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THE COMPLETE ALGEBRAIC STRUCTURE FOR SOME FINITE METABELIAN GROUP ALGEBRAS

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ABSTRACT. The objective of this paper is to find the complete algebraic structure of finite semisimple group algebra $\mathbb{F}_q[G]$, where $\mathbb{F}_q$ is a finite field with q elements and G is a finite group such that $G/Z(G)$ is isomorphic to direct product of two cyclic groups of order p, p odd prime.

1. INTRODUCTION

Let $\mathbb{F}_q$ be a finite field with q elements and let G be a finite group of order coprime to q. Thus $\mathbb{F}_q[G]$ is finite semisimple group algebra. The problem of finding the complete algebraic structure of a semisimple group algebra $\mathbb{F}_q[G]$ in terms of primitive central idempotents and Wedderburn decomposition is known as the fundamental problem in the area of group algebras. It has applications in pure and applied areas of mathematics, mainly in coding theory. Primitive central idempotents are required to find the minimal components of Wedderburn decomposition of $\mathbb{F}_q[G]$. Thus for finding the complete algebraic structure of $\mathbb{F}_q[G]$, our focus is on finding the complete set of primitive central idempotents of $\mathbb{F}_q[G]$.

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Let G be a group such that $G/Z(G) \cong C_p \times C_p$, where C_p is a cyclic group of order p. Cornelissen et al. [2] have classified these groups into nine classes. Out of these nine classes, only five are finite metabelian groups. Gupta et al. [3, 4] have computed the complete algebraic structure of $\mathbb{F}_q[G]$ for G lying in three of these classes.

In this paper the complete algebraic structure of finite semisimple group algebra $\mathbb{F}_q[G]$ is given, where G is finite metabelian group lying in one of the remaining classes, with the following presentation:

$$G =< a, b, x, y | a^p = 1, b^p = y, x^{p^{m_1}} = y^{p^{m_2}} = 1, a^{-1}b^{-1}ab = x^{p^{m_1-1}}, x, y \in Z(G) >$$

2. Notations and Preliminaries

Let G be a finite group with $\gcd(q, |G|) = 1$. Let M and L be subgroups of G with $M \trianglelefteq L$ and L/M cyclic of order n. Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ and ζ be a primitive nth root of unity in $\overline{\mathbb{F}}_q$. Let $\mathcal{C}(L/M)$ be the set of q-cyclotomic cosets of $Irr(L/M)$ which contains the generators of $Irr(L/M)$, where $Irr(G)$ is the set of irreducible characters of G. Consider the action of $T = N_G(M) \cap N_G(L)$ on $\mathcal{C}(L/M)$ by $t * C = t^{-1}Ct$, $t \in T$, $C \in \mathcal{C}(L/M)$. Let $E_G(L/M)$ be the stabilizer of any $C \in \mathcal{C}(L/M)$ and $\mathfrak{R}(L/M)$ be the set of distinct orbits of $\mathcal{C}(L/M)$, under this action. Let $e_C(G, L, M)$ be the sum of distinct conjugates of $\varepsilon_C(L, M)$, where

$$\varepsilon_C(L, M) = |L|^{-1} \sum_{l \in L} tr_{\mathbb{F}_q(\zeta)/\mathbb{F}_q}(\chi(\bar{l}))l^{-1}, \ C \in \mathcal{C}(L/M).$$

For $\mathcal{K} \trianglelefteq G$, let $A_{\mathcal{K}}/\mathcal{K}$ be the normal subgroup of $G/\mathcal{K}$ which is abelian of maximal order and τ be the set of all subgroups $\mathcal{D}/\mathcal{K}$ of $A_{\mathcal{K}}/\mathcal{K}$ such that $A_{\mathcal{K}}/\mathcal{D}$ is cyclic. Let $\tau_{G/\mathcal{K}}$ be the set of all representatives of equivalence classes of τ under the conjugacy relation. Set

$$\mathcal{S}_{G/\mathcal{K}} = \{(\mathcal{D}/\mathcal{K}, A_{\mathcal{K}}/\mathcal{K}) \mid \mathcal{D}/\mathcal{K} \in \tau_{G/\mathcal{K}} \text{ and } \mathrm{Core}_G(\mathcal{D}) = \mathcal{K}\}$$

and

$$\mathcal{S} = \{(\mathcal{K}, \mathcal{D}/\mathcal{K}, A_{\mathcal{K}}/\mathcal{K}) \mid \mathcal{K} \trianglelefteq G, \ \mathcal{S}_{G/\mathcal{K}} \neq \phi, \ (\mathcal{D}/\mathcal{K}, A_{\mathcal{K}}/\mathcal{K}) \in \mathcal{S}_{G/\mathcal{K}}\}.$$

The structure of finite semisimple metabelian group algebra is given by Bakshi et al. [1] as follows:

Theorem 2.1. *[1, Theorem 2] Let $\mathbb{F}_q$ be a field containing q elements and G be a finite metabelian group such that $\gcd(q, |G|) = 1$. Then*

$$\{e_C(G, A_{\mathcal{K}}, \mathcal{D}) \mid (\mathcal{K}, \mathcal{D}/\mathcal{K}, A_{\mathcal{K}}/\mathcal{K}) \in \mathcal{S},\ C \in \mathfrak{R}(A_{\mathcal{K}}/\mathcal{D})\}$$

is the complete set of primitive central idempotents of semisimple group algebra $\mathbb{F}_q[G]$.

