)$. *It is obvious that*

$$\Phi^2(u,v) = (u,\ v) \quad \textit{and} \ \ \Phi^3(u,v) = \Phi(u,\ v) = (v,u);\ \forall\ (u,v) \in \mathbb{C}^2.$$

*It Follows that*

$$\sum_{0\le k\le 3} (-1)^{3-k}\binom{3}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^k x,\ \Phi^{3-k} y\big)^q = 0.$$

*Therefore* $S$ *is a* $(3,q)$*-symmetric map.*

Let $\Phi$ be a self transformation on a complex-valued metric space $(\mathbf{E}, \mathbf{d}_{\mathbb{C}})$. Four $u, v \in \mathbf{E}$, set

$$\mathcal{Z}_l^{(q)}(\Phi;\ u, v) := \sum_{0 \leq k \leq l} (-1)^{l-k} \binom{m}{k} \mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{l-k} v\big)^q. \tag{2.2}$$

**Proposition 2.1.** *Let $\Phi$ be a self transformation on a complex-valued metric space $(\mathbb{E}, \mathbf{d}_{\mathbb{C}})$.*
*(1) Then*

$$\mathcal{Z}_{m+1}^{(q)}(\Phi;\ u, v) = \mathcal{Z}_m^{(q)}(\Phi;\ \Phi u,\ v) - \mathcal{Z}_m^{(q)}(\Phi;\ u,\ \Phi v)\ ; \forall\ u,\ v \in \mathbf{E}. \tag{2.3}$$

*(2) If $\Phi$ is an $(m,q)$-complex symmetric mapping, then $S$ is an $(n,q)$-complex symmetric mapping for all positive integer $n \geq m$.*

*Proof.* (1) From equation (2.2) we have

$$\begin{aligned}
&\mathcal{Z}_{m+1}^{(q)}(\Phi;\ u\ , v) \\
= &\sum_{0 \leq k \leq l} (-1)^{m+1-k} \binom{m+1}{k} \mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{m+1-k} v)^q \\
= &(-1)^{m+1} \mathbf{d}_{\mathbb{C}}\big(u,\ \Phi^{m+1} v)^q - \sum_{1 \leq k \leq m} (-1)^{m-k} \binom{m+1}{k} \mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{m+1-k} v)^q \\
&+ \mathbf{d}_{\mathbb{C}}\big(S^{m+1} u,\ v)^q \\
= &(-1)^{m+1} \mathbf{d}_{\mathbb{C}}\big(u,\ \Phi^{m+1} v)^q - \sum_{1 \leq k \leq m} (-1)^{m-k} \left( \binom{m}{k} + \binom{m}{k-1} \right) \\
&\cdot \mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{m+1-k} v)^q + \mathbf{d}_{\mathbb{C}}\big(\Phi^{m+1} u,\ y)^q \\
= &\Phi_m^{(q)}(\Phi;\ \Phi u,\ v) - \Phi_m^{(q)}(\Phi;\ u, \Phi v).
\end{aligned}$$

The statement (2) is a direct consequence of the statement (1). □

**Remark 2.2.** *From (2.3) we deduce that for all $u,\ v \in \mathbf{E}$,*

$$\Re e\bigg( \mathcal{Z}_{m+1}^{(q)}(\Phi;\ u, v) \bigg) = \Re e\bigg( \mathcal{Z}_m^{(q)}(\Phi;\ \Phi u,\ v) \bigg) - \Re e\bigg( \mathcal{Z}m^{(q)}(\Phi;\ u,\ \Phi v) \bigg)$$

*and*

$$\Im m\bigg( \mathcal{Z}_{m+1}^{(q)}(\Phi;\ u, v) \bigg) = \Im m\bigg( \mathcal{Z}_m^{(q)}(\Phi;\ \Phi u,\ v) \bigg) - \Im m\bigg( \mathcal{Z}_m^{(q)}(\Phi;\ u,\ \Phi v) \bigg).$$

**Lemma 2.1.** *Let $\Phi$ be a self transformation on a complex-valued metric space $(\mathbf{E}, \mathbf{d}_{\mathbb{C}})$. If $S$ is an $(m,q)$-complex symmetric transformation, then for all $u,\ v \in \mathbf{E}$, $p = 0, 1, 2, \cdots$ , and $m = 1, 2 \ldots$,*

$$\mathcal{Z}_{m-1}^{(q)}(\Phi;\ \Phi^p u,\ v) = \mathcal{Z}_{m-1}^{(q)}(\Phi;\ u,\ \Phi^p v).$$

