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REVERSE EDGE MAGIC LABELING OF A CYCLE WITH CHORDS, UNIONS OF CYCLES AND UNIONS OF PATHS

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ABSTRACT. Reverse edge magic (REM) labeling of the graph $G = (V, E)$ is a bijection of vertices and edges to a set of numbers from the set, defined by$\lambda : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$ with the property that for every $x, y \in E$, constant k is the weight of equals to a xy, that is $\lambda(xy) - \lambda(x) + \lambda(y) = k$ for some integer k. In this paper, we given the construction of REM labeling for a cycle with chords ${}^{[c]t}c_n$, unions of paths ${}^{m}P_n$, and unions of cycles and paths $m(C_{n_1(2r+1)} \cup (2r+1)P_{n_2})$.

1. INTRODUCTION

Let G be a simple graph with vertex set V and edge set E. Labeling of G is a bijection $f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$. If $x, y \in V$and if $e = xy \in E$, then the weight $w(e)$ of the edge e is given by $w\,(e) = f\,(x) + f\,(y) + f\,(e)$. The total labeling f is said to be reverse edge-magic (REM) labeling if the weight of each edge is a constant, and this constant is called the magic constant of the REM labeling. REM labeling is called reverse super edge magic (RSEM) labeling if the vertices are labeled using the smallest $|V|$integers.

In [6] the result for REM labeling of a complete bipartite graph stated by Kotzig and Rosa. They used the terminology $M-$valuation, which is now known as EMT labeling and also stated the preservation an EMT labeling for the odd number of

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copies of certain graphs. They used the term edge-magic to describe a graph that has REM labeling.

In [9], The method to expand the result in EMT labeling for some families of graphs is introduced by I. Singgih. In [1] used the results for EMT labeling of 2-regular graphs for the method of generalize. In this article, we apply the method to construct a REM labeling for several other families of graphs and section 2 the main theorem consisting the expansion method is given. The new results for each familie of graphs are given by the following sections, we given some examples and describe how the method works. In [8] Marr and Wallis give a definition of a Kotzig array as $d * m$ grid, each row being a permutation of $\{0, 1, \ldots, m-1\}$ and each column having the same sum. The Kotzig array used in this paper is the $3*(2r+1)$ Kotzig arrayk that is given as an example in [6] after adding each the entry of the array by one:

$$k = \begin{bmatrix} 1 & 2 & \cdots & r+1 & r+2 & \ldots & 2r & 2r+1 \\ r+1 & r+2 & \ldots & 2r+1 & 1 & \ldots & r-1 & r \\ 2r+1 & 2r-1 & \ldots & 1 & 2r & \ldots & 4 & 2 \end{bmatrix}.$$

If we write the first two rows of k as a permutation cycle τ, we have $\tau = (1, r+1, 2r+1, r, \ldots, 3, r+3, 2, r+2)$. The difference between two consecutive elements in τ is equal to τ has taken modulo $(2r\ +\ 1)$. Note that $\tau$is a$(2r\ +\ 1)-$cycle. Since $(2r+1)$ is an odd number for every nonnegative integer r,then $gcd\,(2,\ 2r\ +\ 1)\ =\ 1$, and so we have τ^2 also a $(2r\ +\ 1)-$cycle. This fact plays an important role in preserving the properties of magic labeling of our EMT and SEMT labeling as we extend the length of cycles. Let k' be the modified k, where we switched the first and second row of k' :

$$k = \begin{bmatrix} r+1 & r+ & \ldots & 2r+1 & 1 & \ldots & r-1 & r \\ 1 & 2 & \ldots & r+1 & r+2 & \ldots & 2r & 2r+1 \\ 2r+1 & 2r-1 & \ldots & 1 & 2r & \ldots & 4 & 2 \end{bmatrix}$$

It is clear that if we write the first two rows of as a permutation cycle, we have τ^{-1}. In this section, we will describe first the method that later applied to construct a REM labeling. This method preserves the REM (RSEM) properties as we extend the length of cycles, or multiplying the number of paths, by a factor of an odd number.

