ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.1, 377–384
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.1.37

A NOTE ON SUBSEQUENCES OF EQUIDISTRIBUTED AND WELL DISTRIBUTED SEQUENCES

Leila Miller-Van Wieren

To my father Harry I. Miller, we keep going

ABSTRACT. Many authors studied properties related to distribution and summability of sequences of real numbers and the same properties of their subsequences. In these studies, Lebesgue measure and category were used as gauges of size of the classes of subsequences of a sequence with a certain property. In this paper, we aim to prove some new results on subsequences of a sequence, this time connected to the notions of equidistributed and well distributed sequences.

1. INTRODUCTION

In recent years, many mathematicians studied properties of real valued sequences and the relationship of a sequence and its subsequences regarding some property. In these studies two different gauges of size were used: Lebesgue measure and category, yielding different, interesting results.

Buck [4] has initiated the study of the relationship between the convergence of a given sequence and the summability of its subsequences. Agnew [1], Buck [5], Buck and Pollard [6], Dawson [7], Miller [11], Miller and Orhan [13], Zeager [20] have studied this relation changing the concept of convergence.

2020 *Mathematics Subject Classification.* 40G99, 28A12.
Key words and phrases. Well distributed sequences, equidistributed sequences, uniform statistical density, subsequences.

Later on, in [2], [8], [9], [14], [15], [18], [19] different types of convergence of a sequence and the related summability of its subsequences were studied, using Lebesgue measure as a gauge of the size of convergent subsequences. Also similar relations between sequences and their subsequences were studied, using category, by several authors, [3], [10], [16].

Recently, in one of his last papers [12], Miller specifically studied equidistributed sequences and their subsequences, using both measure and category as gauges of size, and came up with some new and interesting results.

In this paper, we wish to further elaborate on some properties of equidistributed sequences and subsequences. Additionally we study analogous properties of well distributed sequences and subsequences.

2. Preliminaries

Now let us recall some known notions. Let $K \subseteq \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers. If $m, n \in \mathbb{N}$, by $K(m,n)$ we denote the cardinality of the set of numbers i in K such that $m \leq i \leq n$. The numbers

$$\underline{d}(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n},\ \bar{d}(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n}$$

are called the lower and the upper asymptotic density of the set K, respectively. If $\underline{d}(K) = \bar{d}(K)$ then it is said that $d(K) = \underline{d}(K) = \bar{d}(K)$ is the asymptotic density of K. The uniform density of $K \subseteq \mathbb{N}$ has been introduced as follows:

$$\underline{u}(K) = \lim_{n\to\infty} \frac{\min\limits_{i\geq 0} K\left(i+1, i+n\right)}{n}, \qquad \bar{u}(K) = \lim_{n\to\infty} \frac{\max\limits_{i\geq 0} K\left(i+1, i+n\right)}{n}$$

are respectively called the lower and the upper uniform density of the set K (the existence of these bounds is also mentioned in [2]). If $\underline{u}(K) = \bar{u}(K)$, then $u(K) = \underline{u}(K)$ is called the uniform density of K. It is clear that for each $K \subseteq \mathbb{N}$ we have

$$\underline{u}(K) \leq \underline{d}(K) \leq \bar{d}(K) \leq \bar{u}(K).$$

Subsequences of a sequence x can be naturally identified with numbers $t \in (0,1]$ written by a binary expansion with infinitely many 1's. Thus we can denote by $\{x(t)\}$ the subsequence of x corresponding to t.

3. Equidistributed sequences

Here we give our attention to equidistributed sequences, first introduced by Herman Weyl, 100 years ago (see [17]).

Definition 3.1. *A sequence $x = \{x_n\}$ contained in $(0,1]$ is said to be equally distributed if for every $[a,b]$, subinterval of $(0,1]$,*

$$\lim_{n\to\infty} \frac{|\{1 \le i \le n, x_i \in [a,b]\}|}{n} = m([a,b]).$$

(Trivially closed intervals in $(0,1]$ can be replaced with all intervals in $(0,1]$.)

Example 1. *Let $x = \{x_n\}$ where $x_n = [cn]$, where c is irrational and [r] is the fractional part of r. It is well known that this sequence is equidistributed.*

In his paper, [12], Miller studied the relationship between equidistributed sequences and their subsequences. He proved the following results.

Theorem 3.1. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0,1]$. If x is equidistributed, then the set of $t \in (0,1]$ for which $x(t)$ is equidistributed has Lebesgue measure 1.*

Concerning category, in place of measure, Miller proved the next theorem.

