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A NOTE ON THE NON-TRIVIAL ELEMENTS IN THE COHOMOLOGY GROUPS OF THE STEENROD ALGEBRA

Đặng Võ Phúc

ABSTRACT. Let F_2 be the prime field of two elements and let $GL_s := GL(s, F_2)$ be the general linear group of rank s. Denote by $\mathscr{A}$ the Steenrod algebra over F_2. The (mod-2) Lambda algebra, Λ, is one of the tools to describe those mysterious "Ext-groups". In addition, the s-th algebraic transfer of William Singer [19] is also expected to be a useful tool in the study of them. It is a homomorphism $Tr_s : F_2 \otimes_{GL_s} P_{\mathscr{A}}(H_*(B\mathbb{V}_s)) \to \mathrm{Ext}_{\mathscr{A}}^{s,s+*}(F_2, F_2)$, where $\mathbb{V}_s$ denotes an elementary abelian 2-group of rank s, and $H_*(B\mathbb{V}_s)$ is the (mod-2) homology of a classifying space of $\mathbb{V}_s$, while $P_{\mathscr{A}}(H_*(B\mathbb{V}_s))$ means the primitive part of $H_*(B\mathbb{V}_s)$ under the action of $\mathscr{A}$. It has been shown that Tr_s is highly non-trivial and, more precisely, that Tr_s is an isomorphism for $s \leq 3$. In addition, Singer proved that Tr_4 is an isomorphism in some internal degrees. He also investigated the image of the fifth transfer by using the invariant theory.

In this note, we use another method to study the image of Tr_5. More precisely, by direct computations using a representation of Tr_5 over the algebra Λ, we show that Tr_5 detects the non-zero elements $h_0d_0 \in \mathrm{Ext}_{\mathscr{A}}^{5,5+14}(F_2, F_2)$, $h_2e_0 = h_0g \in \mathrm{Ext}_{\mathscr{A}}^{5,5+20}(F_2, F_2)$ and $h_3e_0 = h_4h_1c_0 \in \mathrm{Ext}_{\mathscr{A}}^{5,5+24}(F_2, F_2)$. The same argument can be used for homological degrees $s \geq 6$ under certain conditions.

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1. The Singer algebraic transfer and the (mod-2) Lambda algebra

Let $\mathbb{V}_s$ be a rank s elementary abelian 2-group and let $B\mathbb{V}_s$ denote the classifying space of $\mathbb{V}_s$. We may equally well view $\mathbb{V}_s$ as an s-dimensional F_2-vector space. As well-known, $H^*(\mathbb{V}_s, F_2) \cong S(\mathbb{V}_s^*)$, the symmetric algebra over the dual space $\mathbb{V}_s^* = H^1(\mathbb{V}_s, F_2)$. Pick $x_1, \ldots, x_s$ to be a basis of $H^1(\mathbb{V}_s, F_2)$. Then, we have $H^*(B\mathbb{V}_s) = H^*(\mathbb{V}_s) \cong \mathcal{P}_s := F_2[x_1, \ldots, x_s]$, the polynomial algebra on s generators of dimension one. It is considered as an unstable $\mathscr{A}$-module. The canonical $\mathscr{A}$-action on $\mathcal{P}_1$ is extended to an $\mathscr{A}$-action on $F_2[x_1, x_1^{-1}]$, the ring of finite Laurent series (see [2], [22]). Then, $\overline{\mathcal{P}} = \langle\{x_1^t |\ t \geq -1\}\rangle$ is $\mathscr{A}$-submodule of $F_2[x_1, x_1^{-1}]$. One has a short-exact sequence: $0 \to \mathcal{P}_1 \xrightarrow{q} \overline{\mathcal{P}} \xrightarrow{\pi} \Sigma^{-1}F_2$, where q is the inclusion and π is given by $\pi(x_1^t) = 0$ if $t \neq -1$ and $\pi(x_1^{-1}) = 1$. We now denote by e_1 the corresponding element in $\mathrm{Ext}^1_{\mathscr{A}}(\Sigma^{-1}F_2, \mathcal{P}_1)$. Based on the cross and Yoneda products, Singer considered the element

$$e_s = (e_1 \times \mathcal{P}_{s-1}) \circ (e_1 \times \mathcal{P}_{s-2}) \circ \cdots \circ (e_1 \times \mathcal{P}_1) \circ e_1 \in \mathrm{Ext}^s_{\mathscr{A}}(\Sigma^{-s}F_2, \mathcal{P}_s).$$

and defined $\overline{\varphi}_s : \mathrm{Tor}^{\mathscr{A}}_s(F_2, \Sigma^{-s}F_2) \to \mathrm{Tor}^{\mathscr{A}}_0(F_2, \mathcal{P}_s)$ by $\overline{\varphi}_s(z) = e_s \cap z$. Its image is a submodule of the variant $(F_2 \otimes_{\mathscr{A}} \mathcal{P}_s)^{GL_s}$, where GL_s denotes the general linear group of rank s over F_2. Hence, $\overline{\varphi}_s$ induces homomorphism

$$\varphi_s : \mathrm{Tor}^{\mathscr{A}}_s(F_2, \Sigma^{-s}F_2) \to (F_2 \otimes_{\mathscr{A}} \mathcal{P}_s)^{GL_s}.$$