Theorem 1.1. *[[8], Lemma 2.3] The complete bipartite graph $K_{p,q}$ exists for all $p, q \geq 1,$ for $M-$valuation [EMT labeling].*

Theorem 1.2. *[8] Say G is a 3-colorable edge-magic graph and $H$is the union of t disjoint copies of$G$, $t$odd. Then H is edge magic.*

Theorem 1.3. *Let G be a 2-regular graph that has a REM labeling γ. Let G' be a 2-regular graph obtained by extending the length of each component of G by an odd factor. Then there exists an REM labeling for G' that can be obtained by modifying the REM labeling of G.*

Proof. Let γ be a REM labeling for any 2-regular graph G. For every vertex and edge of G, let λ be the labeling obtained by decreasing the original label by 1, that is, let $\lambda(v) = \gamma(v) - 1$ and $\lambda(e) = \gamma(e) - 1$. For each cycle C_n in G, construct a $n \times 3$ table with entries as follows.

In the first column: For $i = 1, 2, \ldots, n$, the entry in the i^{th} row is the 3×1 matrix

$$\Lambda = \begin{bmatrix} \lambda(v_i) \\ \lambda(v_{i+1}) \\ \lambda(e_{i+1}) \end{bmatrix}.$$

In the second column: For $y = 1,\ 2,\ 3$ and $z = 1, 2, 3, \ldots, (2r+1)$ the entry in the i^{th} row is either k or k' depending on the value of i, namely $k = [k_{yz}]$, if $i \leq \left[\frac{n}{2}\right] + 1$, and k'_{yz}, if $\left[\frac{n}{2}\right] + 1 < i \leq n$, where k_{yz} denotes the element on the y^{th} row and z^{th} column of k. In the third column: for $i = 1, 2, \ldots., n$, the entry in the i^{th} row is the matrix

$$\Theta_i = \begin{cases} k_{yz} + (2r+1)\Lambda_{y1}, & \text{if } i \leq \left[\frac{n}{2}\right] + 1 \\ k'_{yz} + (2r+1)\Lambda_{y1}, & \text{if } \left[\frac{n}{2}\right] + 1 < i \leq n \end{cases}$$

If we multiply the permutation cycles of k and k' in the second column, we obtain $\tau^{\frac{n}{2}} + 1\tau^{n-\left(\left[\frac{n}{2}\right]-n+2\right)} = \tau^{2\left[\frac{n}{2}\right]-n+2}$ If n is odd we have $\tau^{(n-1)-n+2} = \tau$ and if n is even we have $\tau^{n-n+2} = \tau^2$.

The cycle $C_{n(22r+1)}$ is obtained by tracking the numbers on Θ. Let θ^i_{yz} denote the elements of Θ_i in the y^{th} row and z^{th} column. In each Θ_i, the two numbers θ^i_{1z} and θ^i_{2z} will be the labels of teo adjacent vertices on $C_{n(22r+1)}$, and θ^i_{3z} will be the label of the edge they share. For each i, $1 \leq i \leq n$, each pair of θ^{i+1}_{1z} and θ^{i+2}_{2z} that are equal denotes the same vertex on $C_{n(22r+1)}$ and all pairs θ^i_{1z} and θ^i_{2z} represent labels of adjacent vertices.

Recall that in the second column, τ is a permutation cycle of length $2r + 1$. Both 1 and 2 are relatively prime to $2r + 1$ for any integer r, so $\tau = \tau^1$ and τ^2 are also permutation cycles of length $2r + 1$. Consequently, we can track the labeling of $C_n(2r+1)$ by connecting these vertices from the third column continuously until we get a full circle of longer length (not stopping until all numbers in the third column are used). Since $1 \leq z \leq 2r + 1$, the result from this process is the labeled extended cycle $C_n(2r+1)$. For path component of G we create the same table, but since there is no relation between the endpoints, when tracking adjacent vertices in Θ_i from $i = 1$ until $i = m$, we will not be able to go back to $i = 1$. Every time we track adjacent vertices from $i = 1$ until $i = m$, we will get one copy of P_m instead. Since we have $(2r + 1)$ columns in each Θ_i, we end up with $(2r + 1)$ copies of P_m instead of $P_m(2r+1)$. Combining all extended components, we obtain an EMT labeling for G'. □

2. Main results

Cycles with Chords tCn_n.

