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FACTORISATION AND LABELING IN SEMIGRAPHS

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ABSTRACT. In graph theory, a factor of a graph G is a spanning subgraph that is the subgraph that has the same vertex set as G. In this paper we study about factorization and labeling in semigraphs with the construction of semigraphs obtained from graphs.

1. INTRODUCTION

Graphs have applications in theoretical and practical fields. Graphs can describe only binary relations but not always sufficient for modeling problems or data which involves relations higher than binary. For this reason we have the concept of hypergraphs and semigraphs. Factorisation in semigraphs will have more applications in tackling modeling problems. A factor F of graph G is an r factor if the degree of each vertex in F is r. The most dealt degree factors are those in which r equals 1 that is each component is a single edge. Bichitra Kalita has discussed about different types of factorization of complete graphs of some particular forms. He developed the algorithm for the solution of TSP. He discussed theoretical investigations related to 3-factors, 2-factors and 1-factor and cited experimental results. There is a vast study of work on factors and factorization and this topic has much in common with other areas of study in

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graph theory. For example factorization significantly overlaps the topic of edge coloring.

A graph labeling is the assignment of labels which are integers to vertices or edges or to both vertices and edges of a graph. There are two types of labeling with respect to vertices and edges. If there is a function from V to a set of labels, then the graph with such a function is defined as vertex labeled graph. Similarly if there is a function from E to a set of labels, then the graph with such a function is defined as edge labeled graph. If the labels on the edges are elements of an ordered set which are real numbers, then it is called as the weighted graph. The author is motivated to study about factorisation and labeling with the work done by different authors on graph factorization and graph labeling [1–6].

2. Preliminaries

In this section we recall the definitions of edge factor semigraphs and labeling in semigraphs and we introduce new concept edge factor in a semigraph.

2.1. **Edge factors in a semigraph.** If a semigraph is split into edge suhgraphs whose union is the original graph then we define edge subgraphs as edge factors. A semigraph can be split into n edge subgraphs which are called as n edge factor graphs. If n equals 1 then we define them as one edge factor graphs. If n equals two then we define them as two edge factor subgraphs.

2.2. **1-factor in a semigraph.** 1-factor of a semigraph G is defined as the subgraph of G where each of the end vertices is of degree one and union of the graphs with respect to the end vertices forms the original graph.

2.3. **2-factor in a semigraph.** 2-Factor of a semigraph is a sub-graph of a graph G where each of the end vertices are of degree two and their union forms the original graph. In the same way we can define 3 factors, 4 factors, . . . , in a semigraph.

2.4. **Graph labeling.** If vertices and edges of a graph G are assigned natural numbers from 1 to n, where n is the number of vertices and edges then such a mapping $f; V \cup E \to \{1, 2, 3, \ldots, |V \cup E|\}$ is called as graph labeling. Graph is said to be a labeled graph according to Rosa.

3. Factorization and labeling in semigraphs

In this section we construct semigraphs from different graphs to study factorization and labeling in semigraphs.

3.1. **Construction of semigraphs from standard graphs with insertion of middle vertices.** Insert middle vertices in an edge. An edge with one middle vertex has 2 one factor edgegraphs. An edge with two middle vertices has three 1 edge factor graphs. An edge with m number of end vertices will have $m + 1$ number of one edge factor graphs.

3.2. **Uniform cycle semigraph.** We introduce the notion uniform cycle semigraph and then we study edge factor graphs obtained from semigraph.

We classify the semigraphs as follows:

If there are equal number of middle vertices in all the edges of C_n we call it as the uniform cycle semigraph. If there is 1 middle vertex in every edge, it is called as $1m$ cycle semigraph. If there are n number of middle vertices in every edge, then it is called as nm cycle semigraph. If there is one middle vertex in every edge of the cycle C_n, then the obtained semigraph will have $2n$ number of 1 edge factor graphs or the number of 1 edge factors equals the cardinality of the edge set. If there are two middle vertices in every edge of the cycle C_n, then the obtained semigraph will have $3n$ number of 1 edge factor graphs or the number of 1 edge factors equals the cardinality of the edge set. If there are p number of middle vertices in every edge of the cycle C_n, then the obtained semigraph will have pn number of 1 edge factor graphs or the number of 1 edge factors equals the cardinality of the edge set. Similarly if there are p number of middle vertices and q number of middle end vertices then the number of one edge factor graphs will be equal to the cardinality of the edge set of the semigraph. For any n, the cycle C_n itself is the two factor graph. Also the semigraph obtained from C_n with any number of middle vertices is the two factor graph. The cycle C_n with same or different number of middle vertices and same or different number of middle end vertices in all the edges is the two factor graph.