The simple component corresponding to primitive central idempotent $e_C(G, A_{\mathcal{K}}, \mathcal{D})$ is $\mathbb{F}_q[G]e_C(G, A_{\mathcal{K}}, \mathcal{D}) \cong M_{[G:A_{\mathcal{K}}]}\left(\mathbb{F}_{q^{o(A_{\mathcal{K}},\mathcal{D})}}\right)$, the algebra of $[G : A_{\mathcal{K}}] \times [G : A_{\mathcal{K}}]$ matrices over the field $\mathbb{F}_{q^{o(A_{\mathcal{K}},\mathcal{D})}}$, where $o(A_{\mathcal{K}}, \mathcal{D}) = \frac{ord_{[A_{\mathcal{K}}:\mathcal{D}]}(q)}{[E_G(A_{\mathcal{K}}/\mathcal{D}):A_{\mathcal{K}}]}$ and the number of simple components corresponding to $e_C(G, A_{\mathcal{K}}, \mathcal{D})$ is $|\mathfrak{R}(A_{\mathcal{K}}/\mathcal{D})|$.

3. Algebraic Structure of $\mathbb{F}_q[G]$

Let G be a group with the presentation:

$$G =< a, b, x, y | a^p = 1, b^p = y, x^{p^{m_1}} = y^{p^{m_2}} = 1, a^{-1}b^{-1}ab = x^{p^{m_1-1}}, \quad x, y \text{ central in } G > .$$

It can be easily seen that

$$G =< a, b, x | a^p = 1, b^{p^{m_2+1}} = 1, x^{p^{m_1}} = 1, a^{-1}b^{-1}ab = x^{p^{m_1-1}}, \quad x, b^p \text{ central in } G > .$$

Theorem 3.1. *Let $m_1, m_2 > 1$. For $m_1 \leq m_2$, the complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$, G as defined above, is given as follows:*

Primitive Central Idempotents for $m_1 \leq m_2$.

$e_C(G, G, < x, a, b >),\ C \in \mathfrak{R}(G/ < x, a, b >);$
$e_C(G, G, < x, a >),\ C \in \mathfrak{R}(G/ < x, a >);$
$e_C(G, G, < x, b >),\ C \in \mathfrak{R}(G/ < x, b >);$
$e_C(G, G, < x, a, b^{p^j} >),\ C \in \mathfrak{R}(G/ < x, a, b^{p^j} >)\ 1 \leq j \leq m_2;$
$e_C(G, G, < x, a^i b^{p^j} >),\ C \in \mathfrak{R}(G/ < x, a^i b^{p^j} >),\ 1 \leq i \leq p-1,\ 0 \leq j \leq m_2;$
$e_C(G, G, < x^{p^v}, x^{ip^{v-1}}a, b >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}}a, b >),$
$\quad 0 \leq i \leq p-1,\ 1 \leq v \leq m_1 - 1;$
$e_C(G, G, < x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j} >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j} >),$
$\quad 1 \leq j \leq m_2 + 1 - v,\ , 0 \leq i \leq p-1,\ 1 \leq v \leq m_1 - 1,\ \gcd(k, p^v) = 1;$
$e_C(G, G, < x^{p^v}, x^{ip^{v-1}}a, x^k b >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}}a, x^k b >),$

$$
\begin{aligned}
&\quad 0 \le i \le p-1,\ \gcd(k,p^v)=p^\alpha,\ 0\le\alpha\le m_1-1;\\
&e_C(G,<b,x>,<b>),\ \ C\in\mathfrak{R}(<b,x>/<b>);\\
&e_C(G,<a,x,y>,<a,x^jy>),\ \ C\in\mathfrak{R}(<a,x,y>/<a,x^jy>),\\
&\quad \gcd(j,p^{m_1})=p^\alpha,\ 0\le\alpha\le m_1-1;\\
&e_C(G,<a,x,y>,<a,x^jy^{p^\beta}>),\ \ C\in\mathfrak{R}(<a,x,y>/<a,x^jy^{p^\beta}>),\\
&\quad \gcd(j,p^{m_1})=1,\ 1\le\beta\le m_2-m_1.
\end{aligned}
$$

Wedderburn Decomposition for $m_1 \le m_2$.