Let $\mathcal{A}_2^+$ be the augmentation ideal of $\mathcal{A}$ and let $P_{\mathscr{A}}(H_*(B\mathbb{V}_s))$ be the subspace of $H_*(B\mathbb{V}_s)$ consisting of all elements that are $\mathcal{A}_2^+$-annihilated. Then, the dual homomorphism

$$Tr_s : F_2 \otimes_{GL_s} P_{\mathscr{A}}(H_*(B\mathbb{V}_s)) \to \mathrm{Ext}^{s,s+*}_{\mathscr{A}}(F_2, F_2)$$

of φ_s is called *the s-th Singer algebraic transfer*. It is expected to be a useful tool in describing the cohomology groups of $\mathscr{A}$ by means of the invariant theory and the Peterson "hit problem" of determining the minimal set of $\mathscr{A}$-generatos for $\mathcal{P}_s$ (see [3], [7], [12], [13–16, 18], [21]). It was then investigated by many authors (see Boardman [3], Chơn and Hà [5], Crossley [6], Lê [8], Minami [11], the present author [13, 14, 16–18], and others). By Boardman [3] and Singer [19], Tr_s is an isomorphism for $s \leq 3$. In [19], Singer also gave computations to show that Tr_4 is an isomorphism in a range of internal degrees.

As it is known, the homological algebra $\{H_d(B\mathbb{V}_s)|\ d \geq 0\}$ is dual to $\mathcal{P}_s$. Moreover, it is isomorphic to $\Gamma(a_1, \ldots, a_s)$, the divided power algebra generated by $a_1, \dots, a_s$, each of degree one, where $a_j = a_j^{(1)}$ is dual to x_j. Here the duality is taken with respect to the basis of $\mathcal{P}_s$ consisting of all monomials in $x_1, \dots, x_s$. The multiplication on $\{H_d(B\mathbb{V}_s)|\ d \geq 0\}$ defined by

$$(\prod_{1\leq i\leq n} a_i^{(j_i)})(\prod_{1\leq i\leq m} a_i^{(j_{n+i})}) = \prod_{1\leq i\leq n+m} a_i^{(j_i)} \in H_{n+m}(B\mathbb{V}_s)$$

for all $\prod_{1\leq i\leq n} a_i^{(j_i)} \in H_n(B\mathbb{V}_s)$, and $\prod_{1\leq i\leq m} a_i^{(j_{n+i})} \in H_m(B\mathbb{V}_s)$. This algebra is a right $\mathscr{A}$-module. The right $\mathscr{A}$-action on this algebra is given by $(a_t^{(j)})Sq^s = \binom{j-s}{s}a_t^{(j-s)} = Sq_*^s(a_t^{(j)})$ and Cartan's formula. Note that Sq_*^s is the dual Steenrod operation.

We know that the (mod 2) Lambda algebra Λ (see Bousfield et al. [4]) is also one of the tools to compute the cohomology groups of $\mathscr{A}$. Recall that Λ is defined as a differential, bigraded, associative algebra with unit over $\mathbb{Z}/2$, is generated by $\lambda_k \in \Lambda^{1,k}$, satisfying the Adem relations

$$\lambda_k\lambda_{2k+s+1} = \sum_{j\geq 0}\binom{s-j-1}{j}\lambda_{k+s-j}\lambda_{2k+1+j}\ \ (k\geq 0,\ s\geq 0) \tag{1.1}$$

and the differential

$$\partial(\lambda_{s-1}) = \sum_{j\geq 1}\binom{s-j-1}{j}\lambda_{s-j-1}\lambda_{j-1}\ \ (s\geq 1), \tag{1.2}$$

where $\binom{s-j-1}{j}$ denotes the modulo 2 value of the binomial coefficient, with the usual convention $\binom{s-j-1}{j} = 0$ for $j > s-j-1$. In addition,

$$H^{s,*}(\Lambda) = \mathrm{Ext}_{\mathscr{A}}^{s,s+*}(F_2, F_2).$$

Now, for non-negative integers $t_1, \dots, t_s$, a monomial $\lambda_{t_1}\dots\lambda_{t_s} \in \Lambda$ is called *the monomial of the length* s. We shall write $\lambda_T,\ T = (t_1, \dots, t_s)$ for $\prod_{1\leq k\leq s}\lambda_{t_k}$ and refer to $\ell(T) = s$ as the length of T. Note that the algebra Λ is not commutative and that the bigrading of a monomial indexed by T may be written (s, d), where the homological degree s, as above, is the length of T, and $d = \sum_{1\leq k\leq s} t_k$. A monomial λ_T is called *admissible* if $t_k \leq 2t_{k+1}$ for all $1 \leq k \leq s-1$. By the relations (1.1), the F_2-vector subspace