Theorem 3.2. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0,1]$. The set of $t \in (0,1]$ for which $x(t)$ is equidistributed is meager.*

Here we prove the converse of Theorem 3.1.

Theorem 3.3. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0,1]$. If the set of $t \in (0,1]$ for which $x(t)$ is equidistributed has Lebesgue measure 1, then x is equidistributed.*

Proof. Let X denote the set of $t \in (0,1]$ for which $x(t)$ is equidistributed. Suppose that X has measure 1. Let N denote the set normal numbers in $(0,1]$, i.e. the set of $t \in (0,1]$, $t = 0, t_1, t_2, \ldots, t_n, \ldots$ (binary representation with infinitely many 1's) for which the asymptotic density of 1's (0's) is exactly $\frac{1}{2}$. It is well known that $m(N) = 1$.

Since $m(X) = 1$ implies that $m(1-X) = 1$ where $1 - X = \{1 - t : t \in (0,1]\}$, we can fix some $t \in X \bigcap (1-X) \bigcap N$.

Suppose $[a,b] \subseteq (0,1]$ is arbitrarily fixed.

Now suppose n is arbitrarily fixed. Let n_1 denote the number of 1's among the first n indices of t, and n_2 the number of 0's among the first n indices of t.

Then:

$$\begin{aligned}\frac{|\{1 \le i \le n, x_i \in [a,b]\}|}{n} &= \frac{|\{1 \le i \le n_1, (x(t))_i \in [a,b]\}|}{n} \\ &+ \frac{|\{1 \le i \le n_2, (x(1-t))_i \in [a,b]\}|}{n} \\ &= \frac{|\{1 \le i \le n_1, (x(t))_i \in [a,b]\}|}{n_1} \cdot \frac{n_1}{n} \\ &+ \frac{|\{1 \le i \le n_2, (x(1-t))_i \in [a,b]\}|}{n_2} \cdot \frac{n_2}{n}.\end{aligned}$$

Now if we let $n \to \infty$, we have that $n_1 \to \infty$, $n_2 \to \infty$, and that $\frac{n_1}{n} \to \frac{1}{2}$, $\frac{n_2}{n} \to \frac{1}{2}$.

Hence, since $x(t)$ and $x(1-t)$ are equidistributed, from the above we can conclude that

$$\lim_{n\to\infty} \frac{|\{1 \le i \le n, x_i \in [a,b]\}|}{n} = m([a,b]).$$

Since $[a,b] \subseteq (0,1]$ was arbitrary, the proof is complete. $\square$

Finally we can unify Theorems 3.1 and 3.3 as the following.

Theorem 3.4. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0,1]$. Then the set of $t \in (0,1]$ for which $x(t)$ is equidistributed has Lebesgue measure 1 or 0. The measure is 1 if x is equidistributed, and 0 if x is not equidistributed.*

Proof. Let X denote the set of $t \in (0,1]$ for which $x(t)$ is equidistributed. If x is equidistributed, from Theorem 3.1 we know that $m(X) = 1$.

Suppose x is not equidistributed. The set X is a tail set, and therefore has measure 0 or 1, or is nonmeasurable. We will verify that X is measurable.

For $0 < a < b \le 1$ arbitrary, let

$$X_{a,b} = \{t \in (0,1] : \lim_{n\to\infty} \frac{|\{1 \le i \le n, (x(t))_i \in [a,b]\}|}{n} = m([a,b])\}.$$

Then

$$X_{a,b} = \bigcap_l \bigcup_N \bigcap_{n \ge N} \left\{ t \in (0,1] : \left| \frac{|\{1 \le i \le n : (xt)_i \in [a,b]\}|}{n} - m([a,b]) \right| < \frac{1}{l} \right\}.$$

Now it is easy to see that $\left\{t \in (0,1] : \left|\{\frac{|\{1\leq i\leq n:(xt)_i\in[a,b]\}|}{n} - m([a,b])\right|\} < \frac{1}{l}\right\}$ is of the form $G \setminus M$ where G is open, M countable and therefore $X_{a,b}$ is measurable. Consequently

$$X = \bigcap_{a\leq b:a,b\in Q\bigcap(0,1]} X_{a,b}$$

is measurable. Now since x is not equidistributed, from Theorem 3.3 we conclude that $m(X) = 0$. □

4. Results on well distributed sequences

In summability the concept of uniform statistical density and convergence are introduced as more strict than asymptotic density and statistical convergence. Parallel with these notions, from the concept of equidistributed sequences, we move to the more strict notion of well distributed sequences.