$$
\begin{aligned}
\mathbb{F}_q[G] \cong & \mathbb{F}_q \oplus \left(\mathbb{F}_{q^{f_1}}\right)^{\frac{p-1}{f_1}} \oplus \left(\mathbb{F}_{q^{f_{m_2+1}}}\right)^{\frac{p^{m_2+1}-p^{m_2}}{f_{m_2+1}}} \oplus_{j=1}^{m_2} \left(\mathbb{F}_{q^{f_j}}\right)^{\frac{p^j-p^{j-1}}{f_j}}\\
& \oplus_{j=0}^{m_2} \left(\mathbb{F}_{q^{f_{j+1}}}\right)^{\frac{p^j(p-1)^2}{f_{j+1}}} \oplus_{v=1}^{m_1-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{v+1}-p^v}{f_v}}\\
& \oplus_{v=1}^{m_1-1} \oplus_{j=1}^{m_2+1-v} \left(\mathbb{F}_{q^{f_{j+v}}}\right)^{\frac{p^{2v+j-1}(p-1)^2}{f_{j+v}}} \oplus_{v=1}^{m_1-1} \oplus_{\alpha=0}^{v-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{2v-\alpha-1}(p-1)^2}{f_v}}\\
& \oplus M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}} \oplus_{\alpha=0}^{m_1-1} M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{2m_1-\alpha-2}(p-1)^2}{f_{m_1}}}\\
& \oplus_{\beta=1}^{m_2-m_1} M_p\left(\mathbb{F}_{q^{f_{m_1+\beta}}}\right)^{\frac{p^{2m_1+\beta-2}(p-1)^2}{f_{m_1+\beta}}}
\end{aligned}
$$

Proof. We will first find all the normal subgroups of G.

Let $\mathcal{K}\trianglelefteq G$ such that $\mathcal{K}\cap <x>\neq\{e\}$, then $\mathcal{K}\cap <x>=<x^{p^v}>, 0\le v\le m_1-1$. For $\mathcal{K}\cap <x>=<x>$, it can be easily seen that $\mathcal{K}$ is either $<x>$ or $<x,a>$ or $<x,b^{p^j}>$ or $<x,a,b^{p^j}>$ or $<x,a^ib^{p^j}>$, $1\le i\le p-1$, $0\le j\le m_2$.

Assume that $\mathcal{K}\cap <x>=<x^{p^v}>$, $1\le v\le m_1-1$. Now $\mathcal{K}/<x^{p^v}>$ is isomorphic to one of the following $<x>$, $<a<x>>$, $<b^{p^j}<x>>$, $<a^ib^{p^j}<x>>$, $<a<x>,b^{p^j}<x>>$, $1\le i\le p-1$, $0\le j\le m_2$. Let $\mathcal{K}/<x^{p^v}>\cong<x>$, then $\mathcal{K}=<x^{p^v}>$. Let $\mathcal{K}/<x^{p^v}>\cong<a<x>>$, then $\mathcal{K}=<x^{p^v},x^ia>$, which implies for $i=p^v$, $\mathcal{K}=<x^{p^v},a>$ and if $\gcd(i,p^v)=p^\alpha$, then $(x^{p^\alpha}a)^p=x^{p^{\alpha+1}}\in\mathcal{K}$ if and only if $\alpha+1\ge v$, i.e., $\alpha\ge v-1$. Hence in this case $\mathcal{K}=<x^{p^v},a>$ or $<x^{p^v},x^{ip^{v-1}}a>, 1\le i\le p-1$.

Let $\mathcal{K}/<x^{p^v}>\cong<b^{p^j}<x>>$, $0\le j\le m_2$. Then $\mathcal{K}=<x^{p^v},x^kb^{p^j}>$. If $k=p^v$, then $\mathcal{K}=<x^{p^v},b^{p^j}>$ and if $k<p^v$, then $\mathcal{K}=<x^{p^v},x^kb^{p^j}>$, $\gcd(k,p^v)=p^\alpha$, $0\le\alpha\le m_1-1$. Consider $(x^{p^\alpha}b^{p^j})^{p^{m_2+1-j}}=x^{p^{\alpha+m_2+1-j}}\in\mathcal{K}$ if and only if $\alpha+m_2+1-j\ge v$, i.e., $j\le\alpha+m_2+1-v$. Thus in this case $\mathcal{K}=<x^{p^v},x^kb^{p^j}>$, $\gcd(k,p^v)=p^\alpha$, $0\le\alpha\le m_1-1$, $0\le j\le\alpha+m_2+1-v$. Further if $j=0$, then $(x^{p^\alpha}b)^{p^{m_2+1}}=x^{p^{\alpha+m_2+1}}\in\mathcal{K}$ if, and only if, $\alpha+m_2+1\ge v$, i.e., $v-m_2-1\le\alpha$. Thus

$\mathcal{K} =< x^{p^v}, b >$ or $< x^{p^v}, x^k b >$, $\gcd(k, p^v) = p^\alpha$, $\max\{0, v - m_2 - 1\} \leq \alpha \leq m_1 - 1$. If $1 \leq j \leq m_2 + 1 - v + \alpha$, then $\mathcal{K} =< x^{p^v}, b^{p^j} >$ or $< x^{p^v}, x^k b^{p^j} >$, $\gcd(k, p^v) = 1$, $1 \leq j \leq m_2 + 1 - v$.