$$\Lambda^{s,*} = \langle\{\lambda_T|\ T = (t_1, \dots, t_s),\ t_k \geq 0, 1\leq k\leq s\}\rangle$$

of Λ has an additive basis consisting of all admissible monomials of the length t. In [5], Chơn and Hà defined a homomorphism $\psi_s : H_*(B\mathbb{V}_s) \longrightarrow \Lambda^{s,*}$, which is considered as a presentation in the algebra Λ of the Singer transfer Tr_s and determined by the following inductive formula:

$$\psi_s(a^T) = \begin{cases} \lambda_{t_1} & \text{if } \ell(T) = 1, \\ \sum_{j \geq t_s} \psi_{s-1}(\prod_{1 \leq k \leq s-1} a_k^{(t_k)} Sq^{j-t_s})\lambda_j & \text{if } \ell(T) > 1, \end{cases}$$

for any $a^T := \prod_{1 \leq k \leq s} a_k^{(t_k)} \in H_*(B\mathbb{V}_s)$ and $T = (t_1, t_2, \ldots, t_s)$. Note that ψ_s is not an $\mathscr{A}$-homomorphism. The authors showed in [5] that if $a^T \in P_{\mathscr{A}}(H_*(B\mathbb{V}_s))$, then $\psi_s(a^T)$ is a cycle in $\Lambda^{s,*}$ and $Tr_s([a^T]) = [\psi_s(a^T)]$. Therefrom, we have a homomorphism of algebras

$$\psi = \{\psi_s : \ s \geqslant 0\} : \{H_*(B\mathbb{V}_s) : \ s \geqslant 0\} \to \{\Lambda^{s,*} : \ s \geqslant 0\} = \Lambda,$$

which induces the Singer transfer.

By using the invariant theory, Singer showed in [19] that Tr_5 is not an epimorphism when acting on $F_2 \otimes_{GL_5} P_{\mathscr{A}}(H_9(B\mathbb{V}_5))$. This means that the non-zero element $Ph_1 \in \mathrm{Ext}_{\mathscr{A}}^{5,5+9}(F_2, F_2)$ was not detected by Tr_5. The aim of the present note is also to study the image of Tr_5. Our main result is the following.

2. Main Theorem

Theorem 2.1. *The non-zero elements*

(i) $h_0 d_0 \in \mathrm{Ext}_{\mathscr{A}}^{5,5+14}(F_2, F_2)$,
(ii) $h_2 e_0 = h_0 g_1 \in \mathrm{Ext}_{\mathscr{A}}^{5,5+20}(F_2, F_2)$,
(iii) $h_3 e_0 = h_4 h_1 c_0 \in \mathrm{Ext}_{\mathscr{A}}^{5,5+24}(F_2, F_2)$

are in the image of the fifth algebraic transfer.

A proof can be based on the following events: The algebraic transfer Tr_1 detects the family $\{h_t | \, t \geq 0\}$ (see [19]). In the case of rank 4, the Singer transfer detects the elements d_0 and e_0 (see [8]). Moreover, $\bigoplus_{s \geq 0} Tr_s$ is a homomorphism of algebras (see [19]). However, in this note, we adopt another technique to prove this result. This technique can be applied to homological degrees higher under certain conditions. More precisely, we prove the main theorem by direct computations using a representation in the Lambda algebra of the fifth algebraic transfer.

Proof. It is well-known that there exists an endomorphism Sq^0 of the lambda algebra Λ, determined by $Sq^0(\lambda_T = \prod_{1 \leq k \leq s} \lambda_{t_k}) = \prod_{1 \leq k \leq s} \lambda_{2t_k+1}$, where λ_T is not necessarily admissible. It respects the Adem relations in (1.1) and commutes the differential in (1.2) above. Then, Sq^0 induces the classical squaring operation in the Ext groups

$$Sq^0 : H^{s,*}(\Lambda) = \mathrm{Ext}_{\mathscr{A}}^{s,s+*}(F_2, F_2) \to H^{s,s+2*}(\Lambda) = \mathrm{Ext}_{\mathscr{A}}^{s,2(s+*)}(F_2, F_2).$$

According to Liulevicius [10], Sq^0 is not the identity map. Moreover, it commutes with Kameko's squaring operation (see [7])

$$\widetilde{Sq^0} : F_2 \otimes_{GL_s} P_{\mathscr{A}}(H_*(B\mathbb{V}_s)) \to F_2 \otimes_{GL_s} P_{\mathscr{A}}(H_{s+2*}(B\mathbb{V}_s))$$

through the s-th Singer transfer, i.e., the following diagram is commutative:

$$\begin{tikzcd}
F_2 \otimes_{GL_s} P_{\mathscr{A}}(H_*(B\mathbb{V}_s)) \arrow[r, "Tr_s"] \arrow[d, "\widetilde{Sq^0}"] & \mathrm{Ext}_{\mathscr{A}}^{s,s+*}(F_2, F_2) \arrow[d, "Sq^0"] \\
F_2 \otimes_{GL_s} P_{\mathscr{A}}(H_{s+2*}(B\mathbb{V}_s)) \arrow[r, "Tr_s"] & \mathrm{Ext}_{\mathscr{A}}^{s,2(s+*)}(F_2, F_2).
\end{tikzcd}$$

