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THE COEFFICIENTS OF THE SKEW DISTANCE CHARACTERISTIC POLYNOMIAL OF A DIRECTED PATH WITH DEGENERATE ORIENTATION

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ABSTRACT. A directed graph G^ϕ is a finite simple undirected graph G with an orientation ϕ, which assigns to each edge a direction so that G^ϕ becomes a directed graph. G is called the underlying graph of G^ϕ and we denote by $SD(G^\phi)$, the Skew-Distance matrix of G^ϕ. The eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ of the $SD(G^\phi)$ are said to be the skew distance eigen values or the SD-Eigen values of G^ϕ. The Skew Distance Energy, $E_{SD}(G^\phi) = \sum_{i=1}^{n} |\lambda_i - \overline{\lambda}|$, where $\overline{\lambda} = \left[\frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{n}\right]$. In this paper, the coefficients of the skew distance characteristic polynomial of a directed path P_n with degenerate orientation Ψ is obtained from the skew distance matrix $SD(P_n^\Psi)$.

1. INTRODUCTION

Let G be a finite simple connected graph with n vertices and m edges. Let G^ϕ be a graph with an orientation ϕ, which assigns to each edge of G a direction so that G^ϕ becomes a directed graph. The skew adjacency matrix of the directed graph G^ϕ is the $n \times n$ matrix, $S(G^\phi) = (S_{ij})$, where $S_{ij} = 1 = -S_{ji}$ if $v_i \to v_j$ is an arc of G^ϕ, otherwise $S_{ij} = S_{ji} = 0$.

Let d_{ij}^ϕ be the distance between the vertices v_i and v_j in G^ϕ. The Skew Distance Matrix (SD-matrix) of G^ϕ, $SD(G^\phi) = (Sd_{ij})$ is real skew symmetric matrix,

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where

$$Sd_{ij} = \begin{cases} d^{\phi}_{ij} & \text{if } d^{\phi}_{ij} \leq d^{\phi}_{ji} \\ -d^{\phi}_{ji} & \text{if } d^{\phi}_{ij} > d^{\phi}_{ji} \\ 0 & \text{if no path between } v_i \text{ and } v_j \end{cases}$$

$$\text{and } \; Sd_{ij} = -Sd_{ji}.$$

Suppose there is only one path from v_i to v_j or v_j to v_i, then $Sd_{ij} = d^{\phi}_{ij}$ or $Sd_{ij} = -d^{\phi}_{ji}$.

In this paper we find the coefficients of the skew distance characteristic polynomial of a directed path with degenerate orientation Ψ.

2. The coefficients of the skew distance characteristic polynomial of the path P_n with degenerate orientation

Definition 2.1. *[1] Let $G(V, E)$ be a bipartite graph with bi-partition (X, Y). An orientation is said to be* ***canonical*** *if it orients all the edges from one partition set to the other. It is immaterial if it is from X to Y or from Y to X.From this point onwards, σ stands for the canonical orientation with respect to a bipartite graph G with a fixed bi-partition (X,Y).*

Theorem 2.1. *[1] Let ${P_n}^{\sigma}$ be the path on n vertices with the canonical orientation σ and let $P[SD({P_n}^{\sigma}); x] = C_0x^n + c_1x^{n-1} + C_2x^{n-2} + \cdots + C_ix^{n-i} + \cdots + C_n$ be the characteristic polynomial of $SD({P_n}^{\sigma})$. Then (i) $C_0 = 1$, (ii) $C_1 = 0$, (iii) $C_i = 0, \forall$ odd i, (iv) $C_n = 1 \forall$ even $n \geq 2$, and (v) $C_{2i} =$ the coefficient of $x^{n-2i} = \begin{bmatrix} n-i \\ i \end{bmatrix}$.*

Definition 2.2. *[1] Let G be a finite simple connected graph with n vertices and m edges. An orientation is said to be a* ***reachable orientation*** *in P_n if all the edges in P_n are in one direction. An orientation is said to be a reachable orientation in G if any two vertices in G has at least one path with reachable orientation. Reachable orientation is denoted by R.*

Theorem 2.2. *[1] Let P^R_n be the path on n-vertices with the reachable orientation R. Then its skew distance characteristic polynomial is*

$$P[SD({P_n}^{R}); x] = x^n + \frac{n^2(n^2-1)}{12}x^{n-2} \qquad \forall n \geq 2.$$

Definition 2.3. *[1] An orientation in P_n is said to be* ***degenerate*** *if it is neither reachable nor canonical in P_n and is denoted by Ψ.*