By following the same procedure in all the above cases, we will get that the normal subgroups $\mathcal{K}$ such that $\mathcal{K} \cap G \neq \{e\}$, are as follows:

$$\begin{aligned}
&< x >,\ < x, a >,\ < x, b^{p^j} >,\ < x, a^i b^{p^j} >,\ < x, a, b^{p^j} >,\\
&\quad 1 \leq i \leq p - 1,\ 0 \leq j \leq m_2,\\
&< x^{p^v}, a >,\ < x^{p^v}, x^{ip^{v-1}} a >,\ 1 \leq i \leq p - 1,\ 1 \leq v \leq m_1 - 1,\\
&< x^{p^v}, b >,\ < x^{p^v}, x^k b >,\ \gcd(k, p^v) = p^\alpha,\ \max\{0, v - m_2 - 1\} \leq \alpha \leq m_1 - 1,\\
&\quad 1 \leq v \leq m_1 - 1,\\
&< x^{p^v}, b^{p^j} >,\ < x^{p^v}, x^k b^{p^j} >,\ \gcd(k, p^v) = 1,\ 1 \leq j \leq m_2 + 1 - v,\\
&< x^{p^v}, a^i b^{p^j} >,\ < x^{p^v}, x^k a^i b^{p^j} >,\ \gcd(k, p^v) = p^\alpha,\ 0 \leq \alpha \leq m_1 - 1,\\
&\quad 1 \leq v \leq m_1 - 1\ 0 \leq j \leq m_2 + 1 - v + \alpha,\\
&< x^{p^v}, x^{ip^{v-1}} a, b^{p^j} >,\ 0 \leq i \leq p - 1,\ 0 \leq j \leq m_2,\ 1 \leq v \leq m_1 - 1,\\
&< x^{p^v}, x^{ip^{v-1}} a, x^k b >,\ 0 \leq i \leq p - 1,\ \gcd(k, p^v) = p^\alpha,\ 1 \leq v \leq m_1 - 1\\
&\quad \max\{0, v - m_2 - 1\} \leq \alpha \leq m_1 - 1,\\
&< x^{p^v}, x^{ip^{v-1}} a, x^k b^{p^j} >,\ 0 \leq i \leq p - 1,\ \gcd(k, p^v) = 1,\\
&\quad 1 \leq j \leq m_2 + 1 - v,\ 1 \leq v \leq m_1 - 1.
\end{aligned}$$

Observe that if $\mathcal{K} \cap G =< x^{p^v} >,\ 0 \leq v \leq m_1 - 1$, then $G' \subseteq \mathcal{K}$ and hence $G/\mathcal{K}$ is abelian. Thus

$$S_{G/\mathcal{K}} = \begin{cases} (< 1 >, G/\mathcal{K}), & \text{if } G/\mathcal{K} \text{ is cyclic,} \\ \phi & \text{otherwise.} \end{cases} .$$

Out of these normal subgroups following have cyclic quotient with G:

$$\begin{aligned}
&< x, a >,\ < x, b >,\ < x, a, b >,\ < x, a, b^{p^j} >,\ 1 \leq j \leq m_2,\\
&< x, a^i b^{p^j} >,\ 1 \leq i \leq p - 1,\ 0 \leq j \leq m_2,\\
&< x^{p^v}, x^{ip^{v-1}} a, b >,\ 0 \leq i, \leq p - 1,\ 1 \leq v \leq m_1 - 1,\\
&< x^{p^v}, x^{ip^{v-1}} a, x^k b >,\ 1 \leq v \leq m_1 - 1,\ 0 \leq i \leq p - 1,\\
&\quad \gcd(k, p^v) = p^\alpha,\ \max\{0, v - m_2 - 1\} \leq \alpha \leq m_1 - 1,\\
&< x^{p^v}, x^{ip^{v-1}} a, x^k b^{p^j} >,\ 1 \leq v \leq m_1 - 1,\ 0 \leq i \leq p - 1,\\
&\quad \gcd(k, p^v) = 1,\ 0 \leq j \leq m_2 + 1 - v.
\end{aligned}$$

Now, assume that $\mathcal{K} \cap G = \{e\}$, then $\mathcal{K} = \{e\}$ or $< y^{p^j} >,\ 0 \leq j \leq p - 1$ or $< x^j y^{p^\beta} >,\ \gcd(j, p^{m_1}) = p^\alpha,\ 0 \leq \alpha \leq m_1 - 1,\ 0 \leq \beta \leq m_2 - m_1$.

If $\mathcal{K} = \{e\}$ or $< y^{p^j} >,\ 1 \leq j \leq p - 1$, then $S_{G/\mathcal{K}} = \phi$.

If $\mathcal{K} = \{y\}$, then $S_{G/\mathcal{K}} = \{< b > /\mathcal{K}, < b, x > /\mathcal{K}\}$.