In what follows, $(Sq^0)^t : \mathrm{Ext}_{\mathscr{A}}^{*,*}(F_2, F_2) \to \mathrm{Ext}_{\mathscr{A}}^{*,*}(F_2, F_2)$ denotes the composite $Sq^0 \ldots Sq^0$ (t times of Sq^0) if $t > 1$, is Sq^0 if $t = 1$, and is the identity map if $t = 0$. A family $\{a_t : t \geq 0\} \subset \mathrm{Ext}_{\mathscr{A}}^{s,s+*}(F_2, F_2)$ is called a Sq^0-family if $a_t = (Sq^0)^t(a_0)$ for $t \geq 0$. According to Lin [9], the algebra $\{\mathrm{Ext}_{\mathscr{A}}^{s,s+*}(\mathbb{F}_2, \mathbb{F}_2) : s \geqslant 0\}$ for $s \leqslant 4$ is generated by $h_t,\ c_t,\ d_t,\ e_t,\ f_t,\ g_{t+1},\ p_t,\ D_3(t),\ p'_t$ for $i \geqslant 0$ and subject to the Adem relations in (1.1) together with the relations $h_t^2 h_{t+3}^2 = 0, h_j c_t = 0$ for $j \in \{t-1,\ t,\ t+2,\ t+3\}$. Furthermore, the set of the elements $d_t,\ e_t,\ f_t,\ g_{t+1},\ p_t,\ D_3(t)$ and p'_t for $t \geqslant 0,$ is an F_2-basis for the indecomposable elements in $\mathrm{Ext}_{\mathscr{A}}^{4,4+*}(F_2, F_2)$.

By Lin [9], we have

$$\begin{aligned}
h_t &= \{\lambda_{2^t-1} = (Sq^0)^t(\lambda_0)\} \in \mathrm{Ext}_{\mathscr{A}}^{1,2^t} \text{ for } t \geq 0,\\
c_t &= \{(Sq^0)^t(\lambda_2\lambda_3^2)\} \in \mathrm{Ext}_{\mathscr{A}}^{3,2^{t+3}+2^{t+1}+2^t} \text{ for } t \geq 0,\\
d_t &= \{(Sq^0)^t(\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3)\} \in \mathrm{Ext}_{\mathscr{A}}^{4,2^{t+4}+2^{t+1}} \text{ for } t \geq 0,\\
e_t &= \{(Sq^0)^t(\lambda_8\lambda_3^3 + \lambda_4(\lambda_5^2\lambda_3 + \lambda_7\lambda_3^2) + \lambda_2(\lambda_9\lambda_3^2 + \lambda_3^2\lambda_9))\}\\
&\quad \in \mathrm{Ext}_{\mathscr{A}}^{4,2^{t+4}+2^{t+2}+2^t} \text{ for } t \geq 0,
\end{aligned}$$

$$
\begin{aligned}
g_{t+1} &= \{(Sq^0)^t(\lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3^2\lambda_9) + \lambda_3(\lambda_{11}\lambda_3^2 + \lambda_5\lambda_9\lambda_3))\} \\
&\in \mathrm{Ext}_{\mathscr{A}}^{4,2^{t+4}+2^{t+3}} \text{ for } t \geq 0.
\end{aligned}
$$

Combining this and the previous results by Tangora [20], we get

$$
\mathrm{Ext}_{\mathscr{A}}^{5,5+n} = \begin{cases} \langle h_0 d_0 \rangle & \text{if } n = 14, \\ \langle h_2 e_0 \rangle & \text{if } n = 20, \\ \langle h_3 e_0 \rangle & \text{if } n = 24, \end{cases}
$$

and $h_0 d_0 \neq 0$, $h_2 e_0 = h_0 g_1 \neq 0$, $h_3 e_0 = h_4 h_1 c_0 \neq 0$. Here, $h_t = [\lambda_{2^t-1}]$ is the Adams element (see [1]) in $\mathrm{Ext}_{\mathcal{A}_2}^{1,2^t}(F_2, F_2)$, for $0 \leq t \leq 4$.