*Proof.* The proof is deduced from the formula (2.3). □

**Theorem 2.1.** *Let $\Phi$ be a bijective self transformation on a complex-valued metric space $(\mathbf{E}, \mathbf{d}_{\mathbb{C}})$. Then, $\Phi$ is an $(m,q)$-complex symmetric if and only if $\Phi^{-1}$ is an $(m,q)$-complex symmetric transformation.*

*Proof.* Assume that $\Phi$ is an bijective $(m,q)$-complex symmetric, it follows that $\mathcal{Z}_l^{(q)}(\Phi;\ u,v) = 0\ \ \forall\ \ u,v \in \mathbf{E}$. In particular, if we replace $u$ by $\Phi^{-m}u$ and $v$ by $\Phi^{-m}v$ we obtain

$$\begin{aligned} 0 &= \mathcal{Z}_m^{(q)}(\Phi;\ \Phi^{-m}u, \Phi^{-m}v) = \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^{k-m}u,\ \phi^{-k}v\big)^q \\ &= \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big((\Phi^{-1})^k u,\ (\Phi^{-1})^{m-k}v\big)^q \\ &= (-1)^m \mathcal{Z}_m^{(q)}(\Phi^{-1};\ u,v). \end{aligned}$$

Therefore $\Phi^{-1}$ is an $(m,q)$-complex symmetric transformation. □

**Theorem 2.2.** *Let $\Phi$ be a self transformation on a complex valued metric space $\mathbf{E}$. The following statements hold:*

*(1) If $\Phi$ is an $(m,q)$-complex symmetric transformation, then $\Phi^2$ is an $(m,q)$-complex symmetric transformation.*

*(2) If $\Phi$ is an $(2,q)$-complex symmetric, then*

$$\mathcal{Z}_2^{(q)}(\Phi^{n+2},\ u,v) = 2\mathcal{Z}_2^{(q)}(\Phi^{n+1},\ \Phi u, \Phi v) - \mathcal{Z}_2^{(q)}(\Phi^n,\ \Phi^2 u, \Phi^2 v),\ \ \forall\ \ n \in \mathbb{N}.$$

*(3) If $\Phi$ is an $(2,q)$-complex symmetric, then $\Phi^n$ is an $(2,q)$-complex symmetric transformation for all $n \in \mathbb{N}$.*

*Proof.* (1) Since $\Phi$ is an $(m,q)$-complex symmetric transformation, then

$$\mathcal{Z}_m^{(q)}(\Phi,\ u,v) = 0, \quad \forall\, u,v \in \mathbf{E}.$$

This means that $\mathcal{Z}_m^{(q)}(\Phi,\ \Phi^i u, \Phi^{m-i}v) = 0, \quad \forall\, i \in \{0, \cdots, m\}\ \ u, v \in \mathbf{E}$, and

$$\sum_{0\le i\le m} \binom{m}{i} \mathcal{Z}_m^{(q)}(\Phi,\ \Phi^i u, \Phi^{m-i}v) = 0, \forall\, u, v \in \mathbf{E}.$$

Moreover

$$\sum_{0\le i\le m}\sum_{0\le k\le m} (-1)^{m-k}\binom{m}{i}\binom{m}{k} \mathbf{d}_{\mathbb{C}}\big(\ \Phi^{k+i}u,\ \Phi^{m-(k+i)}v\big)^q = 0, \forall\, u, v \in \mathbf{E}.$$

From this equation we get

$$\begin{aligned} 0 &= \sum_{0\le i\le m}\sum_{0\le k\le m} (-1)^{m-k}\binom{m}{i}\binom{m}{k} \mathbf{d}_{\mathbb{C}}\big(\ \Phi^{k+i}u,\ \Phi^{m-(k+i)}v\big)^q \\ &= \sum_{0\le l\le 2m}\sum_{0\le k\le l} (-1)^{m-k}\binom{2m}{l}\binom{m}{l-k} \mathbf{d}_{\mathbb{C}}\big(\ \Phi^{l}u,\ \Phi^{2m-l}v\big)^q, \forall\, u, v \in \mathbf{E}. \end{aligned}$$

Notice that $\displaystyle\sum_{0\le k\le l} (-1)^{m-k}\binom{2m}{l}\binom{m}{l-k} = 0$ for $l$ odd integer.