3.3. **Labeling in a cycle C_n with one middle vertex in all of its edges.** Consider a cycle C_3 with a middle vertex in each of its edges. The obtained semigraph will have 6 vertices and 6 edges. Let the vertices be named as 1, 2, 3, 4, 5, 6 where 2, 4, 6 are the middle vertices and 1, 3, 5 are the end vertices. The labeling in the constructed semigraph is defined as

$$f(v_i) = i, \qquad 1 \le i \le 6;$$
$$f(e_i) = i + 6, \quad 1 \le i \le 6.$$

In general, the cycle C_n with a middle vertex in each of its edges that is the uniform cycle semigraph will have n vertices and n edges. Let the vertices be named as $1, 2, 3, 4, 5, 6, \ldots, n$, which has both middle vertices and end vertices. The labeling in the constructed semigraph is defined as

$$f(v_i) = i, \qquad 1 \le i \le n;$$
$$f(e_j) = j + n, \quad 1 \le j \le n.$$

3.4. **Labeling of semigraph obtained from a complete graph.** Consider a complete graph K_n on n number of vertices. A complete graph with equal number of middle vertices in all of its edges is called as uniform complete semigraph. The number of 1 edge factors in a uniform complete semigraph is equal to the cardinality of the vertex set.

Consider a complete graph with a middle vertex in all of its edges. The constructed semigraph will have $n = |V| + |E|$ vertices and $m = 2\,|E|$ edges. The labeling for the constructed semigraph is as follows:

$$f(v_i) = i, \qquad 1 \le i \le n$$
$$f(ej) = n + j, \quad 1 \le j \le m$$

3.5. **Factor semigraph obtained from particular forms of complete semigraph.** In this section we discuss about the number of 1 factor, 2 factor and 3 factor semigraphs obtained from particular forms of semigraphs.

Theorem 3.1. *The complete graphs, $K_4, K_{10}, K_{16}, \ldots$, that is K_{6p-2} with same or different number of middle vertices and/or middle end vertices in some or all of its edges, forms a semigraph with 1, 3, 5, … number of 3 factor semigraphs that is 2p + 1 number of 3 factor semigraphs respectively, by considering only with respect to the end vertices.*

Proof. We will discuss the proof by mathematical induction. The semigraphs constructed with same or different number of middle vertices or middle end vertices in every edge will have $6p - 2$ number of vertices and $18p^2 - 15p + 3$ number of edges. For $p = 1$, we have the complete graph K_4 with 4 number of end vertices and 6 edges with respect to the end vertices (excluding the middle vertices). This graph K_4 itself is the 3 regular subgraph, that is the 3 factor graph. Let us assume that the statement is true for $p = k$. That is the semigraph K_{6k-2} has $6k - 2$ number of end vertices and $18k^2 - 15k + 3$ number of edges with respect to end vertices. For p equals $k + 1$, the complete graph with middle vertices which gives rise to the semigraphs will have $6(k + 1) - 2$ number of end vertices and $18(k + 1)^2 - 15(k + 1) + 3$ number of edges with respect to end vertices. We find that whenver p is greater than or equal to 1, $k + 1$ is greater than or equal to 1 means k is greater than or equal to zero. These semigraphs will have $2p + 1$ number of 3 factor graphs. By Mathematical Induction, the constructed semigraphs will have $2p + 1$ number of 3 factor semigraphs. □

Theorem 3.2. *The semigraph obtained from complete graph K_{6p} with insertion of same or different number of middle vertices in the edges and/or middle end vertices in some or all of its edges, will have $2p - 1$ number of 3 factor graphs and one 2 factor graph by considering only with respect to the end vertices.*

Proof. We will discuss the proof by mathematical induction. For p equals 1, the semigraph K_6 has one 3 factor graph and one 2 factor graph. Let us assume that the theorem is true for $p = k$.