The case (i). Obviously, $\lambda_0 \in \Lambda^{1,0}$, and

$$
\overline{d}_0 = \lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3 + \lambda_1\lambda_5\lambda_1\lambda_7 \in \Lambda^{4,14}
$$

are the cycles in the algebra Λ. Recall that

$$
h_0 = [\lambda_0] \in \mathrm{Ext}_{\mathscr{A}}^{1,1}(F_2, F_2), \quad d_0 = [\overline{d}_0] \in \mathrm{Ext}_{\mathscr{A}}^{4,18}(F_2, F_2).
$$

By direct computations, we find that the following element is $\mathscr{A}^+$-annihilated in $H_{14}(B\mathbb{V}_5)$

$$
\begin{aligned}
u_{14} \; &= a_1^{(0)}a_2^{(1)}a_3^{(1)}a_4^{(6)}a_5^{(6)} + a_1^{(0)}a_2^{(1)}a_3^{(2)}a_4^{(5)}a_5^{(6)} + a_1^{(0)}a_2^{(1)}a_3^{(3)}a_4^{(4)}a_5^{(6)} \\
&+a_1^{(0)}a_2^{(1)}a_3^{(4)}a_4^{(3)}a_5^{(6)} + a_1^{(0)}a_2^{(1)}a_3^{(5)}a_4^{(2)}a_5^{(6)} + a_1^{(0)}a_2^{(1)}a_3^{(6)}a_4^{(1)}a_5^{(6)} \\
&+a_1^{(0)}a_2^{(2)}a_3^{(1)}a_4^{(6)}a_5^{(5)} + a_1^{(0)}a_2^{(2)}a_3^{(2)}a_4^{(5)}a_5^{(5)} + a_1^{(0)}a_2^{(2)}a_3^{(3)}a_4^{(4)}a_5^{(5)} \\
&+a_1^{(0)}a_2^{(2)}a_3^{(4)}a_4^{(3)}a_5^{(5)} + a_1^{(0)}a_2^{(2)}a_3^{(5)}a_4^{(2)}a_5^{(5)} + a_1^{(0)}a_2^{(2)}a_3^{(6)}a_4^{(1)}a_5^{(5)} \\
&+a_1^{(0)}a_2^{(3)}a_3^{(1)}a_4^{(5)}a_5^{(5)} + a_1^{(0)}a_2^{(3)}a_3^{(2)}a_4^{(6)}a_5^{(3)} + a_1^{(0)}a_2^{(3)}a_3^{(3)}a_4^{(2)}a_5^{(6)} \\
&+a_1^{(0)}a_2^{(3)}a_3^{(4)}a_4^{(1)}a_5^{(6)} + a_1^{(0)}a_2^{(3)}a_3^{(4)}a_4^{(2)}a_5^{(5)} + a_1^{(0)}a_2^{(3)}a_3^{(4)}a_4^{(4)}a_5^{(3)} \\
&+a_1^{(0)}a_2^{(3)}a_3^{(6)}a_4^{(2)}a_5^{(3)} + a_1^{(0)}a_2^{(4)}a_3^{(1)}a_4^{(6)}a_5^{(3)} + a_1^{(0)}a_2^{(4)}a_3^{(2)}a_4^{(5)}a_5^{(3)} \\
&+a_1^{(0)}a_2^{(4)}a_3^{(3)}a_4^{(4)}a_5^{(3)} + a_1^{(0)}a_2^{(4)}a_3^{(4)}a_4^{(3)}a_5^{(3)} + a_1^{(0)}a_2^{(4)}a_3^{(5)}a_4^{(2)}a_5^{(3)} \\
&+a_1^{(0)}a_2^{(4)}a_3^{(6)}a_4^{(1)}a_5^{(3)} + a_1^{(0)}a_2^{(5)}a_3^{(1)}a_4^{(3)}a_5^{(5)} + a_1^{(0)}a_2^{(5)}a_3^{(2)}a_4^{(1)}a_5^{(6)} \\
&+a_1^{(0)}a_2^{(5)}a_3^{(2)}a_4^{(2)}a_5^{(5)} + a_1^{(0)}a_2^{(5)}a_3^{(2)}a_4^{(4)}a_5^{(3)} + a_1^{(0)}a_2^{(5)}a_3^{(3)}a_4^{(1)}a_5^{(5)} \\
&+a_1^{(0)}a_2^{(5)}a_3^{(3)}a_4^{(3)}a_5^{(3)} + a_1^{(0)}a_2^{(5)}a_3^{(5)}a_4^{(1)}a_5^{(3)} + a_1^{(0)}a_2^{(6)}a_3^{(1)}a_4^{(1)}a_5^{(6)} \\
&+a_1^{(0)}a_2^{(6)}a_3^{(1)}a_4^{(2)}a_5^{(5)} + a_1^{(0)}a_2^{(6)}a_3^{(1)}a_4^{(4)}a_5^{(3)} + a_1^{(0)}a_2^{(6)}a_3^{(2)}a_4^{(3)}a_5^{(3)}.
\end{aligned}
$$

Then, using the differential (1.2) and the presentation of Tr_5 over Λ, we see that

$$\begin{aligned}\psi_5(u_{14}) &= \lambda_0\lambda_6\lambda_2\lambda_3^2 + \lambda_0\lambda_4^2\lambda_3^2 + \lambda_0\lambda_2\lambda_4\lambda_5\lambda_3\\ &\qquad\qquad +\lambda_0\lambda_1\lambda_5\lambda_1\lambda_7 + \partial(\lambda_0\lambda_9\lambda_3^2 + \lambda_0\lambda_3\lambda_9\lambda_3)\\ &= \lambda_0\overline{d}_0 + \partial(\lambda_0\lambda_9\lambda_3^2 + \lambda_0\lambda_3\lambda_9\lambda_3).\end{aligned}$$

is a cycle in $\Lambda^{5,14}$. Therefore, h_0d_0 is in the image of Tr_5.