Hence, for all $u, v \in E$,

$$\sum_{0\le l\le 2m}\sum_{0\le k\le l} (-1)^{m-k}\binom{2m}{l}\binom{m}{l-k} \mathbf{d}_{\mathbb{C}}\big(\ \Phi^{l}u,\ \Phi^{2m-l}v\big)^q = \mathcal{Z}_m^{(q)}(\Phi^2,\ u, v).$$

Therefore $S^2$ is a $(m,q)$-complex symmetry.

(2) We prove the statement (2) by induction on $n$. For $n = 1$ we have

$$\begin{aligned} \mathcal{Z}_2^{(q)}(\Phi^3,\ u, v) &= \sum_{0\le k\le 2} (-1)^k \binom{2}{k} d_{\mathbb{C}}\big((\Phi^3)^k u,\ (\Phi^3)^{2-k}v\big)^q \\ &= \mathbf{d}_{\mathbb{C}}\big(\Phi^6 u,\ v\big)^q - 2\mathbf{d}_{\mathbb{C}}\big(\Phi^3 u,\ \Phi^3 v\big)^q + \mathbf{d}_{\mathbb{C}}\big(u,\ \Phi^6 v\big)^q \\ &= 2\mathbf{d}_{\mathbb{C}}\big(\Phi^5 u,\ \Phi v\big)^q - 2\mathbf{d}_{\mathbb{C}}\big(\Phi^4 u,\ \Phi^2 v\big)^q - 2\mathbf{d}_{\mathbb{C}}\big(\Phi^3 u,\ \phi^3 v\big)^q \\ &\quad +2d\big(\Phi u,\ \Phi^5 v\big)^q - \mathbf{d}_{\mathbb{C}}\big(\Phi^2 u,\ \Phi^4 v\big)^q \\ &= \mathbf{d}_{\mathbb{C}}\big(\Phi u,\ \Phi^4 S v\big)^q - 2\mathbf{d}_{\mathbb{C}}\big(\Phi^3 u,\ \Phi^3 v\big)^q + \mathbf{d}_{\mathbb{C}}\big(\Phi^4\Phi u, \Phi v\big)^q \\ &\quad -\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^2 u,\ \Phi^2\Phi^2 v\big)^q - 2\mathbf{d}_{\mathbb{C}}\big(\Phi^3 u, \Phi^3 v\big)^q + \mathbf{d}_{\mathbb{C}}\big(\Phi^4 u,\ \Phi^2 v\big)^q\Big) \\ &= 2\mathcal{Z}_2^{(q)}(\Phi^2;\ \Phi u, \Phi v) - \mathcal{Z}_2^{(q)}(\Phi;\ \Phi^2 u, \Phi^2 v). \end{aligned}$$

Hence, the statement (2) is true for $n = 1$. We prove it for $n \geq 2$,

$$\begin{aligned}
& \mathcal{Z}_2^{(q)}(\Phi^{n+2},\ u,v) \\
= \ & \mathbf{d}_{\mathbb{C}}\big(u,\ \Phi^{2n+4}v\big)^q - 2\mathbf{d}_{\mathbb{C}}\big(\Phi^{n+2}u,\ \Phi^{n+2}v\big)^q + \mathbf{d}_{\mathbb{C}}\big(\Phi^{2n+4}u,\ v\big)^q \\
= \ & 2\bigg(\mathbf{d}_{\mathbb{C}}\big(\Phi u,\ \Phi^{2n+3}v\big)^q\bigg) - \mathbf{d}_{\mathbb{C}}\big(\Phi^{n+2}u,\ \Phi^{n+2}v\big)^q + \mathbf{d}_{\mathbb{C}}\big(\Phi^{2n+3}u,\ \Phi v\big)^q\bigg) \\
& -\bigg(\mathbf{d}_{\mathbb{C}}\big(\Phi^2u,\ \Phi^{2n+2}v\big)^q - 2\mathbf{d}_{\mathbb{C}}\big(\Phi^{n+2}u, \Phi^{n+2}v\big)^q + \mathbf{d}_{\mathbb{C}}\big(\Phi^{2n+2}u, \Phi^2v\big)^q\bigg) \\
= \ & 2\mathcal{Z}_2^{(q)}(\Phi^{n+1};\ \Phi u, \Phi v) - \mathcal{Z}_2^{(q)}(\Phi^n;\ \Phi^2u, \Phi^2v).
\end{aligned}$$