The complete graph K_{6p} for p equals k, has $6k$ vertices and $18k^2 - 3k$, has $2k - 1$ number of 3 factor graphs and one 2 factor graph. For p equals $k + 1$, the complete graph $K_6(p + 1)$ has $6(p + 1)$ vertices and $18(k + 1)^2 - 3(k + 1)$ edges will have $2(p + 1) - 1$ number of 3 factor graphs and one 2 factor graph. We find that whenever p is greater than or equal to 1, $k + 1$ is greater than or equal to 1 means k is greater than or equal to zero. By Mathematical Induction, the constructed semigraphs will have $2p - 1$ number of 3 factor semigraphs and only one 2 factor graph. □

Theorem 3.3. *The semigraph obtained from complete graph K_{6p+2} for $p \geq 1$ with insertion of same or different number of middle vertices in the edges and/or middle end vertices in some or all of its edges, has $2p$ number of 3 factor graphs and one 1 factor graph by considering only with respect to the end vertices.*

Proof. We are going to prove the theorem with the help of method of mathematical induction. For $p = 1$, the complete graph K_6 has 6 vertices and 15 edges. Let us assume that the theorem is true for $p = k$. The semigraph K_{6k} with $6k + 2$

vertices and $18k^2+9k+1$ edges, will have $2k-1$ number of 3 factor graphs and one 2 factor graph.

For $p=k+1$, the complete graph $K_6(k+1)+2$ with $6(k+1)+2$ vertices and $18k^2+45k+28$ will have $2(k+1)$ number of 3 factor graphs and one 2 factor graph. It is clear that whenever p is greater than or equal to 1, $k+1$ is greater than or equal to 1 means k is greater than or equal to zero. The theorem is true for m equals $k+1$. By the principle of mathematical induction the constructed semigraphs will have $2m$ number of 3 factor graphs and one 1 factor graph. □

4. One edge factor graphs and labeling in semigraphs obtained from special graphs

4.1. Fan semigraph. The fan semigraph F_n for $n \geq 2$ is obtained by joining all vertices of the semigraph—a path of n vertices with middle vertices to a further vertex called as the middle vertex and $2n-1$ edges. $F_n = P_n + K_1$. The labeling in the constructed fan semigraph is

$$\begin{aligned} f(v_i) &= i, \quad 1 \leq i \leq p; \\ f(e_i) &= j, \quad p+1 \leq j \leq p+m, \end{aligned}$$

where p is $n+1$ and m is the number of edges.

The number of one edge factor graphs is equal to the cardinality of the edge set of the fan semigraph. In the constructed fan semigraph, the degree of end vertices is 2, degree of middle end vertices is 3 and the degree of the central or the middle vertex is n of the path P_n.

4.2. Friendship semigraph. A friendship semigraph is the semigraph which consist of n triangles, which are semigraphs with a common vertex. The labeling in the constructed friendship semigraph is defined as

$$\begin{aligned} f(v_i) &= i, & 1 \leq i \leq 2p, \\ f(e_j) &= 2p+j, & 1 \leq j \leq 3q, \end{aligned}$$

where p is the number of triangles and q is the total number of edges of all triangles.

The number of one factor graphs is 3 times the number of triangles.

The degree of the central vertex or the middle vertex of the friendship semigraph is the number of triangles incident with the vertex and the degree of other vertices is 2.

4.3. **Wheel semigraph.** The wheel semigraph is defined as the join of the complete graph with one edge with the cycle of n vertices that is C_n. The middle vertex is the intersection of a vertex from K_1 with every vertex of the cycle C_n. The labeling in the constructed wheel semigraph is defined as

$$\begin{aligned} f(v_i) &= i, & 1 \leq i \leq p, \\ f(e_j) &= p + j, & 1 \leq j \leq m, \end{aligned}$$

where p is $n+1$ and m is the number of edges.

The degree of the middle vertex is n which is same as the n vertices of the cycle. The degree of other vertices is 3. The number of one edge factor semigraphs is same as the cardinality of the edge set of the wheel semigraph.

4.4. **Helm semigraph.** The helm semigraph is constructed from a wheel semigraph by attaching a pendant vertex at each vertex of the n cycle. The labeling in newly constructed helm semigraph is defined as

$$\begin{aligned} f(v_i) &= i, & 1 \leq i \leq p \\ f(e_j) &= j + p, & 1 \leq j \leq m + s, \end{aligned}$$

where m is the number of edges of the constructed semigraph and s is the number of pendant vertices and p is $2n+1$.

The degree of the middle vertex of the constructed helm semigraph is n which is same as the cardinality of the cycle with n vertices and the degree of the other vertices is 4. The number of one edge factor graphs is equal to the cardinality of the edge set of the helm semigraph.

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