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ϕ-REPRESENTATION OF HYPERSEMILATTICES

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ABSTRACT. A hypersemilattice is a structure possessing a hyperoperation $\otimes$ on a nonempty set, subject to a lot of aphorisms. In the current article we characterize the results of ϕ-products of hypersemilattices and then study the necessary and sufficient condition for ϕ-Representation of hypersemilattice.

1. INTRODUCTION

Ever since F. Marty [2] initiated the study of hyperstructures (multi algebras or poly algebras) in 1934, the study of hyperalgebras gained much importance as these have many applications in several areas of pure and applied sciences. Hypergroups studied by J. Janstosciak in 1997, hyperrings studied by R. Rosaria [3] in 1996, hyper BCK-algebras studied by Xin Xiao-long [4] in 2001, hyperlattices and hypersemilattices studied by Zhao Bin and others [1] are some important hyperstructures studied so far. Zhao Bin [1] introduced the notions of ideal, hyperorder and observed several properties of these.

Walendziak in 1992 presented the idea of $\phi-$ representation of algebras and got the fundamental and enough condition for $< (S_i : i \in I) >$ to have a $\phi-$

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representation. As the operations in a hypersemilattice are not usual binary operations, the results of universal algebra cannot be applied directly to the hypersemilattices. In this paper we study the $\phi-$ representation of hypersemilattices.

2. Preliminaries

This section presents the definitions on the concept and the results which have been already proven for our use to prove our main theorem.

Definition 2.1. *Let S alone a non void set and let $P(S)$ means the power set of $S, P^*(S) = P(S)\backslash\{\emptyset\}$. A binary hyperoperation " $\otimes$ " on S is a function from $S \times S \rightarrow P^*(S)$ satisfying the following conditions:*

(1) $a \otimes B = \bigcup_{b\in B}(a \otimes b)$

(2) $\mathrm{B} \otimes a = \bigcup_{b\in B}(\mathrm{~b} \otimes a)$

(3) $A \otimes B = \bigcup_{a\in A, b\in B}(\mathrm{a} \otimes b)$ *for* $a, b \in S$ *and* $A, B \in P^*(\mathrm{~S})$

Definition 2.2. *A hypersemilattice* $(\mathrm{S}, \otimes)$ *is a nonempty set with a hyperoperation* $\otimes$ *satisfying*

(1) $x \in x \otimes x$

(2) $x \otimes y = x \otimes y$

(3) $(\mathrm{x} \otimes y) \otimes \mathrm{z} = x \otimes (y \otimes \mathrm{z})$ *for any* $x, \mathrm{y}, \mathrm{z} \in \mathrm{S}$

Example 1. *Let S = x,y and characterize the operation $\otimes$ as follow:*

$\otimes$	*x*	*y*
x	{*x*,*y*}	{*x*,*y*}
y	{*x*,*y*}	{*y*}

At that point (S,$\otimes$) is a hypersemilattice.

Theorem 2.1. *The direct product of two hypersemilattices is likewise a hypersemilattice.*

Definition 2.3. *Let S_1,S_2 be two hypersemilattices. Then a function $h : S_1 \rightarrow S_2$ is known as a homomorphism if and just if $h(x \otimes y) = h(x) \otimes h(y)$.*

Theorem 2.2. *The homomorphic image of a hypersemilattice is likewise a hypersemilattice.*

Definition 2.4. *Let* $\{\mathrm{S}_k \mid k \in I\}$ *be a family of hypersemilattices. Then the direct product of* $\mathrm{S}_k, k \in I$ *is the Cartesian product* $\mathrm{II}\,(\mathrm{S}_k \mid k \in I) = \{(a_k)_k \in_I \mid a_i \in S_i\}$.

Theorem 2.3. *The direct product of a family of hypersemilattices is again a hypersemilattice.*

Definition 2.5. *Let* $S = (\mathrm{S}, \otimes)$ *be a hypersemilattice and* θ *be an equivalence relation on* S. *We stat that,* $A\Theta B$ *if and just if for every one of the a in* A *there exists* b *in* B *such that* a *O* b *for any* $A, B \subseteq S.\theta$ *is said to be a congruence relation if for any* $p, q, r, s \in S$, $p\theta q$ *and* $r\theta$ *s suggest* $(\mathrm{p} \otimes r)\theta(\mathrm{q} \otimes s)$ *It signifies the equivalence class* $\{\mathrm{p} \in S \mid \mathrm{z}\theta\mathrm{p}\}$ *by* $\mathrm{C_z}$ *for any* $\mathrm{z} \in \mathrm{S}$.

Definition 2.6. *Suppose* $\mathbf{S}_1, S_2$ *be two hypersemilattice and* $f : S_1 \to S_2$ *be a homomorphism. Then ker* $f = \{(\mathrm{p}, q) \in S_1 \times S_2 \mid f(\mathrm{p}) = f(q)\}$ *Therefore, it is comprehended that ker* f *is a congruence relation on* S.

