ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.1, 311–317
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.1.31

UNIQUE METRO DOMINATION OF POWER OF PATHS

Kishori P. Narayankar, Denzil Jason Saldanha[1], and John Sherra

ABSTRACT. A dominating set D of G which is also a resolving set of G is called a *metro dominating set*. A metro dominating set D of a graph $G(V, E)$ is a *unique metro dominating set* (in short an *UMD-set*) if $\mid N(v) \cap D \mid= 1$ for each vertex $v \in V - D$ and the minimum of cardinalities of an *UMD-sets of G* is the *unique metro domination number of G* denoted by $\gamma_{\mu\beta}(G)$. In this paper we determine unique metro domination number of power of paths.

1. INTRODUCTION

All the graphs considered in this paper are simple, connected and undirected. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d(u, v)$. For a vertex v of a graph, $N(v)$ denote the set of all vertices adjacent to v and is called open neighborhood of v. Similarly, the closed neighborhood of v is defined as $N[v] = N(v) \cup \{v\}$.

Let $G(V, E)$ be a graph. For each ordered subset $S = \{v_1, v_2, \dots, v_k\}$ of V, each vertex $v \in V$ can be associated with a vector of distances denoted by $\Gamma(v/S) = (d(v_1, v), d(v_2, v), \dots, d(v_k, v))$. The set S is said to be a *resolving set* of G if $\Gamma(v/S) \neq \Gamma(u/S)$ for every $u, v \in V - S$, see [1]. A resolving set of minimum cardinality is a *metric basis* and cardinality of a metric basis is the *metric dimension* of G, see [2]. The k-tuple, $\Gamma(v/S)$ associated to the vertex

---

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 05C69.

Key words and phrases. Domination, metric dimension, metro domination, unique metro domination.

$v \in V$ with respect to a metric basis S, is referred as a *code generated by S* for that vertex v. If $\Gamma(v/S) = (c_1, c_2, \ldots, c_k)$, then $c_1, c_2, c_3, \ldots, c_k$ are called components of the code of v generated by S and in particular $c_i, 1 \leq i \leq k$, is called i^{th}-component of the code of v generated by S.

A dominating set D of a graph $G(V, E)$ is the subset of V having the property that for each vertex $v \in V - D$, there exists a vertex $u \in D$ such that uv is in E, see [3]. A dominating set D of G which is also a resolving set of G is called a *metro dominating set*.

A metro dominating set D of a graph $G(V, E)$ is a *unique metro dominating set* (in short an *UMD set*) if $|N(v) \cap D| = 1$ for each vertex $v \in V - D$. Generally, if $|N(v) \cap D| = k$ for each vertex $v \in V - D, k \geq 1$, such a metro dominating set D is called a *Smarandachely distance k dominating set* (Smarandachely k DD-sets of G) and the minimum of cardinalities of the Smarandachely DD-sets of G is the number of Smarandachely k UDD-sets of G, denoted by $\gamma^k_{S\mu\beta}(G)$. Particularly, if $k = 1$, i.e., the *unique metro domination number of G* denoted by $\gamma_{\mu\beta}(G)$, see [4–6].

2. Main Results

Take $P_n, n > k$. If $k < i \leq n - k$, join v_i to $v_{i-2}, v_{i-3}, \ldots v_{i-k}$ and $v_{i+2}, \ldots, v_{i+k}$. If $i > n - k$, then join v_i to $v_{i-2}, v_{i-3}, \ldots, v_{i-k}$ and all $v_j, j > i + 1$. Similarly if $i \leq k$, then join v_i to $v_j, j < i - 1$ and to $v_{i+2}, \ldots, v_{i+k}$. The resulting graph is called P^k_n.

If $k < i \leq n - k$, then v_i dominates $v_i, v_{i-1}, v_{i-2}, \ldots, v_{i-k}, v_{i+1}, v_{i+2}, \ldots v_{i+k}$. If $|i - j| \leq 2k + 1$, then vertex $v_{i+1}, v_{i+2}, \ldots v_{j-1}$ are dominated by v_i and v_j.