(3) Assume that $\Phi$ is an $(2,q)$-complex symmetric transformation. We prove by induction on $n \geq 2$ that $\Phi^n$ is also $(2,q)$-complex symmetric. In fact, for $n = 2$, $\Phi^2$ is a $(2,q)$ complex symmetric by the statement (1). Assume that $\Phi^n$ is an $(2,q)$-complex symmetric for $n$ and prove it for $n+1$. In view of the identity

$$\mathcal{Z}_2^{(q)}(\Phi^{n+1},\ u,v) = 2\mathcal{Z}_2^{(q)}(\Phi^n;\ \Phi u, \Phi v) - \mathcal{Z}_2^{(q)}(\Phi^{n-1}; \Phi^2u, \Phi^2v)$$

and the assumption that $\Phi^n$ is a $(2,q)$-complex symmetric, we get $\mathcal{Z}_2^{(q)}(\Phi^{n+2},\ u,v)$ $= 0$. So that $\Phi^{n+1}$ is a $(2,q)$-complex symmetric. □

**Proposition 2.2.** *Let $\Psi$ be a self transformation on a complex-valued metric space $(\mathbf{E}, \mathbf{d}_{\mathbb{C}})$ and let $\Phi$ be a self map on $(\mathbf{E}, \mathbf{d}_{\mathbb{C}})$.*

*(1) If $\Psi$ is an $(m,q)$-complex symmetric and $\Phi$ is a bijective isometric transformation, then $\Phi\Psi\Phi^{-1}$ and $\Phi^{-1}\Psi\Phi$ are $(m,q)$-complex symmetric.*

*(2) If $\Psi$ is an $(m,q)$-complex symmetric transformation and $\Phi$ is a complex symmetric transformation such that $\Psi\Phi = \Phi\Psi$, then $\Phi\Psi$ is an $(m,q)$-complex symmetric.*

*Proof.* (1) Assume that $\Psi$ is an $(m,q)$-complex symmetric and $\Phi$ is a bijective isometry, then

$$\begin{aligned}
& \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big((\Phi\Psi\Phi^{-1})^{k}u,\ (\Phi\Psi\Phi^{-1})^{m-k}v\big)^{q}\\
= & \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi\Psi^{k}\Phi^{-1}u,\ \Phi\Psi^{m-k}\Phi^{-1})v\big)^{q}\\
= & \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Psi^{k}\Phi^{-1}u,\ \Psi^{m-k}\Phi^{-1}v\big)^{q}\\
= & \ 0.
\end{aligned}$$

(2) Under the assumption that $\Phi$ is a complex symmetric transformation and $\Psi\Phi=\Phi\Psi$, we have

$$\mathbf{d}_{\mathbb{C}}\big(\Phi u,\ v\big)=\mathbf{d}_{\mathbb{C}}\big(u,\ \Phi v\big);$$

for all $u,v\in\mathbf{E}$ and

$$\mathbf{d}_{\mathbb{C}}\big((\Phi\Psi)^{k}u,\ (\Phi\Psi)^{m-k}v\big)^{q}=\mathbf{d}_{\mathbb{C}}\big(\Phi^{k}\Psi^{k}u,\ \Phi^{m-k}\Psi^{m-k}v\big)^{q}=d\big(\Psi^{k}u,\ \Psi^{m-k}v\big)^{q},$$

for all $u,v\in\mathbf{E}$. Moreover

$$\begin{aligned}
& \sum_{0\leq k\leq m}(-1)^{m-k}\mathbf{d}_{\mathbb{C}}\big((\Phi\Psi)^{k}u,\ (\Phi\Psi)^{m-k}v\big)^{q}\\
= & \sum_{0\leq k\leq m}(-1)^{m-k}\mathbf{d}_{\mathbb{C}}\big(\Phi^{k}\Psi^{k}u,\ \Phi^{m-k}\Psi^{m-k}v\big)^{q}\\
= & \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Psi^{k}u,\ \Psi^{m-k}v\big)^{q}\\
= & \ 0
\end{aligned}$$

for all $u,v\in\mathbf{E}$. □

**Definition 2.3.** *( [5]) Let $(\mathbf{E},\ \mathbf{d}_{\mathbb{C}})$ be a complex valued metric space. A sequence $(u_n)_n$ of elements of $\mathbf{E}$ is said to be convergent to $u$ in $\mathbb{E}$ if*

$$\forall\ a\in\mathbb{C}:0\prec a\ \ \exists\ n_0\in\mathbb{N}\ /\ \mathbf{d}_{\mathbb{C}}\big(u_n,\ u\big)\prec a\ \ \forall\ n\geq n_0.$$