Definition 2.7. *Let* $(\mathrm{S}, \otimes)$ *be a hypersemilattice and* O *be a congruence relation taking place* S. *It is defined as* $C_x\otimes_y = \{C_t \mid t \in x \otimes y\}$

Definition 2.8. *Let* $\Pi\,(S_i \mid i \in I)$ *be the direct product of hypersemilattices* $S_i, i \in I$. *Then the mapping* P_k : *I I* $(\mathrm{S}_i \mid i \in I) \to L_k$ *It is characterized that* $(a_i)_i \in_1$. x_k *are known as the* k^{th} *projection of the direct product I I (S* $_i$ *)* $i \in I$ *).*

Definition 2.9. *Let* $S = (\mathrm{S}, \otimes)$ *be a hypersemilattice. A non-empty subset* $H \subseteq \mathrm{S}$ *is said to be Subhypersemilattice of* S *if for all* $\mathrm{a}, \mathrm{b} \in \mathrm{H}, a \otimes \mathrm{b} \subseteq \mathrm{P}^*(\mathrm{H})$.

Definition 2.10. *Let* $\{\mathrm{S}_j \mid \mathrm{j} \in I\}$ *be a group of hypersemilattices. At that point a subhypersemilattice* H *of the direct product* $\Pi\mathrm{I}\,(\mathrm{S}_j \mid j \in I)$ *is known as a sub direct product of* $\{\mathrm{S}_j \mid \mathrm{j} \in I\}$ *if* $P_k(H) = S_k \quad \forall k \in I$.

3. ϕ-REPRESENTATION OF HYPERSEMILATTICES

Let ConS means the set of all congruence relations on a hypersemilattice S. At that point ConS shapes a complete semilattice with 0 S and 1 S the smallest and largest congruence relations separately.

Definition 3.1. *Let* $\{S_j/j \in I\}$ *be a group of hypersemilattices. Let* S *alone a subhypersemilattice of* $\Pi\,(\mathrm{S}_j \mid j \in I)$ *and* $\phi \in$ *ConS. Then* S *is supposed to be a* $\phi-$ *product of hypersemilattices* $S_j, j \in I$ *if it fulfills the accompanying:*

(i) *S is a sub direct product of the hypersemilattices $S_j, j \in I$,*

(ii) *for each $\bar{x} = (x_j; j \in I) \in S^I if\,(x_j, x_k) \in \phi$ $\forall$j, k $\in I$ then $(x_j(j) : j \in I) \in S.$*

Remark 3.1. *Let S be a subhypersemilattice of the direct product of hypersemilattices $S_j, j \in I$ then*

(i) *S is a sub direct product of $S_j, j \in I$ iff S is a 0_s - product of $S_j, j \in I$,*

(ii) *S is $a1_s-$ product of $S_j, j \in I$ iff $S = \prod_{j\in I} S_j$.*

Proof. (i) Suppose, S is a 0_s - product of hypersemilattices $S_i, j \in I$. Then S is a sub direct product of $S_j, \in$ I. Conversely, Assume that S be a sub direct product of $S_j, j \in I$, Let $x = (x_j(j) : j \in I) \in$ S is a 0_s - product of $S_j, j \in I$.

(ii) Suppose, S is a 1_S - product of hypersemilattices $S_j, j \in I$.
Let $x = (x_j j \in I) \in 1 \mid S_j$.Since $Pj \mid$ s is a surjective homomorphism, $\exists a_j \in S \supset P_j|_s\,(a_i) = x_j \quad \forall$ j $\in I.\bar{\exists}\bar{a} = (a_j : j \in I) \in S^I \supset a_j(j) = x_j$ for all j $\in I$. Since $1_s = S^I \times S^l, (a_j, a_k) \in 1_s$ for all j, k $\in I_.\ (a_j(j) : j \in I) \in$ S. $x = (x_j : j \in I) \in$ s. Hence $S = \prod_{j\in I} S_j$. Conversely, suppose that $S = \prod_{j\in l} S_j$.
Since each direct product is a subdirect product, S is a subdirect product of the hypersemilattices $S_j, j \in$ I. Let $\bar{x} = (x_j(j) : j \in I) \in S^I$ and suppose $(x_j, x_k) \in 1_s \quad \forall$ j, k $\in I$. Then clearly $(x_j(j) : j \in I) \in \prod_{j\in I} S_j = S.$

Hence S is a 1_s - product of the $S_i, j \in I$. ☐

Theorem 3.1. *Let $\{S_j/j \in I\}$ be a group of hypersemilattices, Let S be a subhypersemilattice of the direct product $\prod_{j\in I} S_j$ and let $\theta \in$ ConS. For $j \in I$, let θ_j be the kernel of the projection at j, confined to S. In the event that S is $a\phi-$ product of hypersemilattices $S_j, j \in I$ then*

(i) $0_s = \bigcap_{j\in I} \theta_j$

(ii) *For each $\bar{y} = (y; j \in I) \in S^I$ if $(x_j, x_k) \in \phi$ for all* j, k $\in I$ *then* $\exists y \in$ S *such that* $(y, y_j) \in \theta_j \quad \forall$j $\in I$.