The set $D = \{v_1, v_6, v_9\}$ is a dominating set in Figure 1 for P^2_9. It is also a metro dominating set. Note that v_7 is dominated by v_6 and v_9. Hence D is not a UMD set.

The set $D = \{v_3, v_8\}$ is a dominating set for P^2_9 in Figure 2. All vertices are dominated uniquely by $\{v_3, v_8\}$. But code generated by D to v_4 and v_5 is same. Hence $\{v_3, v_8\}$ does not resolve the vertex set V of P^2_9 and hence D is not a UMD set. If $v_1 \in D$, it dominates $v_2, v_3, \ldots, v_{k+1}$. If $v_i \in D$ and if $i < 2k + 2$, then v_{k+1} is dominated by v_1 and v_i. If $i > 2k + 2$, then v_{k+2} is not dominated. Further if $i = 2k + 2$ then the vertices $v_2, v_3, \ldots, v_{i-1}$ are uniquely dominated.

A vertex in P^k_n can dominate a maximum of $2k + 1$ vertices.

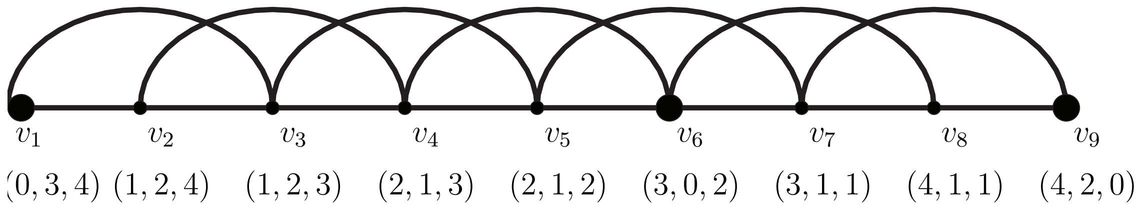


FIGURE 1. P_9^2

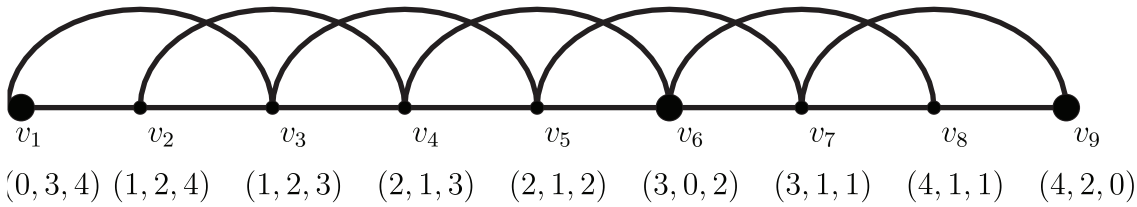


FIGURE 2. P_9^2

Lemma 2.1. *If D is a minimal dominating set then* $|D| \geq \left\lceil \frac{n}{2k+1} \right\rceil$.

However if $n = k+1+(2k+1)p, (p \in \mathbb{N})$, then $D = \{v_{k+1}, v_{3k+2}, v_{5k+3}, ...v_{n-k}\}$ is a minimal dominating set and $|D| = \frac{n}{2k+1}$. Hence we have

Lemma 2.2. *When* $n = k+1+(2k+1)p, p \in \mathbb{N}$*, the domination number* $\gamma(P_n^k) = \frac{n}{2k+1} = \left\lceil \frac{n}{2k+1} \right\rceil$.

Observe that when $n = (k+1)+(2k+1)p, p \in \mathbb{N}$, D=$\{v_{k+1}, v_{3k+2}, \ldots v_n\}$ dominates $V - D$ uniquely.
For any v_i and v_j,

$$d(v_i, v_j) = \left\lceil \frac{|i-j|}{k} \right\rceil.$$

Consider $1 \leq i < j \leq n-k$, such that $j - i \equiv 0$ (mod k). Then we get $d(v_i, v_{j+1}) = d(v_i, v_{j+2}) = \ldots = d(v_i, v_{j+k}) = \frac{j-i+k}{k}$.
If $k+1 \leq t < i < j \leq n-k$, $2i = j+t, j-i \equiv 0$ (mod k) and $i - t \equiv 0$ (mod k), then $d(v_i, v_{t-1}) = d(v_i, v_{i-2}) =, \ldots, = d(v_i, v_{t-k}) = d(v_i, v_{j+1}) = d(v_i, v_{j+2}) = \ldots = d(v_i, v_{j+k}) = \frac{j-t+2k}{2k}$.