**Notation:** $u_n\longrightarrow^{\mathbf{d}_{\mathbb{C}}}u.$

**Proposition 2.3.** *Let $\Phi$ be a self transformation on a complex-valued metric space $(\mathbf{E},\ \mathbf{d}_{\mathbb{C}})$ such is an $(m,q)$-complex symmetric. If $\Phi$ satisfies*

$$\mathbf{d}_{\mathbb{C}}\big(\Phi u,\ \Phi v\big)\preccurlyeq\mathbf{d}_{\mathbb{C}}\big(u,v\big),\ \ \forall\ u,u\in\mathbf{E},$$

*then*

$$\mathcal{Z}^{(q)}_{m-1}(\Phi;\ \Phi^{n+1}u,\ \Phi^n v) = \mathcal{Z}^{(q)}_{m-1}(\Phi;\ \Phi^n u,\ \Phi^{n+1}v) \longrightarrow 0 \ \textit{ as } n \longrightarrow \infty.$$

*Proof.* From the condition that $\Phi$ satisfies $\mathbf{d}_{\mathbb{C}}\big(\Phi u,\ \Phi v\big) \preccurlyeq \mathbf{d}_{\mathbb{C}}\big(u, v\big)$, for all $u, v \in \mathbf{E}$, we deduce that,

$$\mathbf{d}_{\mathbb{C}}\big(\Phi^{n+1}u,\ \Phi^{n+1}v\big)^q \preccurlyeq \mathbf{d}_{\mathbb{C}}\big(\Phi^n u, \Phi^n v\big)^q; \quad \forall\, u, v \in \mathbf{E},\ \text{and}\ \ n \in \mathbb{N}.$$

This means that

$$\Re e\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^{n+1}u,\ \Phi^{n+1}v\big)^q\Big) \leq \Re e\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^n u, \Phi^n v\big)^q\Big); \quad \forall\, u, v \in \mathbf{E},\ \text{and}\ \ n \in \mathbb{N},$$

and

$$\Im m\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^{n+1}u,\ \Phi^{n+1}v\big)^q\Big) \leq \Im m\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^n u, \Phi^n v\big)^q\Big); \ \forall\, u, v \in \mathbf{E},\ \text{and}\ \ n \in \mathbb{N}.$$

Consequently, the real sequences $\Big\{\Re e\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^n u, \Phi^n v\big)^q\Big)\Big\}_n$ and $\Big\{\Im m\Big(\mathbf{d}_{\mathbb{C}}\big(S^n u, S^n v\big)^q\Big)\Big\}_n$ are decreasing and bounded from lower. So convergent, and this means that

$$\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^n u, \Phi^n v\big)^q = \Re e\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^n u, \Phi^n v\big)^q\Big) + i\ \Im m\Big(\mathbf{d}_{\mathbb{C}}\big(\Phi^n u, \Phi^n v\big)^q\Big)\Big)_{n\in\mathbb{N}},$$

is convergent sequence.

From the hypothesis that $\Phi$ is an $(m, q)$-complex symmetric and together (2.3), we get

$$\mathcal{Z}^{(q)}_{m-1}(\Phi, \Phi u, v) = \mathcal{Z}^{(q)}_{m-1}(\Phi;\ u, \Phi v),$$

or more generally,

$$\mathcal{Z}^{(q)}_{m-1}(\Phi,\ \Phi^{n+1}u,\ \Phi^n v) = \mathcal{Z}^{(q)}_{m-1}(\Psi;\ \Psi^n u,\ \Psi^{n+1}v),$$

However

$$\mathcal{Z}^{(q)}_{m-1}(\Phi;\ \Phi^{n+1}u, \Phi^n v) = \mathcal{Z}^{(q)}_{m-2}(\Phi;\ \Phi^{n+2}u, \Phi^n v) - \Phi^{(q)}_{m-2}(S;\ \Phi^{n+1}u,\ \Phi^{n+1}v),$$

so that

$$\mathcal{Z}^{(q)}_{m-1}(\Phi;\ \Phi^{n+1}u, \Phi^n v) = \sum_{0\leq j\leq m-2}(-1)^{m-j}\binom{m-2}{j}$$
$$\cdot\Big[\mathbf{d}_{\mathbb{C}}\big(\Phi^{n+2+j}u, \Phi^{n+m-j}v\big)^q - \mathbf{d}_{\mathbb{C}}\big(\Phi^{n+1+j}u,\ \Phi^{n+1+m-j}v\big)^q\Big].$$