(iii) $S/\theta_j \cong$ S$_j$ *for all* j $\in I$.

Proof. (i) Let $(a, b) \in \bigcap_{j\in I} \theta_j$ for some a, b $\in$ S. Then P_j Is $(a) = P_j|_S\,(b) \quad \forall$j $\in I.P_j(a) = P_j(b) \quad \forall$j $\in I.$a $= b.(a\ , b) \in 0$ s. There fore $0_S = \bigcap_{j\in I} \theta_j$.

(ii) Let $\bar{y} = (y_j : j \in I) \in S'$ such that $(y_j, y_k) \in \phi \forall\ iy = \langle y_j(j) : j \in I >\in S$.y$(j) = y_j(j)$ for all $\mathrm{y}, \mathrm{y_j} \in S, \forall \mathrm{j} \in I. P_{jlS}(y) = P_j \mid s\,(y_j) \quad \forall \mathrm{j} \in I.\,(y, y_j) \in \theta_{\mathrm{j}}$ for all j $\in I$. Subsequently there exists $y \in S$ to such an extent that $(y, y_j) \in \theta_j$ for all j $\in I$.

(iii) Define $f : S/\theta_{\mathrm{j}} \to \mathrm{S_j}$ by $[u]_{\theta j} \mapsto u(j)$. Note that for any $[u]_{\theta_i}, [v]_{\theta_j} \in S/\theta_j$, $[u]_{\theta_j} = [v]_{\theta_j} \Leftrightarrow (u, v) \in \theta_j \Leftrightarrow P_j\big|_s(u) = P_j\big|_s(v) \Leftrightarrow u(j) = v(j) \Leftrightarrow f\left([u]_{\theta_j}\right) = f\left([v]_{\theta_j}\right)$. Thus f is well defined and one-one. For any $u_j \in S_j$ there exists $u \in S$ such that $u(j) = P_j/s(u) = u_{j.} \Rightarrow \exists\ [u]_{\theta_j} \in S/\theta_j$ such that $f\left([u]_{\theta_j}\right) = u(j) = u_j$. Thus f is onto. Let $[u_1]_{\theta_j}, [u_2]_{\theta_j} \in S/\theta_j$, then $f\left([u_1]_{\theta_j} \otimes [u_2]_{\theta_j}\right) = f\left(\left\{[u_t]_{\theta_j}/u_t \in u_1 \otimes u_2\right\}\right) = \{\mathrm{u}_t(j) \mid u_t \in \mathrm{u}_l \otimes u_2\} = \mathrm{u}_l(j) \otimes u_2(j) = f\left([u_1]_{\theta_j}\right) \otimes f\left([u_2]_{\theta_j}\right)$. Thus f is homomorphism. $S/\theta_{\mathrm{j}} \cong \mathrm{S_j}$ for all j $\in I$.

□

Definition 3.2. *Let S be a hypersemilattice and let $\phi \in ConS$. For any system $\{\theta_i/\mathrm{i} \in I\}$ of congruences of S, we compose $0_S = \prod_\phi (\theta_i : i \in I)$ iff the conditions (i) and (ii) of above theorem are fulfilled.*

Remark 3.2. *Let* S *alone a hypersemilattice and let $\{\theta_i/i \in I\}$ be an arrangement of congruences of* S. *At that point*

(i) $O_S = \prod_{0_s} (\theta_i : i \in I)$ *if and just if* $0_S = \bigcap_{i \in I} \theta_i$

(ii) $I_S = \prod_{1_s} (\theta_i : i \in I)$ *if and just if* $0_s = \bigcap_{i \in I} \theta_i$ *and for each* $(y_i : i \in I) \in S^I \quad \exists\, y \in S \ni (y, y_i) \in \theta_i \forall i \in I$

Definition 3.3. *Let $f : S \to S'$ be an epimorphism of hypersemilattices* S, S' *and α be any congruence on* S. *At that point we characterize* $f(\alpha) = \{(f(u), f(v))/(u, v) \in \alpha\}$.