Definition 4.1. *A sequence $x = \{x_n\}$ contained in $(0,1]$ is said to be well distributed if for every $[a,b]$, subinterval of $(0,1]$,*

$$\lim_{n\to\infty} \frac{|\{m+1 \leq i \leq m+n, x_i \in [a,b]\}|}{n} = m([a,b])$$

uniformly in m. (Trivially closed intervals in $(0,1]$ can be replaced with all intervals in $(0,1]$.)

Now we can formulate some results concerning the relationships of sequences and their subsequences and the property of well distributiveness, analogous to the results in the previous section.

First, we show that the class of well distributed subsequences of a sequence is small in category (meager). The following theorem is a corollary of Theorem 3.2.

Theorem 4.1. *If $x = \{x_n\}$ is a sequence of reals in $(0,1]$, the set of $t \in (0,1]$ for which $x(t)$ is well distributed is meager.*

Proof. From Theorem 3.2, we know that the set of $t \in (0,1]$ for which $x(t)$ is equidistributed is meager. Since every well distributed sequence is also equidistributed, the conclusion follows. □

Now, we study the Lebesgue measure of the set of well distributed sequences of a given sequence. We have the following result.

Theorem 4.2. *If $x = \{x_n\}$ is a sequence of reals in $(0,1]$, the set of $t \in (0,1]$ for which $x(t)$ is well distributed has Lebesgue measure 0.*

Proof. Clearly $\bar{u}(\{i, x_i \in (0, \frac{1}{2}]\} \geq \frac{1}{2}$ or $\bar{u}(\{i, x_i \in (\frac{1}{2}, 1]\} \geq \frac{1}{2}$. Without loss of generality we can assume that $\bar{u}(\{i, x_i \in (0, \frac{1}{2}]\} \geq \frac{1}{2}$.

We will show that, for $n \in \mathbf{N}$,

$$X_n = \{t \in (0,1], x(t)\text{contains } n \text{ consecutive terms in } (0, \frac{1}{2}]\}$$

has measure 1.

Assume that n is fixed. Since $\bar{u}(\{i, x_i \in (0, \frac{1}{2}]\} \geq \frac{1}{2}$, there exist positive integers $N_1 < N_2 < N_3 < \cdots < N_{2k-1} < N_{2k} < \cdots$ such that $(N_{2k} - N_{2k-1}) \to \infty$ and

$$\frac{|\{i : N_{2k-1} \leq i \leq N_{2k} : x_i \in (0, \frac{1}{2}]\}|}{N_{2k} - N_{2k-1}} \geq \frac{1}{3}.$$

Without loss of generality we can assume that $N_{2k} - N_{2k-1} > 6n$ for $k = 1, 2, \ldots$. Setting $m = 6n$, from the above it is easy to show that we can find m_k, $k = 1, 2, \ldots$ so that $N_{2k-1} \leq m_k < m_k + m - 1 \leq N_{2k}$ such that

$$\frac{|\{i : m_k \leq i \leq m_k + m - 1 : x_i \in (0, \frac{1}{2}]\}|}{m} \geq \frac{1}{6}.$$

Now for $k = 1, 2 \ldots$ let T_k denote the set of all $t \in (0,1]$, $t = 0.t(1)t(2)\cdots t(n)\cdots$ (binary representation with infinitely many 1's) such that for $m_k \leq i \leq m_k + m - 1$:

$$t_i = \begin{cases} 1 & ; \quad x_i \in (0, \frac{1}{2}] \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(and t_i is 0 or 1 for $i \notin [m_k, m_k + m - 1]$).

Then for all k, $m(T_k) = \frac{1}{2^m}$ and $m((0,1] \setminus T_k) = 1 - \frac{1}{2^m}$.

Since $[m_k, m_k + m - 1]$ are mutually disjoint,

$$m(\bigcap_k ((0,1] \setminus T_k) = (1 - \frac{1}{2^m})(1 - \frac{1}{2^m}) \cdots = 0$$

and hence

$$m(\bigcup_k T_k) = 1.$$

Since $m = 6n$, from the definition of T_k it is clear that if $t \in T_k$, then $x(t)$ contains n (or more) consecutive terms from $(0, \frac{1}{2}]$, for $k = 1, 2 \ldots$.