If $\mathcal{K} =< x^jy >$, $\gcd(j, p^{m_1}) = p^\alpha$, $0 \le \alpha \le m_1 - 1$, then $S_{G/\mathcal{K}} = \{< a, x^jy > /\mathcal{K}, < a, x, y > /\mathcal{K}\}$.

If $\mathcal{K} =< x^jy^{p^\beta} >$, $\gcd(j, p^{m_1}) = 1$, $1 \le \beta \le m_2 - m_1$, then $S_{G/\mathcal{K}} = \{< a, x^jy^{p^\beta} > /\mathcal{K}, < a, x, y > /\mathcal{K}\}$.

Corresponding to these normal subgroups, $o(A_\mathcal{K}, D)$ and $|\mathfrak{R}(A_{,\mathcal{K}}, D)|$ have been given in Table 1.

TABLE 1.

$\mathcal{K}$	$(D, A_\mathcal{K})$	$o(A_\mathcal{K}, D)$	$\lvert\mathfrak{R}(A_\mathcal{K}/D)\rvert$
$< x, a, b >$	(G, G)	1	1
$< x, a >$	$(\mathcal{K}, G)$	f_{m_2+1}	$\frac{p^{m_2+1}-p^{m_2}}{f_{m_2+1}}$
$< x, b >$	$(\mathcal{K}, G)$	f_1	$\frac{p-1}{f_1}$
$< x, a, b^{p^j} >$	$(\mathcal{K}, G)$	f_j	$\frac{p^j-p^{j-1}}{f_j}$
$< x, a^ib^{p^j} >, 1 \le i \le p-1$ $0 \le j \le m_2$	$(\mathcal{K}, G)$	f_{j+1}	$\frac{p^{j+1}-p^j}{f_{j+1}}$
$< x^{p^v}, x^{ip^{v-1}}a, b >, 1 \le v \le m_1 - 1$ $0 \le i \le p-1$	$(\mathcal{K}, G)$	f_v	$\frac{p^v-p^{v-1})}{f_v}$
$< x^{p^v}, x^{ip^{v-1}}a, x^kb >,$ $0 \le i \le p-1,$ $1 \le v \le m_1 - 1,\ \gcd(k, p^v) = p^\alpha,$ $\max\{0, v - m_2 - 1\} \le \alpha \le m_1 - 1$	$(\mathcal{K}, G)$	f_v	$\frac{p^v-p^{v-1}}{f_v}$
$< x^{p^v}, x^{ip^{v-1}}a, x^kb^{p^j} >,$ $0 \le i \le p-1,$ $1 \le v \le m_1 - 1,\ \gcd(k, p^v) = 1$ $1 \le j \le m_2 + 1 - v$	$(\mathcal{K}, G)$	f_{j+v}	$\frac{p^{j+v}-p^{j+v-1}}{f_{j+v}}$
$< y >$	$(< b >, < b, x >)$	f_{m_1}	$\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}$
$< x^jy >$ $\gcd(j, p^{m_1}) = p^\alpha,\ 0 \le \alpha \le m_1 - 1$	$(< a, x^jy >,$ $< a, x, y >)$	f_{m_1}	$\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}$
$< x^jy^{p^\beta} >$ $\gcd(j, p^{m_1}) = 1,\ 1 \le \beta \le m_2 - m_1$	$(< a, x^jy^{p^\beta} >,$ $< a, x, y >)$	$f_{m_1+\beta}$	$\frac{p^{m_1+\beta}-p^{m_1+\beta-1}}{f_{m_1+\beta}}$

Thus by using this table primitive central idempotents and Wedderburn decomposition given in Theorem 3.1 can be easily obtained. □

Theorem 3.2. *Let $m_1, m_2 > 1$. For $m_1 > m_2$, the complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$, G as defined above, is given as follows:*

Primitive Central Idempotents for $m_1 > m_2$.

$$e_C(G, G, <x, a, b>),\ C \in \mathfrak{R}(G/<x, a, b>);$$
$$e_C(G, G, <x, a>),\ C \in \mathfrak{R}(G/<x, a>);$$
$$e_C(G, G, <x, b>),\ C \in \mathfrak{R}(G/<x, b>);$$
$$e_C(G, G, <x, a, b^{p^j}>),\ C \in \mathfrak{R}(G/<x, a, b^{p^j}>)\ 1 \le j \le m_2;$$
$$e_C(G, G, <x, a^i b^{p^j}>),\ C \in \mathfrak{R}(G/<x, a^i b^{p^j}>),\ 1 \le i \le p-1,\ 0 \le j \le m_2;$$
$$e_C(G, G, <x^{p^v}, x^{ip^{v-1}}a, b>),\ C \in \mathfrak{R}(G/<x^{p^v}, x^{ip^{v-1}}a, b>),$$
$$0 \le i \le p-1,\ 1 \le v \le m_1 - 1;$$
$$e_C(G, G, <x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j}>),\ C \in \mathfrak{R}(G/<x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j}>),$$
$$1 \le j \le m_2 + 1 - v,\ ,0 \le i \le p-1,\ 1 \le v \le m_2,\ \gcd(k, p^v) = 1;$$
$$e_C(G, G, <x^{p^v}, x^{ip^{v-1}}a, x^k b>),\ C \in \mathfrak{R}(G/<x^{p^v}, x^{ip^{v-1}}a, x^k b>),$$
$$0 \le i \le p-1,\ \gcd(k, p^v) = p^\alpha,\ \max\{0, v - m_2 - 1\} \le \alpha \le v - 1;$$
$$e_C(G, <b, x>, <b>),\ C \in \mathfrak{R}(<b, x> / <b>);$$
$$e_C(G, <a, x, y>, <a, x^j y>),\ C \in \mathfrak{R}(<a, x, y> / <a, x^j y>),$$
$$\gcd(j, p^{m_1}) = p^\alpha,\ m_1 - m_2 \le \alpha \le m_1 - 1.$$