Remark 3.3. *$f(\alpha)$ be congruence on S'.*

Lemma 3.1. *Let S and S' be hypersemilattices and $\phi, \theta_i (i \in I)$ be congruences of S. On the off chance that f is an isomorphism from* S *onto S' at that point $O_S = \prod_\phi (\theta_i : i \in I)$ if and just if $0_{S'} = \prod_{f(\phi)} (f(\theta_i) : i \in I)$.*

Theorem 3.2. *Let S be a hypersemilattice, $\phi \in \operatorname{con} S$ and $(\theta_i : i \in I)$ be a system of congruences of S such that $O_S = \prod_\phi (\theta_i : i \in I)$. We set $S_i = S/\theta_i$ for* i $\in I$. *If the mapping $f : S \to \prod S_i$ is characterized by $f(x) = ([x]_\theta : i \in I) \forall x \in S$ then $f(S)$ is a $f(\phi)-$ product of hypersemilattices $S_i, i \in I$.*

Proof. Note that $x, y \in S$ such that $f(x) = f(y) \Leftrightarrow [x]_{\theta_i} = [y]_{\theta_i} \forall \mathrm{i} \in I \Leftrightarrow (x, y) \in \theta_i \quad \forall \mathrm{i} \in I \Leftrightarrow (x, y) \in \bigcap \theta_i = 0_S. \Leftrightarrow x = y$. Thus f is well-defined and one-one. To demonstrate f is homomorphism, let $x, y \in$ S. Then $f(x \otimes y) = (\{t/t \in x \otimes y\})d = \{([t]_{\theta_i} : i \in I)/t \in x \otimes y\} = ([x]_{\theta_i} : i \in I) \otimes ([y]_{\theta_i} : i \in I) = f(x) \otimes f(y)$. Subsequently f homomorphism.

$\Rightarrow S \cong f(S)$. Obviously $f(S)$ is a subhypersemilattice of $\prod S_i$. For any j $\in I$ $P_j(f(S) = \{P_j(f(x)) : x \in S)\} = \left\{[x]_{\theta_j}/x \in S\right\} = \mathrm{S}_j$. Consequently $f(\mathrm{~S})$ is a subdirect product of hypersemilattices $\mathrm{S_i}, \mathrm{i} \in$ I. Let $(y_i : i \in I) \in f(S)^I \ni (y_i, y_j) \in f(\phi) \forall \mathrm{i}, \mathrm{j} \in I$ then $\exists\, x_i \in S \forall i \in I \ni f(x_i) = y_i$ and $(x_i, x_j) \in \phi \forall \mathrm{i}, \mathrm{j} \in I$. $\Rightarrow \exists x \in S$ such that $(x, x_i) \in \theta_i \forall i \in I \Rightarrow [x]_{\theta_i} = [x_i]_{\theta_i} \forall \mathrm{i} \in I. \Rightarrow f(x)(i) = y_i(i) \forall \mathrm{i} \in I$.

Hence $(y_i : i \in I) = f(x) \in f(S)$. Consequently $f(S)$ is a $f(\phi)-$ product of hypersemilattices $S_i, \mathrm{i} \in I$. □

Definition 3.4. *Let S,* $\mathrm{S_i}(i \in I)$ *be hypersemilattices,* $\phi \in \operatorname{con} S$ *and* f *be an embedding of* S *into* $\prod S_i$*, on the off chance that* $f(S)$ *is an* $f(\phi)-$ *product of hypersemilattices* $S_i, \mathrm{i} \in I$ *then the order pair* $< (S_i : i \in I), f >$ *is called a* $\phi-$ *representation of the hypersemilattice* S.

Remark 3.4. *Let S,* $S_i(i \in I)$ *be hypersemilattices. At that point*

(i) $< (S_i : i \in I), f >$ *is an* 0_s *- representation of* S *if and just if* $f(S)$ *is a sub direct product of* $S_i, \mathrm{i} \in I$.

(ii) $< (S_i : i \in I), f >$ *is an* 1_S *- representation of* S *if and only if* $f(S)$ *is the direct product of* $S_i, \mathrm{i} \in I$.

Theorem 3.3. *Let S be a hypersemilattice,* $\phi, \theta_i(i \in I) \in \operatorname{con} S$*. Take* $S_i = S/\theta_i, i \in I$*, and characterize* $f : S \to \prod S_i$ *by* $x \mapsto ([x]_{\theta_i} : i \in I)$ *then* $< (S_i : i \in I, f >$ *is a* $\phi-$ *representation of* S *if and only if* $0_S = \prod_\phi (\theta_i : i \in I)$.

Corollary 3.1. (i) *An arrangement* $(\theta_i : i \in I)$ *of cogruences of hypersemilattice* S *gives a subdirect representation if and just if* $\bigcap \theta_i = 0_S$

(ii) *An arrangement* $(\theta_i : i \in I)$ *of cogruences of hypersemilattice S constitutes a direct representation if and only if* $0_S = \prod_{1_S} (\theta_i : i \in I)$.

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