Whole paper tC_n denotes a cycle C_n with one chord of length t, while $[c]\,tC_n$ denotes a cycle C_n with c chords, each of length t. In [6] several results for RSEM of the cycle with one chord are known and are stated in Theorem 2.1

Theorem 2.1. *[6] The following cycles with one chord has RSEM labeling: (a) tC_{4m+1} for all t other than $t = 5,\ 9,\ 4m-4,\ 4m-\ 8,$ given $m \geq 3$. (b) tC_{4m+1} for all $t \equiv 1$ (mod 4) except $t = 4m-\ 3$. (c) tC_{4m} for any m and $t \equiv 2$ (mod 4). (d) tC_{4m+2}, $m \geq 2$ for $t = 2,\ 6$ and all odd t other than 5. We apply Theorem 1.2 and Theorem 1.3 to general cycles with one chord. The cycle part get extended, while the number of chords multiplied.*

Theorem 2.2. *If the graph tC_n has a REM (RSEM) labeling, then there exist positive integers h and t_h such that for every integer $r \geq 0$, the graph $[(2r+1)\,h]\,thC_{n(2r+1)h}$ also has a REM (RSEM) labeling.*

Proof. Applying Theorem 1.3 to RSEM labeling of tC_n gives SEMT labeling of $[2r+1]\,t_1\ C_{n(2r+1)}$ for some value of $t_1 \in N$. Applying Theorem 1.3 to the SEMT labeling of $[2r+1]\,t_1C_{n(2r+1)}$ gives RSEM labeling of $\left[(2r+1)^2\right]t_2C_{n(2r+1)^2}$ for each of $t_2 \in N$. Performing this h times for any finite number h, we get RSEM labeling for $[(2r+1)]\,t_1C_{n(2r+1)^h}$. The pattern of how the length of the chord changes is still unknown. However, we do know the location of the chords. Suppose tC_n has a chord connecting the vertices labeled $\lambda(v_a)$ and $\lambda(v_b)$ where$v_a,\ v_b \in V$. Denote these vertices with the pair notation $(\lambda(v_a), \lambda(v_b))$. Applying Theorem 1.3 to tC_n, we get $[2r+1]\, t_1C_{(2r+1)n}$. The set of $(2r\ +1)$ chords written in the pair notation is $\{(2r\ +1)(\lambda(v_a)-1)+\kappa_{1j},\ (2r\ +1)(\lambda(v_b)-1)+\kappa_{2j}\}$ Since we can use different κ and κ' combination in the second column of the table, the number of the expanded graph is not unique. If we arrange the second column in the table in such a manner that we have τ^a where $a \neq 1,\ a \neq 2$ and a is relatively prime to n, then we can obtain differently expanded graphs. In the following example, we use both the proposed $\tau^4\left(\tau^{-1}\right)^3 = \tau$ and the possible alternative $\tau^4\left(\tau^{-1}\right)^0 = \tau^7$ for expanding $2C_7$. $\square$

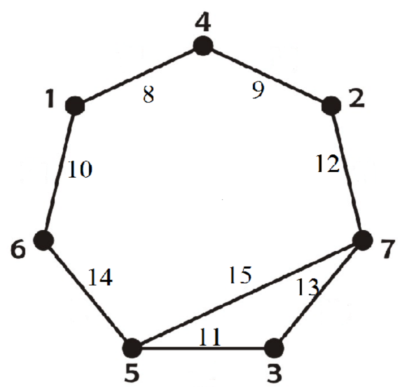

FIGURE 1. RSEM labeling for $2C_7$

Example 1. *$2C_7 \to [5]\, 9C_{35}$ and $2C_7 \to [3]\, 5C_{21}$ The RSEM labeling for $2C_7$ with $k\ = 3$ as given in [6] is shown in Figure 1. We expand using the factor $2r\ +\ 1\ =\ 3$. The table for the chord is given in Table 1.*

TABLE 1. Table for the chord on both $2C_7 \to [5]\, 9C_{35}$ and $2C_7 \to [3]\, 5C_{21}$

Λ	κ or κ'			θ_i		
4	*1*	*2*	*3*	*13*	*14*	*15*
6	*2*	*3*	*1*	*20*	*21*	*19*
14	*3*	*1*	*2*	*45*	*43*	*44*

The distinct tables using different κ and κ' combination of the cycle is given in Table 2. The second column of the left table is as defined in the proof of Theorem

TABLE 2. Table for the chord on both $2C_7 \to [3]\,9C_{21}$ (left) and $2C_7 \to [3]\,5C_{21}$(right)