Wedderburn Decomposition for $m_1 > m_2$.

$$\begin{aligned}\mathbb{F}_q[G] \cong & \mathbb{F}_q \oplus \left(\mathbb{F}_{q^{f_1}}\right)^{\frac{p-1}{f_1}} \oplus \left(\mathbb{F}_{q^{f_{m_2+1}}}\right)^{\frac{p^{m_2+1}-p^{m_2}}{f_{m_2+1}}} \oplus_{j=1}^{m_2} \left(\mathbb{F}_{q^{f_j}}\right)^{\frac{p^j-p^{j-1}}{f_j}} \\ & \oplus_{j=0}^{m_2} \left(\mathbb{F}_{q^{f_{j+1}}}\right)^{\frac{p^j(p-1)^2}{f_{j+1}}} \oplus_{v=1}^{m_1-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{v+1}-p^v}{f_v}} \\ & \oplus_{v=1}^{m_2} \oplus_{j=1}^{m_2+1-v} \left(\mathbb{F}_{q^{f_{j+v}}}\right)^{\frac{p^{2v+j-1}(p-1)^2}{f_{j+v}}} \oplus_{v=1}^{m_2} \oplus_{\alpha=0}^{v-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{2v-\alpha-1}(p-1)^2}{f_v}} \\ & \oplus_{v=m_2+1}^{m_1-1} \oplus_{\alpha=v-m_2-1}^{v-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{2v-\alpha-1}(p-1)^2}{f_v}} \oplus M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}} \\ & \oplus_{\alpha=m_1-m_2}^{m_1-1} M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{2m_1-\alpha-2}(p-1)^2}{f_{m_1}}}\end{aligned}$$

Proof. By following the same procedure as in Theorem 3.1, we will get that the normal subgroups $\mathcal{K}$ such that $\mathcal{K} \cap G \neq \{e\}$, are as follows:

$$<x>,\ <x, a>,\ <x, b^{p^j}>,\ <x, a^i b^{p^j}>,\ <x, a, b^{p^j}>,$$
$$1 \le i \le p-1,\ 0 \le j \le m_2,$$
$$<x^{p^v}, a>,\ <x^{p^v}, x^{ip^{v-1}}a>,\ 1 \le i \le p-1,\ 1 \le v \le m_1 - 1,$$
$$<x^{p^v}, b>,\ <x^{p^v}, x^k b>,\ \gcd(k, p^v) = p^\alpha,$$

$\max\{0, v-m_2-1\} \leq \alpha \leq v-1,\ 1 \leq v \leq m_1-1,$
$< x^{p^v}, b^{p^j} >,\ < x^{p^v}, x^k b^{p^j} >,\ \gcd(k, p^v) = 1,\ 1 \leq j \leq m_2+1-v,\ 1 \leq v \leq m_2,$
$< x^{p^v}, a^i b^j >, < x^{p^v}, x^k a^i b^j >, \gcd(k, p^v) = p^\alpha, \max\{0, v-m_2-1\} \leq \alpha \leq v-1,$
$< x^{p^v}, a^i b^{p^j} >,\ < x^{p^v}, x^k a^i b^{p^j} >,\ \gcd(k, p^v) = 1,$
$1 \leq j \leq m_2+1-v,\ 1 \leq v \leq m_2,$
$< x^{p^v}, x^{ip^{v-1}} a, b^{p^j} >,\ 0 \leq i \leq p-1,\ 0 \leq j \leq m_2,\ 1 \leq v \leq m_1-1,$
$< x^{p^v}, x^{ip^{v-1}} a, x^k b >,\ 0 \leq i \leq p-1,\ \gcd(k, p^v) = p^\alpha,$
$\max\{0, v-m_2-1\} \leq \alpha \leq v-1,\ 1 \leq v \leq m_1-1,$
$< x^{p^v}, x^{ip^{v-1}} a, x^k b^{p^j} >,\ 0 \leq i \leq p-1,\ \gcd(k, p^v) = 1,$
$1 \leq j \leq m_2+1-v,\ 1 \leq v \leq m_2.$

Out of the normal subgroups listed above, only following have cyclic quotient with G:

$< x, a >,\ < x, b >,\ < x, a, b >,\ < x, a, b^{p^j} >,\ 1 \leq j \leq m_2,$
$< x, a^i b^{p^j} >,\ 1 \leq i \leq p-1,\ 0 \leq j \leq m_2,$
$< x^{p^v}, x^{ip^{v-1}} a, b >,\ 0 \leq i, \leq p-1,\ 1 \leq v \leq m_1-1,$
$< x^{p^v}, x^{ip^{v-1}} a, x^k b >,\ 1 \leq v \leq m_1-1,\ 0 \leq i \leq p-1,\ \gcd(k, p^v) = p^\alpha,$
$\max\{0, v-m_2-1\} \leq \alpha \leq v-1,$
$< x^{p^v}, x^{ip^{v-1}} a, x^k b^{p^j} >,\ 0 \leq i \leq p-1,\ \gcd(k, p^v) = 1,$
$0 \leq j \leq m_2+1-v,\ 1 \leq v \leq m_1-1.$

Now, assume that $\mathcal{K} \cap G = \{e\}$. In this case, if $\mathcal{K} = \{e\}$ or $< y^{p^j} >,\ 1 \leq j \leq p-1$, then $S_{G/\mathcal{K}} = \phi$; if $\mathcal{K} = \{y\}$, then $S_{G/\mathcal{K}} = \{< b > /\mathcal{K}, < b, x > /\mathcal{K}\}$; if $\mathcal{K} =< x^j y >,\ \gcd(j, p^{m_1}) = p^\alpha,\ m_1 - m_2 \leq \alpha \leq m_1 - 1$; then $S_{G/\mathcal{K}} = \{< a, x^j y > /\mathcal{K}, < a, x, y > /\mathcal{K}\}$.

Corresponding to these normal subgroups, $o(A_\mathcal{K}, D)$ and $|\mathfrak{R}(A_{,\mathcal{K}}, D)|$ have been given in Table 2.

Table 2:

$\mathcal{K}$	$(D, A_\mathcal{K})$	$o(A_\mathcal{K}, D)$	$\lvert\mathfrak{R}(A_\mathcal{K}/D)\rvert$
$< x, a, b >$	(G, G)	1	1
$< x, a >$	$(\mathcal{K}, G)$	f_{m_2+1}	$\frac{p^{m_2+1}-p^{m_2}}{f_{m_2+1}}$
$< x, b >$	$(\mathcal{K}, G)$	f_1	$\frac{p-1}{f_1}$
$< x, a, b^{p^j} >$	$(\mathcal{K}, G)$	f_j	$\frac{p^j-p^{j-1}}{f_j}$

$< x, a^i b^{p^j} >, 1 \leq i \leq p-1$ $0 \leq j \leq m_2$	$(\mathcal{K}, G)$	f_{j+1}	$\frac{p^{j+1}-p^j}{f_{j+1}}$
$< x^{p^v}, x^{ip^{v-1}}a, b >, 1 \leq v \leq m_1 - 1$ $0 \leq i \leq p-1$	$(\mathcal{K}, G)$	f_v	$\frac{p^v-p^{v-1})}{f_v}$
$< x^{p^v}, x^{ip^{v-1}}a, x^k b >,$ $0 \leq i \leq p-1,\ \gcd(k, p^v) = p^\alpha$	$(\mathcal{K}, G)$	f_v	$\frac{p^v-p^{v-1}}{f_v}$
$< x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j} >,$ $0 \leq i \leq p-1,\ \gcd(k, p^v) = 1,$ $1 \leq j \leq m_2 + 1 - v,$ $1 \leq v \leq m_2$	$(\mathcal{K}, G)$	f_{j+v}	$\frac{p^{j+v}-p^{j+v-1}}{f_{j+v}}$
$< y >$	$(< b >, < b, x >)$	f_{m_1}	$\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}$
$< x^j y >,\ \gcd(j, p^{m_1}) = p^\alpha,$ $m_1 - m_2 \leq \alpha \leq m_1 - 1$	$(< a, x^j y >,$ $< a, x, y >)$	f_{m_1}	$\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}$

Thus by using this table primitive central idempotents and Wedderburn decomposition given in Theorem 3.2 can be easily obtained. $\square$

We have obtained the complete algebraic structure of $\mathbb{F}_q[G]$ for $m_1, m_2 > 1$. In the following theorems, we have obtained the same for cases $m_1 \geq 1, m_2 = 1$ and $m_1 = 1, m_2 \geq 1$.

Theorem 3.3. *Let G be a group defined above. For $m_1 \geq 1, m_2 = 1$ the complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$ is given as follows:*

Primitive Central Idempotents for $m_1 \geq 1, m_2 = 1$.