The case (ii). Obviously, $\lambda_3 \in \Lambda^{1,3}$ and

$$\overline{e}_0 = \lambda_3^3\lambda_8 + (\lambda_3\lambda_5^2 + \lambda_3^2\lambda_7)\lambda_4 + \lambda_7\lambda_5\lambda_3\lambda_2 + \lambda_3^2\lambda_5\lambda_6$$

are the cycles in the algebra Λ. We have

$$h_2 = [\lambda_3] \in \mathrm{Ext}_{\mathscr{A}}^{1,4}(F_2, F_2),\ \ e_0 = [\overline{e}_0] \in \mathrm{Ext}_{\mathscr{A}}^{4,21}(F_2, F_2).$$

Consider the following element in $H_{20}(B\mathbb{V}_5)$:

$$\begin{aligned}u_{20} \ &= a_1^{(3)}a_2^{(5)}a_3^{(5)}a_4^{(5)}a_5^{(2)} + a_1^{(3)}a_2^{(5)}a_3^{(5)}a_4^{(6)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(8)}a_5^{(1)}\\
&+a_1^{(3)}a_2^{(5)}a_3^{(3)}a_4^{(8)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(7)}a_5^{(1)} + a_1^{(3)}a_2^{(6)}a_3^{(3)}a_4^{(7)}a_5^{(1)}\\
&+a_1^{(3)}a_2^{(5)}a_3^{(7)}a_4^{(4)}a_5^{(1)} + a_1^{(3)}a_2^{(7)}a_3^{(5)}a_4^{(4)}a_5^{(1)} + a_1^{(3)}a_2^{(6)}a_3^{(7)}a_4^{(3)}a_5^{(1)}\\
&+a_1^{(3)}a_2^{(7)}a_3^{(6)}a_4^{(3)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(9)}a_4^{(4)}a_5^{(1)} + a_1^{(3)}a_2^{(9)}a_3^{(3)}a_4^{(4)}a_5^{(1)}\\
&+a_1^{(3)}a_2^{(3)}a_3^{(9)}a_4^{(3)}a_5^{(2)} + a_1^{(3)}a_2^{(9)}a_3^{(3)}a_4^{(3)}a_5^{(2)} + a_1^{(3)}a_2^{(5)}a_3^{(9)}a_4^{(2)}a_5^{(1)}\\
&+a_1^{(3)}a_2^{(9)}a_3^{(5)}a_4^{(2)}a_5^{(1)} + a_1^{(3)}a_2^{(5)}a_3^{(10)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(9)}a_3^{(6)}a_4^{(1)}a_5^{(1)}\\
&+a_1^{(3)}a_2^{(3)}a_3^{(11)}a_4^{(2)}a_5^{(1)} + a_1^{(3)}a_2^{(11)}a_3^{(3)}a_4^{(2)}a_5^{(1)} + a_1^{(3)}a_2^{(5)}a_3^{(5)}a_4^{(3)}a_5^{(4)}\\
&+a_1^{(3)}a_2^{(5)}a_3^{(3)}a_4^{(5)}a_5^{(4)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(5)}a_5^{(4)} + a_1^{(3)}a_2^{(3)}a_3^{(12)}a_4^{(1)}a_5^{(1)}\\
&+a_1^{(3)}a_2^{(11)}a_3^{(4)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(7)}a_3^{(8)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(7)}a_3^{(7)}a_4^{(1)}a_5^{(2)}\\
&+a_1^{(3)}a_2^{(13)}a_3^{(2)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(14)}a_3^{(1)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(6)}a_3^{(5)}a_4^{(3)}a_5^{(3)}\\
&+a_1^{(3)}a_2^{(5)}a_3^{(3)}a_4^{(6)}a_5^{(3)} + a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(5)}a_5^{(3)} + a_1^{(3)}a_2^{(6)}a_3^{(3)}a_4^{(3)}a_5^{(5)}\\
&+a_1^{(3)}a_2^{(3)}a_3^{(3)}a_4^{(6)}a_5^{(5)} + a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(3)}a_5^{(5)} + a_1^{(3)}a_2^{(5)}a_3^{(3)}a_4^{(3)}a_5^{(6)}\\
&+a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(3)}a_5^{(6)} + a_1^{(3)}a_2^{(3)}a_3^{(3)}a_4^{(5)}a_5^{(6)} + a_1^{(3)}a_2^{(3)}a_3^{(3)}a_4^{(3)}a_5^{(8)}\\
&+a_1^{(3)}a_2^{(3)}a_3^{(3)}a_4^{(4)}a_5^{(7)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(2)}a_5^{(7)} + a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(1)}a_5^{(7)}\\
&+a_1^{(3)}a_2^{(3)}a_3^{(3)}a_4^{(9)}a_5^{(2)} + a_1^{(3)}a_2^{(3)}a_3^{(3)}a_4^{(10)}a_5^{(1)} + a_1^{(3)}a_2^{(5)}a_3^{(3)}a_4^{(7)}a_5^{(2)}\\
&+a_1^{(3)}a_2^{(5)}a_3^{(7)}a_4^{(3)}a_5^{(2)} + a_1^{(3)}a_2^{(7)}a_3^{(5)}a_4^{(3)}a_5^{(2)} + a_1^{(3)}a_2^{(6)}a_3^{(3)}a_4^{(7)}a_5^{(1)}\\
&+a_1^{(3)}a_2^{(6)}a_3^{(7)}a_4^{(3)}a_5^{(1)} + a_1^{(3)}a_2^{(7)}a_3^{(6)}a_4^{(3)}a_5^{(1)}.\end{aligned}$$