Now consider the unique dominating set $D = \{v_{k+1}, v_{3k+2}, \ldots\}$ for P_n^k. The vertices v_i and $v_j \in D$, $i = (k+1) + (2k+1)l$ and $j = (k+1) + (2k+1)(l+1)$, generate the same code to all the vertices in $U = \{v_{i+1}, v_{i+2}, \ldots, v_{i+k}\}$ and the same code to all the vertices in $W = \{v_{i+k+1}, v_{i+k+2}, \ldots, v_{j-1}\}$.

For example, $\{v_{k+1}, v_{3k+2}\}$ generates the same code (1, 2) for $v_{k+2}, v_{k+3}, \ldots, v_{2k+1}$ and same code (2, 1) for $v_{2k+2}, v_{2k+3} \ldots v_{3k+1}$.

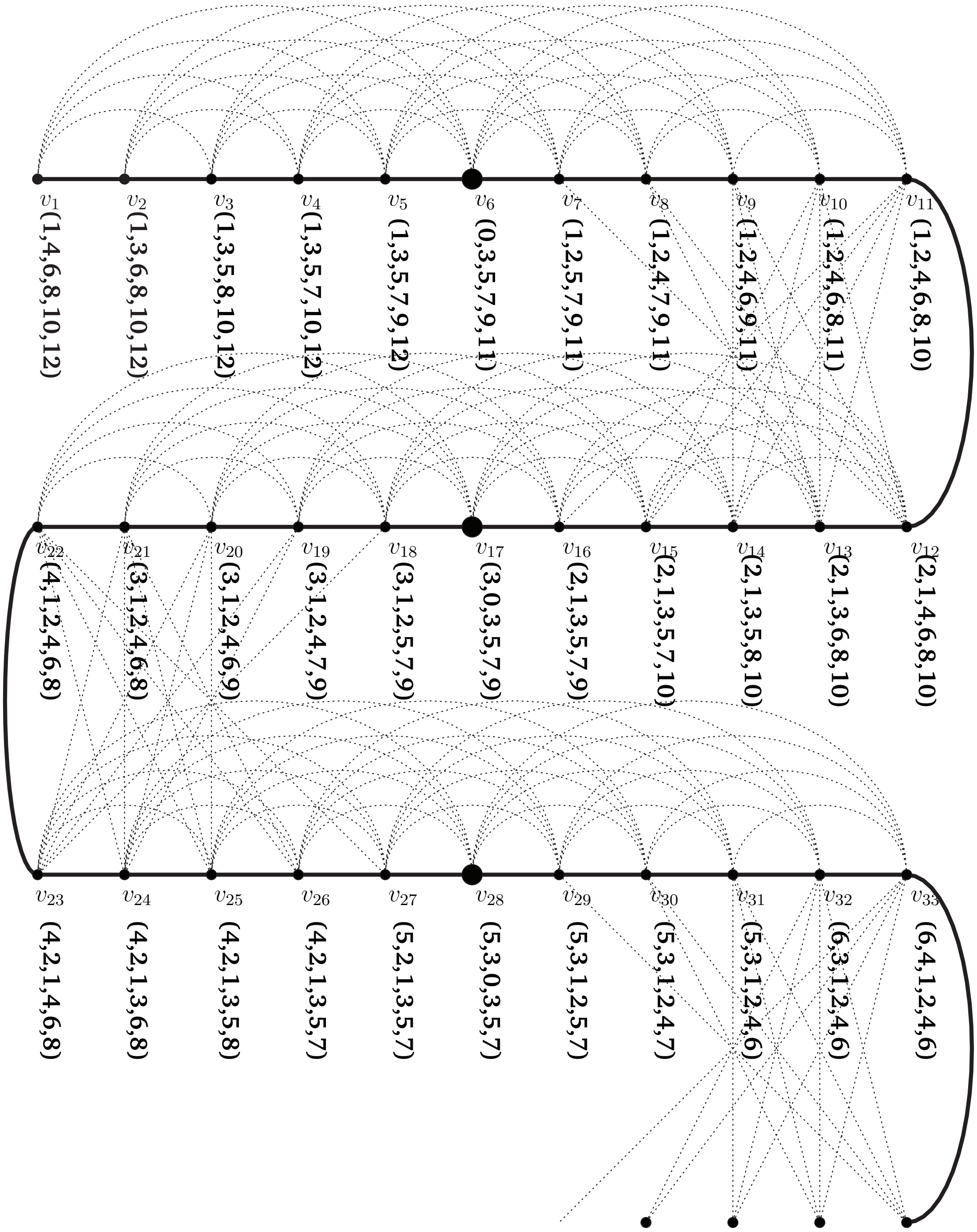
Take $v_h \in D$, $h = (k+1) + (2k+1)(l+2)$. Then $\dfrac{|h-i|}{k} = \dfrac{4k+2}{k} = 4 + \dfrac{2}{k}$. Hence $d(v_h, v_i) = \left\lceil 4 + \dfrac{2}{k} \right\rceil = 5$ and $d(v_k, v_{i+1}) = \left\lceil 4 + \dfrac{1}{k} \right\rceil = 5$.