$e_C(G, G, < x, a, b >),\ C \in \mathfrak{R}(G/ < x, a, b >);$
$e_C(G, G, < x, a >),\ C \in \mathfrak{R}(G/ < x, a >);$
$e_C(G, G, < x, b >),\ C \in \mathfrak{R}(G/ < x, b >);$
$e_C(G, G, < x, a, b^p >),\ C \in \mathfrak{R}(G/ < x, a, b^p >);$
$e_C(G, G, < x, a^i b >),\ C \in \mathfrak{R}(G/ < x, a^i b >),\ 1 \leq i \leq p-1;$
$e_C(G, G, < x, a^i b^p >),\ C \in \mathfrak{R}(G/ < x, a^i b^p >),\ 1 \leq i \leq p-1;$
$e_C(G, G, < x^p, a, x^i b^p >),\ C \in \mathfrak{R}(G/ < x^p, a, x^i b^p >),\ 1 \leq i \leq p-1;$
$e_C(G, G, < x^p, x^i a, x^j b^p >),\ C \in \mathfrak{R}(G/ < x^p, x^i a, x^j b^p >),\ 1 \leq i, j \leq p-1;$
$e_C(G, G, < x^{p^v}, x^{ip^{v-1}}a, x^j b^p >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}}a, x^{jp^{v-1}}b >),$
$0 \leq i, j \leq p-1,\ 1 \leq v \leq m_1 - 1;$

$$e_C(G, G, < x^{p^v}, x^{ip^{v-1}}a, x^{jp^{v-1}}b >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}}a, x^{jp^{v-1}}b >),$$
$$0 \le i \le p-1,\ \ ,1 \le j \le p-1,\ 2 \le v \le m_1 - 1;$$
$$e_C(G, < b, x >, < b >),\ C \in \mathfrak{R}(< b, x > / < b >);$$
$$e_C(G, < a, x, y >, < a, x^{ip^{m_1-1}}y >),\ C \in \mathfrak{R}(< a, x, y > / < a, x^{ip^{m_1-1}}y >),$$
$$1 \le i \le p-1.$$

Wedderburn Decomposition for $m_1 \geq 1, m_2 = 1$.

$$\mathbb{F}_q[G] \cong \mathbb{F}_q \oplus \left(\mathbb{F}_{q^{f_1}}\right)^{\frac{p^2-1}{f_1}} \oplus \left(\mathbb{F}_{q^{f_2}}\right)^{\frac{p^3(p-1)}{f_2}} \oplus_{v=1}^{m_1-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{v+2}-p^{v+1}}{f_v}}$$
$$\oplus_{v=2}^{m_1-1} \left(\mathbb{F}_{q^{f_{v+1}}}\right)^{\frac{p(p-1)(p^{v+1}-p^v)}{f_{v+1}}} \oplus M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{m_1}(p-1)}{f_{m_1}}}$$

Theorem 3.4. *Let G be a group defined above. For $m_1 = 1, m_2 \geq 1$ the complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$ is given as follows:*

Primitive Central Idempotents for $m_1 = 1, m_2 \geq 1$.

$$e_C(G, G, < x, a, b >),\ C \in \mathfrak{R}(G/ < x, a, b >);$$
$$e_C(G, G, < x, a >),\ C \in \mathfrak{R}(G/ < x, a >);$$
$$e_C(G, G, < x, b >),\ C \in \mathfrak{R}(G/ < x, b >);$$
$$e_C(G, G, < x, a, b^{p^j} >),\ C \in \mathfrak{R}(G/ < x, a, b^{p^j} >)\ 1 \le j \le m_2;$$
$$e_C(G, G, < x, a^i b^{p^j} >),\ C \in \mathfrak{R}(G/ < x, a^i b^{p^j} >),\ 1 \le i \le p-1,\ 0 \le j \le m_2;$$
$$e_C(G, < b, x >, < b >),\ C \in \mathfrak{R}(< b, x > / < b >);$$
$$e_C(G, < a, x, y >, < a, x^i y^{p^j} >),\ C \in \mathfrak{R}(< a, x, y > / < a, x^i y^{p^j} >),$$
$$1 \le i \le p-1,\ 0 \le j \le m_2 - 1.$$

Wedderburn Decomposition for $m_1 = 1, m_2 \geq 1$.

$$\mathbb{F}_q[G] \cong \mathbb{F}_q \oplus \left(\mathbb{F}_{q^{f_1}}\right)^{\frac{p-1}{f_1}} \oplus \left(\mathbb{F}_{q^{f_{m_2+1}}}\right)^{\frac{p^{m_2+1}-p^{m_2}}{f_{m_2+1}}} \oplus_{j=1}^{m_2} \left(\mathbb{F}_{q^{f_j}}\right)^{\frac{p^j-p^{j-1}}{f_j}}$$
$$\oplus_{j=0}^{m_2} \left(\mathbb{F}_{q^{f_{j+1}}}\right)^{\frac{p^j(p-1)^2}{f_{j+1}}} \oplus M_p\left(\mathbb{F}_{q^{f_1}}\right)^{\frac{(p-1)}{f_1}} \oplus_{j=0}^{m_2-1} M_p\left(\mathbb{F}_{q^{f_{j+1}}}\right)^{\frac{p^j(p-1)^2}{f_{j+1}}}$$

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