Then, using the differential (1.2) the presentation in the Lambda algebra of Tr_5, we have

$$\begin{aligned}\psi_5(u_{20}) &= \lambda_3(\lambda_3^3\lambda_8 + (\lambda_3\lambda_5^2 + \lambda_3^2\lambda_7)\lambda_4 + \lambda_7\lambda_5\lambda_3\lambda_2 + \lambda_3^2\lambda_5\lambda_6)\\ &\qquad + \lambda_3(\lambda_3^3\lambda_8 + \lambda_3\lambda_5^2\lambda_4 + \lambda_3\lambda_5\lambda_6\lambda_3)\\ &= \lambda_3\overline{e}_0 + \partial(\lambda_3^2\lambda_5\lambda_{10} + \lambda_3^2\lambda_{12}\lambda_3 + \lambda_3\lambda_4\lambda_7^2 + \lambda_3\lambda_0\lambda_{11}\lambda_7).\end{aligned}$$

So, $\psi_5(u_{20})$ is a cycle in $\Lambda^{5,20}$. Moreover, a direct computation shows that u_{20} is $\mathscr{A}^+$-annihilated in $H_{20}(B\mathbb{V}_5)$. These data imply that h_2e_0 is in the image of Tr_5.

The case (iii). We see that $\lambda_7 \in \Lambda^{1,7}$ is a cycle in the algebra Λ and that $h_3 = [\lambda_7] \in \mathrm{Ext}_{\mathscr{A}}^{1,8}(F_2, F_2)$. We consider the following element in $H_{24}(B\mathbb{V}_5)$:

$$\begin{aligned}u_{24} &= a_1^{(7)}a_2^{(5)}a_3^{(5)}a_4^{(5)}a_5^{(2)} + a_1^{(7)}a_2^{(5)}a_3^{(5)}a_4^{(6)}a_5^{(1)} + a_1^{(7)}a_2^{(3)}a_3^{(5)}a_4^{(8)}a_5^{(1)}\\
&+a_1^{(7)}a_2^{(5)}a_3^{(3)}a_4^{(8)}a_5^{(1)} + a_1^{(7)}a_2^{(3)}a_3^{(6)}a_4^{(7)}a_5^{(1)} + a_1^{(7)}a_2^{(6)}a_3^{(3)}a_4^{(7)}a_5^{(1)}\\
&+a_1^{(7)}a_2^{(5)}a_3^{(7)}a_4^{(4)}a_5^{(1)} + a_1^{(7)}a_2^{(7)}a_3^{(5)}a_4^{(4)}a_5^{(1)} + a_1^{(7)}a_2^{(6)}a_3^{(7)}a_4^{(3)}a_5^{(1)}\\
&+a_1^{(7)}a_2^{(7)}a_3^{(6)}a_4^{(3)}a_5^{(1)} + a_1^{(7)}a_2^{(3)}a_3^{(9)}a_4^{(4)}a_5^{(1)} + a_1^{(7)}a_2^{(9)}a_3^{(3)}a_4^{(4)}a_5^{(1)}\\
&+a_1^{(7)}a_2^{(3)}a_3^{(9)}a_4^{(3)}a_5^{(2)} + a_1^{(7)}a_2^{(9)}a_3^{(3)}a_4^{(3)}a_5^{(2)} + a_1^{(7)}a_2^{(5)}a_3^{(9)}a_4^{(2)}a_5^{(1)}\\
&+a_1^{(7)}a_2^{(9)}a_3^{(5)}a_4^{(2)}a_5^{(1)} + a_1^{(7)}a_2^{(5)}a_3^{(10)}a_4^{(1)}a_5^{(1)} + a_1^{(7)}a_2^{(9)}a_3^{(6)}a_4^{(1)}a_5^{(1)}\\
&+a_1^{(7)}a_2^{(3)}a_3^{(11)}a_4^{(2)}a_5^{(1)} + a_1^{(7)}a_2^{(11)}a_3^{(3)}a_4^{(2)}a_5^{(1)} + a_1^{(7)}a_2^{(5)}a_3^{(5)}a_4^{(3)}a_5^{(4)}\\
&+a_1^{(7)}a_2^{(5)}a_3^{(3)}a_4^{(5)}a_5^{(4)} + a_1^{(7)}a_2^{(3)}a_3^{(5)}a_4^{(5)}a_5^{(4)} + a_1^{(7)}a_2^{(3)}a_3^{(12)}a_4^{(1)}a_5^{(1)}\\
&+a_1^{(7)}a_2^{(11)}a_3^{(4)}a_4^{(1)}a_5^{(1)} + a_1^{(7)}a_2^{(7)}a_3^{(8)}a_4^{(1)}a_5^{(1)} + a_1^{(7)}a_2^{(7)}a_3^{(7)}a_4^{(1)}a_5^{(2)}\\
&+a_1^{(7)}a_2^{(13)}a_3^{(2)}a_4^{(1)}a_5^{(1)} + a_1^{(7)}a_2^{(14)}a_3^{(1)}a_4^{(1)}a_5^{(1)} + a_1^{(7)}a_2^{(6)}a_3^{(5)}a_4^{(3)}a_5^{(3)}\\
&+a_1^{(7)}a_2^{(5)}a_3^{(3)}a_4^{(6)}a_5^{(3)} + a_1^{(7)}a_2^{(3)}a_3^{(6)}a_4^{(5)}a_5^{(3)} + a_1^{(7)}a_2^{(6)}a_3^{(3)}a_4^{(3)}a_5^{(5)}\\
&+a_1^{(7)}a_2^{(3)}a_3^{(3)}a_4^{(6)}a_5^{(5)} + a_1^{(7)}a_2^{(3)}a_3^{(6)}a_4^{(3)}a_5^{(5)} + a_1^{(7)}a_2^{(5)}a_3^{(3)}a_4^{(3)}a_5^{(6)}\\
&+a_1^{(7)}a_2^{(3)}a_3^{(5)}a_4^{(3)}a_5^{(6)} + a_1^{(7)}a_2^{(3)}a_3^{(3)}a_4^{(5)}a_5^{(6)} + a_1^{(7)}a_2^{(3)}a_3^{(3)}a_4^{(3)}a_5^{(8)}\\
&+a_1^{(7)}a_2^{(3)}a_3^{(3)}a_4^{(4)}a_5^{(7)} + a_1^{(7)}a_2^{(3)}a_3^{(5)}a_4^{(2)}a_5^{(7)} + a_1^{(7)}a_2^{(3)}a_3^{(6)}a_4^{(1)}a_5^{(7)}\\
&+a_1^{(7)}a_2^{(3)}a_3^{(3)}a_4^{(9)}a_5^{(2)} + a_1^{(7)}a_2^{(3)}a_3^{(3)}a_4^{(10)}a_5^{(1)} + a_1^{(7)}a_2^{(5)}a_3^{(3)}a_4^{(7)}a_5^{(2)}\\
&+a_1^{(7)}a_2^{(5)}a_3^{(7)}a_4^{(3)}a_5^{(2)} + a_1^{(7)}a_2^{(7)}a_3^{(5)}a_4^{(3)}a_5^{(2)} + a_1^{(7)}a_2^{(6)}a_3^{(3)}a_4^{(7)}a_5^{(1)}\\
&+a_1^{(7)}a_2^{(6)}a_3^{(7)}a_4^{(3)}a_5^{(1)} + a_1^{(7)}a_2^{(7)}a_3^{(6)}a_4^{(3)}a_5^{(1)}.\end{aligned}$$

Using the differential (1.2) the presentation over the algebra Λ of Tr_5, we get

$$\begin{aligned}\psi_5(u_{24}) &= \lambda_7(\lambda_3^3\lambda_8 + (\lambda_3\lambda_5^2 + \lambda_3^2\lambda_7)\lambda_4 + \lambda_7\lambda_5\lambda_3\lambda_2 + \lambda_3^2\lambda_5\lambda_6)\\ &\qquad + \lambda_7(\lambda_3^3\lambda_8 + \lambda_3\lambda_5^2\lambda_4 + \lambda_3\lambda_5\lambda_6\lambda_3)\\ &= \lambda_7\overline{e}_0 + \partial(\lambda_7\lambda_3\lambda_5\lambda_{10} + \lambda_7\lambda_3\lambda_{12}\lambda_3 + \lambda_7\lambda_4\lambda_7^2 + \lambda_7\lambda_0\lambda_{11}\lambda_7),\end{aligned}$$

where $\overline{e}_0$ is determined as in the case (ii). Then, $\psi_5(u_{24})$ is a cycle in $\Lambda^{5,24}$. Furthermore, it's easy to verify that $u_{24} \in P_{\mathscr{A}}(H_{24}(B\mathbb{V}_5))$. Hence, h_3e_0 is in the image of Tr_5. The proof of the main theorem is completed. $\square$

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FACULTY OF EDUCATION STUDIES
UNIVERSITY OF KHANH HOA, 01 NGUYEN CHANH, KHANH HOA, VIET NAM
Email address: `dangvophuc@ukh.edu.vn`