Taking the limit as $n \to \infty$ in the preceding equality to give

$$\mathcal{Z}_{m-1}^{(q)}(\Phi;\ \Phi^{n+1}u,\ \Phi^n v) \longrightarrow 0.$$

□

**Proposition 2.4.** *Let* $\Phi$ *be a self transformation on a complex valued metric space* $(\mathbf{E},\ \mathbf{d}_{\mathbb{C}})$*. Then the following identities hold for* $m \geq 1$*:*

$$\mathcal{Z}_m^{(q)}(\Phi;\ u,v) = \mathbf{d}_{\mathbb{C}}\big(\Phi^m u,\ v\big)^q - \sum_{0\leq k\leq m-1}\binom{m}{k}\mathcal{Z}_k^{(q)}(\phi;\ u,\Phi v), \tag{2.4}$$

*where* $\mathcal{Z}_0^{(q)}(S;\ u,v) = \mathbf{d}_{\mathbb{C}}\big(u,\ v\big)^q$.

*Proof.* We use an induction on $m \geq 1$ to prove (2.4). If $m = 1$, we see by (2.4) that

$$\mathcal{Z}_1^{(q)}(\Phi;\ u,v) = \mathbf{d}_{\mathbb{C}}\big(\Phi u,\ v\big)^q - \mathbf{d}_{\mathbb{C}}\big(u,\ \Phi v\big)^q = \mathbf{d}_{\mathbb{C}}\big(\Phi u,\ v\big)^q - \mathcal{Z}_0^{(q)}(\Phi;\ u,v).$$

Therefore (2.4) is obviously true. Suppose that the induction hypothesis holds for $m$. By the induction hypothesis and (2.3), we obtain

$$\begin{aligned}
&\mathcal{Z}_{m+1}^{(q)}(\Phi;\ u,v) = \mathcal{Z}_m^{(q)}(\Phi;\ \Phi u,v) - \mathcal{Z}_m^{(q)}(\Phi;\ u,\Phi v)\\
=\ & \mathbf{d}_{\mathbb{C}}\big(\Phi^{m+1}u,\ v\big)^q - \sum_{0\leq k\leq m-1}\binom{m}{k}\mathcal{Z}_k^{(q)}(\Phi;\Phi u,\Phi v)\\
& -\mathbf{d}_{\mathbb{C}}\big(\Phi^m u,\ \Phi v\big)^q + \sum_{0\leq k\leq m-1}\binom{m}{k}\mathcal{Z}_k^{(q)}\Phi;\ u,\ \Phi^2 v)\Big)\\
=\ & \mathbf{d}_{\mathbb{C}}\big(S^{m+1}u,\ v\big)^q - \mathbf{d}_{\mathbb{C}}\big(\Phi^m u,\ \Phi v\big)^q\\
& - \sum_{0\leq k\leq m-1}\binom{m}{k}\Big(\mathcal{Z}_k^{(q)}(\Phi;\ \Phi u,\ v) - \mathcal{Z}_k^{(q)}(\Phi;\ u,\ \Phi^2 v)\Big)\\
=\ & \mathbf{d}_{\mathbb{C}}\big(\Phi^{m+1}u,\ v\big)^q - \mathbf{d}_{\mathbb{C}}\big(\Phi^m u,\ \Phi v\big)^q - \sum_{0\leq k\leq m-1}\binom{m}{k}\mathcal{Z}_{k+1}^{(q)}(\Phi;\ u,\ \Phi v)\\
=\ & \mathbf{d}_{\mathbb{C}}\big(\Phi^{m+1}u,\ v\big)^q - \mathcal{Z}_m^{(q)}(\Phi,\ u,\Phi v) - \sum_{0\leq k\leq m-1}\binom{m}{k}\mathcal{Z}_k^{(q)}(\Phi;\ u,\Phi v)\\
& - \sum_{0\leq k\leq m-1}\binom{m}{k}\mathcal{Z}_{k+1}^{(q)}(\Phi;\ u,\Phi v)
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{d}_{\mathbb{C}}\big(\Phi^{m+1}u,\ v\big)^q - \mathcal{Z}_m^{(q)}(\Phi,\ u,\Phi v) - \sum_{0\le k\le m-1}\binom{m}{k}\mathcal{Z}_k^{(q)}(\Phi;\ u,\Phi v)\\
&\quad - \sum_{1\le k\le m}\binom{m}{k-1}\mathcal{Z}_k^{(q)}(\Phi;\ u,\Phi v)\\
&= \mathbf{d}_{\mathbb{C}}\big(\Phi^{m+1}u,\ v\big)^q - \mathcal{Z}_m^{(q)}(\Phi;\ u,\Phi v)\\
&\quad -\mathcal{Z}_0^{(q)}(\Phi;\ u,\Phi v) - \sum_{1\le k\le m-1}\left(\binom{m}{k}+\binom{m}{k-1}\right)\mathcal{Z}_k^{(q)}(\Phi;\ u,\Phi v)\\
&\quad -\binom{m}{m-1}\mathcal{Z}_m^{(q)}(\Phi;\ u,\Phi v)\\
&= \mathbf{d}_{\mathbb{C}}\big(\Phi^{m+1}u,\ v\big)^q - \mathcal{Z}_0^{(q)}(\Phi; u,\ \Phi v)\\
&\quad - \sum_{1\le k\le m-1}\binom{m+1}{k}\mathcal{Z}_k^{(q)}(\Phi;\ u,\Phi v) - \binom{m+1}{m}\Phi_m^{(q)}(\Phi;\ u,\Phi v)\\
&= \mathbf{d}_{\mathbb{C}}\big(\Phi^{m+1}u,\ v\big)^q - \sum_{0\le k\le m}\binom{m+1}{k}\mathcal{Z}_k^{(q)}(\Phi;\ u,\Phi v).
\end{aligned}$$