Λ	κ or κ'			θ_i			Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3	0	1	2	3	1	2	3
5	2	3	1	17	18	16	5	2	3	1	17	18	16
9	3	1	2	30	28	29	9	3	1	2	30	28	29
5	1	2	3	16	17	18	5	1	2	3	16	17	18
4	2	3	1	14	15	13	4	2	3	1	14	15	13
13	3	1	2	42	40	41	13	3	1	2	42	40	41
4	1	2	3	13	14	15	4	1	2	3	13	14	15
2	2	3	1	8	9	7	2	2	3	1	8	9	7
10	3	1	2	33	31	32	10	3	1	2	33	31	32
2	1	2	3	7	8	9	2	1	2	3	7	8	9
6	2	3	1	20	21	19	6	2	3	1	20	21	19
12	3	1	2	29	27	28	12	3	1	2	29	27	28
6	1	2	3	19	20	21	6	2	3	1	20	21	19
1	2	3	1	5	6	4	1	1	2	3	4	5	6
11	3	1	2	36	34	35	11	3	1	2	36	34	35
1	1	2	3	4	5	6	1	2	3	1	5	6	3
3	2	3	1	11	12	10	3	1	2	3	10	11	12
8	3	1	2	27	25	26	8	3	1	2	27	25	26
3	1	2	3	10	11	12	3	2	3	1	11	12	10
0	2	3	1	2	3	1	0	1	2	3	1	2	3
7	3	1	2	24	22	23	7	3	1	2	24	22	23

2.2. The original cycle is $C_n = C_7$, so the first $\left[\frac{n}{2}\right] + 1 = 4$ rows use the array κ, while the other rows use the array κ'. The right table, on the other hand, use κ for every row. Both tables work, but they give different expanded graphs.

From the tables get a RSEM for $[3]\,9C_{21}$ and for $[3]\,5C_{21}$ with $k = 10$, as shown in Figure 2 and Figure 3.

3. UNIONS OF CYCLE PATHS mPn

In [2], [3] and [4], several results for SEMT of unions of paths are known:

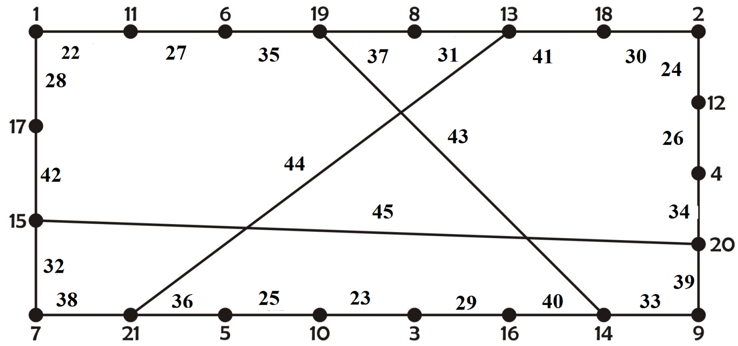

FIGURE 2. RSEM labeling for [3] $9C_{21}$

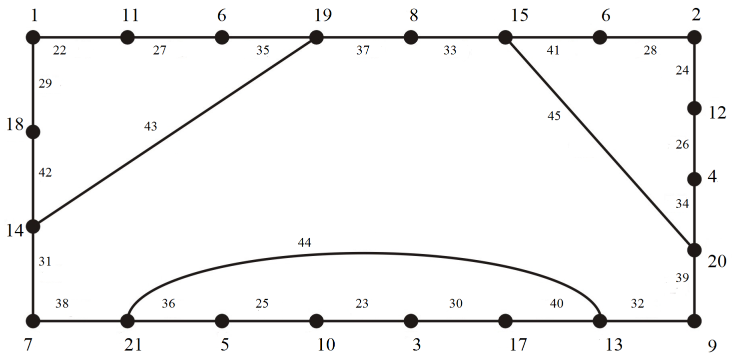

FIGURE 3. RSEM labeling [3] $5C_{21}$

Theorem 3.1. *The following graph has SEMT labeling:*

(a) [2] The graph $F \cong P_m \cup P_n$*, iff* $(m,\ n) \neq (2,\ 2)$ *or* $(3,\ 3)$.

(b) [3] mPn, *if* m *is odd.*

(c) [3] $P_3 \cup mP_2$*, for all* m.

(d) [3] $m(P_2 \cup mP_n)$*, if* m *is odd and.*

(e) [4] $2P_n$ *iff* n *is not.*

(f) [4] $2P_{4n}$ *has SEMT labeling for all* n.