Hence $\bigcup_k T_k \subseteq X_n$ and consequently $m(X_n) = 1$ for $n \in \mathbb{N}$. Therefore $m(\bigcap_n X_n) = 1$. Now $t \in \bigcap_n X_n$ implies that $\bar{u}(\{i, (x(t))_i \in (0, \frac{1}{2}]\}) = 1$.

This means that for almost all $t \in [0, 1)$, $\bar{u}(\{i, (x(t))_i \in (0, \frac{1}{2}]\}) = 1$. Therefore the set of all $t \in (0, 1]$ such that

$$\lim_{n\to\infty} \frac{|\{i : m \le i \le m + n : (x(t))_i \in (0, \frac{1}{2}]\}|}{n} = \frac{1}{2}$$

uniformly in m, has measure 0. Hence the set of all $t \in (0, 1]$ such that $x(t)$ is well distributed has measure 0. $\square$

REFERENCES

[1] R. P. AGNEW: *Summability of subsequences*, Bull. Amer. Math. Soc., **50** (1944), 596–598.

[2] V. BALAZ, T. ŠALÁT: *Uniform density u and corresponding I_u- convergence*, Math. Communications, **11** (2006), 1–7.

[3] M. BALCERZAK, S. GLAB, A. WACHOWICZ: *Qualitative properties of ideal convergent subsequences and rearrangements*, Acta Math. Hungar., **150** (2016), 312–323.

[4] R. C. BUCK: *A note on subsequences*, Bull. Amer. Math. Soc., **49** (1943), 924–931.

[5] R. C. BUCK: *An addentum to "a note on subsequences"*, Proc. Amer. Math. Soc., **7** (1956), 1074–1075.

[6] R. C. BUCK, H. POLLARD: *Convergence and summability properties of Subsequences*, Bull. Amer. Math. Soc., **49** (1943), 924–931.

[7] D. F. DAWSON: *Summability of subsequences and stretchings of Sequences*, Pacific J. Math., **44** (1973), 455–460.

[8] J. A. FRIDY, C. ORHAN: *Statistical limit superior and limit Inferior*, Proc. Amer. Math. Soc., **125** (1997), 3625–3631.

[9] J. A. FRIDY, C. ORHAN: *Statistical core theorems*, J. Math. Anal. Appl., **208** (1997), 520–527.

[10] P. LEONETTI, H. I. MILLER, L. MILLER-VAN WIEREN: *Duality between Measure and Category of Almost All Subsequences of a Given Sequence*, Periodica Math. Hungar., **78(4)** (2018), 1–5.

[11] H. I. MILLER: *A measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc., **347** (1995), 1811–1819.

[12] H. I. MILLER: *More measure and category questions for subsequences of a given sequence*, Rocky Mountain J. Math., **49(3)** (2019), 903–911.

[13] H. I. MILLER, C. ORHAN: *On almost convergence and statistically convergent subsequences*, Acta. Math. Hungar., **93** (2001), 135–151.

[14] H. I. Miller, L. Miller-Van Wieren: *Some statistical cluster point theorems*, Hacet. J. Math.Stat., **44** (2015), 1405–1409.

[15] H. I. Miller, L. Miller-Van Wieren: *Statistical cluster point and statistical limit point sets of subsequences of a given sequence*, Hacet. J. Math., **49(2)** (2020), 494–497.

[16] L. Miller-Van Wieren, E. Tas, T. Yurdakadim: *Category theoretical view of I-cluster and I-limit points of subsequences*, Acta et Comment. Univer. Tartu. De Math., **24(1)** (2020), 103–108.

[17] H. Weyl: *Uber die Gleichverteilung von Zahlen mod. Eins[On the distribution of numbers modulo one]*, Math. Ann. (in German), **77(3)** (1916), 313–352.

[18] T. Yurdakadim, L. Miller-Wan Wieren: *Subsequential results on uniform statistical convergence*, Sarajevo J. Math., **12 (25) 2** (2016), 1–9.

[19] T. Yurdakadim, L. Miller-Van Wieren: *Some results on uniform statistical cluster points*, Turk. J. Math., **41** (2017), 1133—1139.

[20] J. Zeager: *Buck-type theorems for statistical convergence*, Radovi Math., **9** (1999), 59–69.

Faculty of Engineering and Natural Sciences
International University of Sarajevo
Hrasnička 15, Sarajevo, Bosnia and Herzegovina
Email address: lmiller@ius.edu.ba