Hence, (2.4) is proved. □

By observing that

$$\begin{aligned}
&\mathcal{Z}_m^{(q)}(\Phi;\ u,v) = \sum_{0\le k\le m}(-1)^k\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{m-k}v\big)^q\\
&= \sum_{\substack{0\le k\le m\\ k\ (\text{even})}}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{m-k}v\big)^q - \sum_{\substack{0\le k\le m\\ k\ (\text{od})}}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{m-k}v\big)^q\\
&= \sum_{\substack{0\le k\le m\\ k\ (\text{even})}}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^k\Phi u,\ \Phi^{m-1-k}v\big)^q\\
&\quad - \sum_{\substack{0\le k\le m\\ k\ (\text{odd})}}\binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^{k-1}u,\ \Phi^{m-k}\Phi v\big)^q\\
&= \widetilde{\mathbf{d}_{\mathbb{C}}}(u,\Phi v)^q - \widetilde{\widetilde{\mathbf{d}_{\mathbb{C}}}}(\Phi u,\ v)^q,
\end{aligned}$$

where

$$\widetilde{\mathbf{d}_{\mathbb{C}}}(u,v)^{q} = \sum_{\substack{0\leq k\leq m\\ k\ (\text{even})}} \binom{m}{k} \mathbf{d}_{\mathbb{C}}\big(\Phi^{k}u,\ \Phi^{m-1-k}v\big)^{q}\Big),\ \ (u,v)\in \mathbf{E}^2,\ q\geq 1,$$

and

$$\widetilde{\widetilde{\mathbf{d}_{\mathbb{C}}}}(u,\ v)^{q} = \sum_{\substack{0\leq k\leq m\\ k\ (\text{odd})}} \binom{m}{k} \mathbf{d}_{\mathbb{C}}\big(\Phi^{k-1}u,\ \Phi^{m-k}v\big)^{q},\ \ (u,v)\in \mathbf{E}^2,\ q\geq 1.$$

**Proposition 2.5.** *Let $\Phi$ be a self transformation on a complex-valued metric space $(\mathbf{E},\mathbf{d}_{\mathbb{C}})$. Then $(\mathbf{E},\widetilde{\mathbf{d}_{\mathbb{C}}})$ and $(\mathbf{E},\widetilde{\widetilde{\mathbf{d}_{\mathbb{C}}}})$ are complex valued metric spaces, where*

$$\widetilde{\mathbf{d}_{\mathbb{C}}}(u,v) \sum_{\substack{0\leq k\leq m\\ k\ (\textit{even})}} \binom{m}{k} \mathbf{d}_{\mathbb{C}}\big(\Phi^{k}u,\ \Phi^{m-1-k}v\big),\ \ (u,v)\in \mathbf{E}^2,$$

*and*

$$\widetilde{\widetilde{\mathbf{d}_{\mathbb{C}}}}(u,\ v) = \sum_{\substack{0\leq k\leq m\\ k\ (\textit{odd})}} \binom{m}{k} \mathbf{d}_{\mathbb{C}}\big(\Phi^{k-1}u,\ \Phi^{m-k}v\big),\ \ (u,v)\in \mathbf{E}^2.$$

*Proof.* The proof follows from the fact that the map $(u,v)\longmapsto \mathbf{d}_{\mathbb{C}}(u,v)$ is a complex valued metric on $\mathbf{E}$. □