Note that Theorem 3.1 (e) is a special case of Theorem 3.1(a) when $m = n$. *Applying Theorem 1.3 to Theorem 3.1 above we can summarize our new results in Theorem 3.2*

Theorem 3.2. *The following graph has RSEM labeling:*

(a) $(2r + 1)\ (P_m \cup P_n)$*, for any* r*, if* $(m,\ n) \neq (2,\ 2)$, *or*

(b) $(2r + 1)$ $(P_3 \cup P_2)$ *for any* m *and* r*,*
(c) mP_n *for even values of* $m,\ m \equiv 2$ *(mod 4), if*$n \neq 2, 3$*,*
(d) mP_{4n} *for even value of* $m,\ m \equiv 2$ *(mod 4) and all* $n \geq 2$*.*

Proof. Apply Theorem 1.3 to Theorem 3.1. □

Theorem 3.3. *[2] The following graph has SEMT labeling:*
(a) $C_3 \cup P_{n_2}$*, if* $n_2 \geq 6$*.*
(b) $C_4 \cup P_{n_2}$*, if* $n_2 \neq 3$*.*
(c) $C_5 \cup P_{n_2}$*, if* $n_2 \geq 4$*.*
(d) $C_{n1} \cup P_{n_2}$*, if* n_2 *is even and* $n_2 \geq \frac{n_1}{2} + 2$*.*

Theorem 3.4. *(a)* $m(C_{3(2r+1)} \cup (2r+1)\, P_{n_2}$*, for any*$r \geq 0$*, odd* m*, and* $n_2 \geq 2$*.*
(b) $m(C_{3(2r+1)} \cup (2r+1)\, P_{n_2}$*, for any*$r \geq 0$*, odd* m*, and* $n_2 \neq 3$*.*
(c) $m(C_{5(2r+1)} \cup (2r+1)\, P_{n_2}$*, for any*$r \geq 0$*, odd* m*, and* $n_2 \geq 6$*.*
(d) $m(C_{n_1(2r+1)} \cup (2r+1)\, P_{n_2}$*, for any* $r \geq 0$*, odd* m*, and* $n_1 \geq 4$ *and* $n_2 \geq \frac{n_1}{2} + 2$*.*

Proof. Apply Theorem 1.2 to Theorem 3.3 □

Lemma 3.1. *For any non-negative integer* r*, odd* m *and any positive integer* n_2*, the graph*$m(C_{n_1(2r+1)} \cup (2r+1)\, P_{n_2}$ *has an SEMT labeling when* $n_1 = 4, 5, 5, 8$ *or* 10*, unless (*n_1*,* n_2*)=(4,3),(5,1),(10,1).*

Example 2. $C_4 \cup P_2 \rightarrow C_{20} \cup 5P_2$.

RSEM labeling for $C_4 \cup P_2$ as given in [2] is shown in Figure 4.

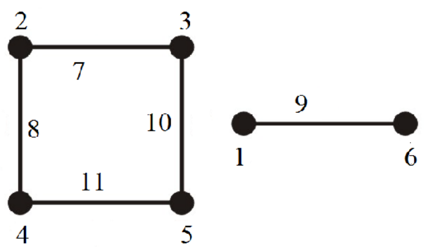

FIGURE 4. RSEM labeling for $C_4 \cup P_2$

We expand using the factor $2r + 1 = 5$. The tables are given in Table 2. For the table for $C_n = C_4$(left table), the first $\left[\frac{n}{2}\right] + 1 = 3$ rows use the array κ, while the fourth row use the array κ'. The table for path P_2(right table) use the only κ

since there is only one row.

More Results.

Theorem 1.3 is, in fact, applicable to any graph that has REM or RSEM labeling. The expanded graphs, however, either overlapped with results that already known, or might have little to none regularity that is of interest in magic labeling.

Example 3. $C_4 \cup P_2 \to C_{20} \cup 5P_2$

Figure 5 shows the SEMT labeling for $C_{20} \cup 5P_2$ with $k = 12$.