All other vertices of U will have the same distance 4 from v_h.
Now $|h-(i+k+1)| = |3k+1|$. Therefore $d(v_h, v_{i+k+1}) = \left\lceil \dfrac{3k+1}{k} \right\rceil = \left\lceil 3 + \dfrac{1}{k} \right\rceil = 4$ and all other vertices of W will have the same distance 3 from v_h. Hence code generated by $\{v_i, v_j, v_h\}$ will be the same for vertices in $U - \{v_{i+1}\}$ and is the same for vertices in $W - \{v_{i+k+1}\}$.

For example, the code generated by $\{v_{k+1}, v_{3k+2}, v_{5k+3}\}$ to v_{k+2} is (1,2,5) and to v_{2k+2} is (2,1,4) where as same code (1,2,4) is generated to $v_{k+3}, \ldots, v_{2k+1}$, same code (2,1,3) is generated to $v_{2k+3}, \ldots, v_{3k+1}$.
Now take $v_h \in D$, $h = (k+1)+(2k+1)(l-1)$. Then $\dfrac{|h-i|}{k} = \dfrac{2k+1}{k} = 2 + \dfrac{1}{k}$ and $\dfrac{|h-(i+k)|}{k} = 3 + \dfrac{1}{k}$. Therefore $d(v_h, v_i) = 3 = d(v_h, v_{i+1}) = \ldots = d(v_h, v_{i+k-1})$ and $d(v_h, v_{i+k}) = 4$. Further $\dfrac{|h-(j-1)|}{k} = \dfrac{|4k+1|}{k} = 4 + \dfrac{1}{k}$, and therefore $d(v_h, v_{j-1}) = 5$ and $d(v_k, v_{j-2}) = 4 = d(v_h, v_{j-3}) = \ldots = d(v_h, v_{i+k+1})$. Therefore code generated by $\{v_i, v_j, v_h\}$ will be same for vertices in $U - \{v_{i+k}\}$ which is different from the code of v_{i+k} and code generated will be the same for vertices in $W - \{v_{i+k+1}\}$, which is different from the code of v_{i+k+1}.

Every vertex of D, when added to $\{v_i, v_j\}$, it produces different code to exactly one vertex of U and exactly one vertex of W. Hence to resolve all the vertices betweeen v_i and v_j, we require $k-1$ vertices of D. Therefore minimum $|D| = k+1$. For each l, $0 \leq l \leq k+1$, there are $2k$ vertices between v_i and v_j. Also we have $v_1, v_2, \ldots v_k$. Thus to resolve $V - D$, it is necessary to have atleast $k(2k) + (k+1) + k = 2k^2 + 2k + 1$ vertices in V. Thus we have