**Theorem 2.3.** *Let $\Psi$ be a self transformation on a complex valued metric space $(\mathbf{E},\mathbf{d}_{\mathbb{C}})$. The following statements are equivalent.*

*(1)* $\Phi : (\mathbf{E},\ \mathbf{d}_{\mathbb{C}}) \to (\mathbf{E},\ \mathbf{d}_{\mathbb{C}})$ *is an* $(m,1)$*-complex symmetric transformation,*

*(2)* $\Phi : \big(\mathbf{E},\ \widetilde{d_{\mathbb{C}}}\big) \to \big(\mathbf{E},\ \widetilde{\widetilde{d_{\mathbb{C}}}}\ \big)$ *is an complex symmetric transformation.*

*Proof.* In view of Definition 2.2, it follows that,

$$\Phi \text{ is an } (m,1)\text{-complex symmetric} \Longleftrightarrow \mathcal{Z}_m^{(1)}(\Phi;\ u,v) = 0\ \ \forall\ u,v\in \mathbf{E}.$$

$$
\begin{aligned}
& \mathcal{Z}_m^{(1)}(\Phi;\ u,v)=0 \\
\Longleftrightarrow\ & \sum_{\substack{0\le k\le m\\ r\ (\text{even})}} \binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^k u,\ \Phi^{m-k}v\big) \\
& \quad = \sum_{\substack{0\le k\le m\\ r\ (\text{odd})}} \binom{m}{k}\mathbf{d}_{\mathbb{C}}\big(\Phi^{k-1}\Phi u,\ \Phi^{m-k-1}\Phi v\big) \\
\Longleftrightarrow\ & \widetilde{d}_{\mathbb{C}}(u,\Phi v)-\widetilde{\widetilde{d}}_{\mathbb{C}}(\ \Phi u,v)=0,\ \forall\ u,v\in\mathbf{E} \\
\Longleftrightarrow\ & \Phi \text{ is complex symmetric.}
\end{aligned}
$$

□

**Proposition 2.6.** *Let $\Phi$ be a self transformation on a real valued metric space $(\mathbf{E},\mathbf{d}_{\mathbb{R}})$. Then $(\mathbf{E},\widetilde{\mathbf{d}_{\mathbb{R}}})$ and $(\mathbf{E},\widetilde{\widetilde{\mathbf{d}_{\mathbb{R}}}})$ are real valued metric spaces, where*

$$
\widetilde{\mathbf{d}_{\mathbb{R}}}(u,v)=\Bigg(\sum_{\substack{0\le k\le m\\ k\ (\textit{even})}} \binom{m}{k}\mathbf{d}_{\mathbb{R}}\big(S^k u,\ S^{m-1-k}v\big)^q\Bigg)^{\frac{1}{q}},\quad (u,v)\in\mathbf{E}^2, q\ge 1,
$$

*and*

$$
\widetilde{\widetilde{\mathbf{d}_{\mathbb{R}}}}(u,\ v)=\Bigg(\sum_{\substack{0\le k\le m\\ k\ (\textit{odd})}} \binom{m}{k}\mathbf{d}_{\mathbb{R}}\big(\Phi^{k-1} u,\ \Phi^{m-k}v\big)^q\Bigg)^{\frac{1}{q}},\quad (u,v)\in\mathbf{E}^2.
$$

**Theorem 2.4.** *Let $\Psi$ be a self transformation on a real valued metric space $(\mathbf{E},\mathbf{d}_{\mathbb{R}})$ and $q\ge 1$. The following statements are equivalent.*

*(1)* $\Phi:(\mathbf{E},\ \mathbf{d}_{\mathbb{R}})\to(\mathbf{E},\ \mathbf{d}_{\mathbb{R}})$ *is an $(m,q)$- symmetric transformation,*

*(2)* $\Phi:\big(\mathbf{E},\ \widetilde{d}_{\mathbb{R}}\big)\to\big(\mathbf{E},\ \widetilde{\widetilde{d}}_{\mathbb{R}}\ \big)$ *is an symmetric transformation.*

*Proof.* By similar technics as in the proof of Theorem 2.3. □

**Remark 2.3.** *In [7], it was observed that if* $\Phi : \mathbf{H} \to \mathbf{H}$ *(*$\mathbf{H}$ *Hilbert space) is an* $2$*-symmetric. then* $\Phi$ *is a symmetric.*

**Question.** Does it true that if $\Phi$ is an $(2, q)$-symmetric transformation on a complex valued metric space $(\mathbf{E}, \mathbf{d}_{\mathbb{C}})$, then $\Phi$ is a symmetric transformation?

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