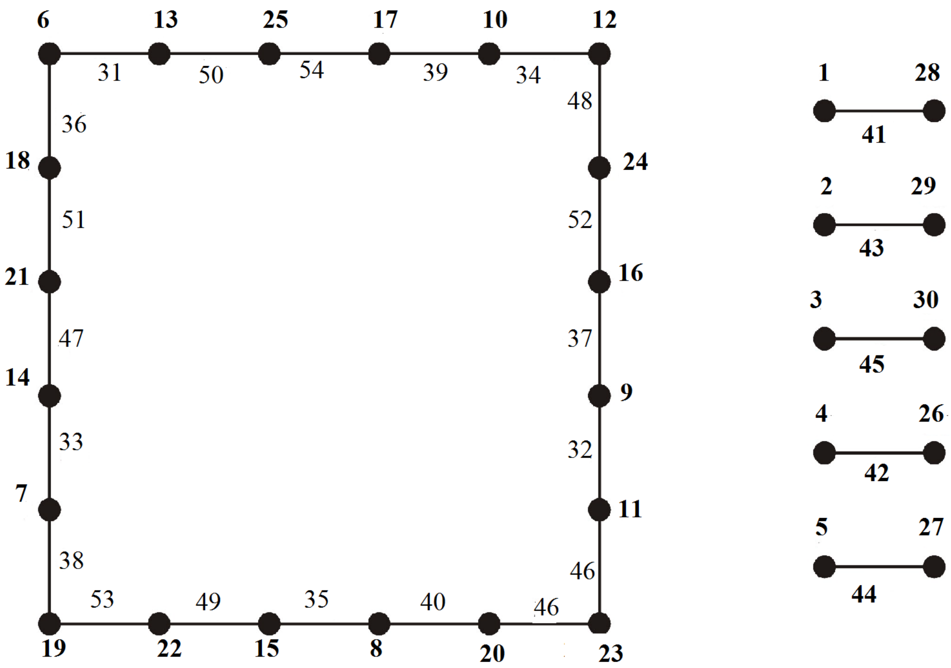

FIGURE 5. RSEM labeling $C_{20} \cup 5P_2$

Applying Theorem 3.1 to a SEMT labeling Cartesian product of $P_2 \odot P_n$ (ladder graph) that is given in [9] gives SEMT labeling of the union of an odd number of ladders $(P_2 \odot P_n)$for any odd values of m. However, ladder graphs are 3-colorable so the preservation of its SEMT labeling is already guaranteed by Theorem 2.2. In [8], application of Theorem 3.1 to other families such as fans, wheels, umbrellas, tadpoles, braids and many other families of graphs are given. One of the interesting possibilities of this work is to find a way to apply Theorem 3.1 using a certain combination ofκ, κ', and probably other variations of Kotzig arrays, to solve the open problem of finding the EMT or SEMT labeling of unions of the odd number of wheels.

In [2], Enomoto et al. checked all wheels up to $n = 29$ and found that W_n has EMT labeling if $n \neq 3$ mod 4. The construction of EMT labeling of W_n for all other cases are given in [5, 7]. When n is even wheels W_n are 3-colorable, so the existence of EMT labeling of tW_n for odd t and even n is guaranteed by Theorem 2.1. Also, it is given in [7] that tW_n does not have EMT labeling when t is odd and $n \equiv 3$ (mod 4).

4. Unions of Paths mP_n

Figure 6 shows the RSEM labeling for $P_4 \cup P_2$ with $k = 1$.

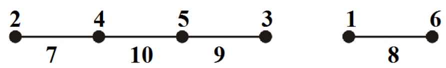

Figure 6. RSEM labeling $P_4 \cup P_2$

Table 3. Tables for $P_4 \cup P_2 \to 3\,(P_4 \cup P_2)$

Λ	κ or κ'			θ_i		
1	1	2	3	4	5	6
3	2	3	1	11	12	10
6	3	1	2	21	19	20
3	1	2	3	10	11	12
4	2	3	1	14	15	13
9	3	1	2	30	28	29
4	1	2	3	13	14	15
2	2	3	1	8	9	7
8	3	1	2	27	25	26

Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3
5	2	3	1	17	18	16
7	3	1	2	24	22	23

From the tables, we get an RSEM labeling for $3\,(P_4 \cup P_2)$ with $k = 4$. As shown in Figure 7.

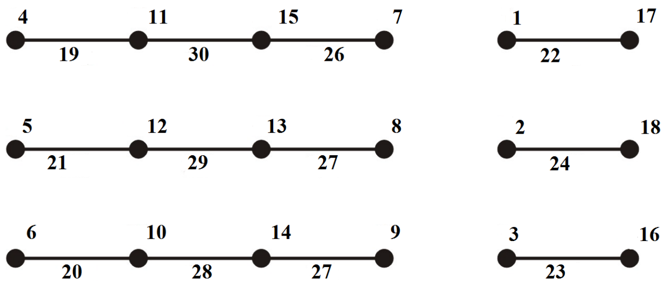

FIGURE 7. RSEMT labeling for $3\,(P_4 \cup P_2)$

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