Lemma 2.3. *If $n = 2k^2 + 2k + 1$, then $D = \{v_{k+1}, v_{3k+2}, \ldots\}$ is a unique metro dominating set.*


v_1 (1,4,6,8,10,12)
v_2 (1,3,6,8,10,12)
v_3 (1,3,5,8,10,12)
v_4 (1,3,5,7,10,12)
v_5 (1,3,5,7,9,12)
v_6 (0,3,5,7,9,11)
v_7 (1,2,5,7,9,11)
v_8 (1,2,4,7,9,11)
v_9 (1,2,4,6,9,11)
v_{10} (1,2,4,6,8,11)
v_{11} (1,2,4,6,8,10)
v_{22} (4,1,2,4,6,8)
v_{21} (3,1,2,4,6,8)
v_{20} (3,1,2,4,6,9)
v_{19} (3,1,2,4,7,9)
v_{18} (3,1,2,5,7,9)
v_{17} (3,0,3,5,7,9)
v_{16} (2,1,3,5,7,9)
v_{15} (2,1,3,5,7,10)
v_{14} (2,1,3,5,8,10)
v_{13} (2,1,3,6,8,10)
v_{12} (2,1,4,6,8,10)
v_{23} (4,2,1,4,6,8)
v_{24} (4,2,1,3,6,8)
v_{25} (4,2,1,3,5,8)
v_{26} (4,2,1,3,5,7)
v_{27} (5,2,1,3,5,7)
v_{28} (5,3,0,3,5,7)
v_{29} (5,3,1,2,5,7)
v_{30} (5,3,1,2,4,7)
v_{31} (5,3,1,2,4,6)
v_{32} (6,3,1,2,4,6)
v_{33} (6,4,1,2,4,6)

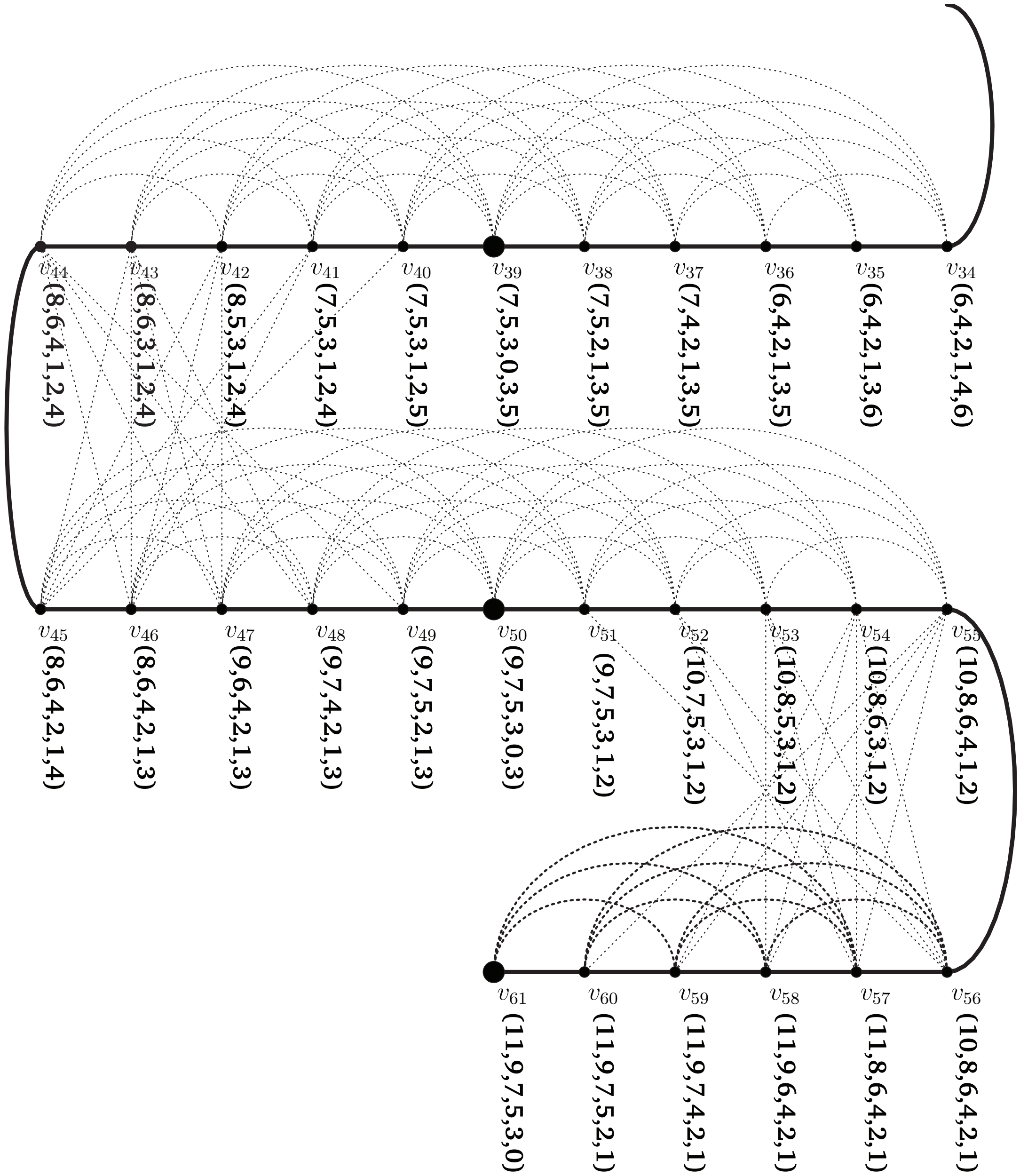


FIGURE 3. P_{61}^{5}

Further we observe, that if $n \geq 2k^2+2k+1$, and $(2k+1)p-k \leq n \leq (2k+1)p-1$, then $D = \{v_{k+1}, v_{3k+2}, \ldots, v_{(2k+1)p-k}\}$ is a UMD set.
Also if $n \geq 2k^2 + 2k + 1$ and $(2k+1)p - 2k \leq n < (2k+1)p - k$, then $D = \{v_1, v_{2k+2}, v_{4k+3}, \ldots v_{(2k+1)p-2k}\}$ is a UMD set.

In any case $|D| = p = \left\lceil \dfrac{n}{2k+1} \right\rceil$. Hence we have the following theorem.

Theorem 2.1. $\gamma_{\mu\beta}(P_n^k) = \begin{cases} \left\lceil \dfrac{n}{2k+1} \right\rceil, & \textit{for } n \geq 2k^2+k+1 \\ n, & \textit{for } n < 2k^2+k+1 \end{cases}$

REFERENCES

[1] F. HARARY, R. MELTER: *On the Metric Dimension of a graph*, Ars Combin., **2** (1976), 191–195.

[2] C. POISSON, P. ZHANG: *The Metric Dimension Of Unicyclic Graphs*, J. Comb. Math Comb. Compu., **40** (2002), 17–32.

[3] P. J. SLATER: *Domination and location in acyclic graphs*, Networks, **17** (1987), 55–64.

[4] B. SOORYANARYANA, J. SHERRA: *Unique metro domination in graphs*, Adv. Appl. Discrete Math., **14**(2) (2014), 125–149.

[5] B. SOORYANARYANA, J. SHERRA: *Unique metro domination of Circulant graphs*, International J. Math. Combin., (1) (2019), 53–61.

[6] H. B. WALIKAR, K. P. NARAYANKAR, S. S.SHIRAKOL: *The number of minimum dominating sets in $P_n \times P_2$* , International J. Math. Combin., (3) (2010), 17–21.

DEPARTMENT OF MATHEMATICS
MANGALORE UNIVERSITY
MANGALAGANGOTHRI, MANGALORE-574199, INDIA
Email address: kishori_pn@yahoo.co.in

DEPARTMENT OF MATHEMATICS
MANGALORE UNIVERSITY
MANGALAGANGOTHRI, MANGALORE-574199, INDIA
Email address: denzil53@gmail.com

DEPARTMENT OF MATHEMATICS
ST ALOYSIUS COLLEGE (AUTONOMOUS)
MANGALORE-575003, INDIA
Email address: johnsherra@gmail.com