Notation 2.1. *[1] Ψ_{n_1,n_2} represents the degenerate orientation on the path P in which the first segment n_1 edges of P has one direction and the second segment n_2 edges of P has an opposite direction to the direction of the edges in the first segment of P.*

Notation 2.2. *[1]*

(i) $P_n^{\Psi_{n_1,n_2}}$ *represents the path on $n = (n_1+1)+n_2$ vertices with the degenerate orientation Ψ_{n_1,n_2}.*
(ii) $P_n^{\Psi_{n-1}} = P_n^R$.
(iii) $P_{n+1}^{\Psi_{1,1,1,\ldots,1(ntimes)}} = P_{n+1}^{\sigma}$.

Theorem 2.3. *[1] Let $P_n^{\Psi_{n_1,n_2}}$ be a path on n-vertices with the degenerate orientation Ψ in which the $i^{th}(i=1,2)$ segment of P_n has $n_i(i=1,2)$ edges and all the edges in the i^{th} segment are in one direction. Then*

$$\begin{aligned}
&P[SD(P_n^{\Psi_{n_1,n_2}});x]\\
&= x^{n_1+n_2+1} + \left[\frac{n_1(n_1+1)^2(n_1+2)}{12} + \frac{n_2(n_2+1)^2(n_2+2)}{12}\right] x^{n_1+n_2-1}\\
&\quad + [\frac{n_1^2(n_1^2-1)}{12}\frac{n_2^2(n_2^2-1)}{12} + \frac{n_1^2(n_1^2-1)}{12}\frac{n_2(n_2+1)(2n_2+1)}{6}\\
&\quad + \frac{n_2^2(n_2^2-1)}{12}\frac{n_1(n_1+1)(2n_1+1)}{6}]x^{n_1+n_2-3},
\end{aligned}$$

for all $n = (n_1+1)+n_2 \geq 4$.

Corollary 2.1. *[1]*

$$\begin{aligned}
P[SD(P_n^{\Psi_{n_1,n_2}});x] = x\ &P[SD(P_{n-1}^{\Psi_{n_1,n_2-1}});x]\\
&+ \frac{n_2(n_2-1)(2n_2-1)}{6}\ x^{n_2-2}\ P[SD(P_{n-n_2}^{\Psi_{n_1}});x]\\
&+ {n_2}^2 x^{n_2-1} P[SD(P_{n-n_2-1}^{\Psi_{n_1-1}});x],
\end{aligned}$$

where $n = (n_1+1)+n_2$.

Theorem 2.4. *[1] Let $P_n^{\Psi_{n_1,n_2,\ldots,n_k}}$ be the path on $n = (n_1+1)+n_2+\cdots+n_k$ vertices with degenerate orientation $\Psi_{n_1,n_2,\ldots,n_k}$. Then*

$$\begin{aligned}
&P[SD(P_n^{\Psi_{n_1,n_2,\ldots,n_k}});\ x] = P[SD(P_{(n_1+1)+n_2+\cdots+n_k}^{\Psi_{n_1,n_2,\ldots,n_k}});\ x]\\
&= xP[SD(P_{(n_1+1)+n_2+\cdots+n_k-1}^{\Psi_{n_1,n_2,\ldots,n_k-1}});\ x]\\
&\quad + \frac{n_k(n_k-1)(2n_k-1)}{6}\, x^{n_k-2}P[SD(P_{(n_1+1)+n_2+\cdots+n_{k-1}}^{\Psi_{n_1,n_2,\ldots,n_{k-1}}});\ x]\\
&\quad + {n_k}^2 x^{n_k-1}P[SD(P_{(n_1+1)+n_2+\cdots+n_{k-1}-1}^{\Psi_{n_1,n_2,\ldots,n_{k-1}-1}});\ x],
\end{aligned}$$

for all $n \geq 4$.

Corollary 2.2. *[1] If $n_1 = n_2 = \cdots = n_k = 1$, then $P[SD(P_{(n_1+1)+n_2+\cdots+n_k}^{\Psi_{n_1,n_2,\ldots,n_k}});\ x] = P[SD(P_{k+1}^{\sigma});x]$.*

Theorem 2.5. *Let C_r be the coefficient of x^{n-r} in $P[SD(P_n^{\Psi});\ x] = C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\cdots+C_rx^{n-r}+\cdots+C_n$, Ψ is a degenerate orientation in P_n. Then for $r \geq 1$ and $k \geq 2$,*

$$\begin{aligned}
&C_{2r} \textit{ in } P[SD(P_{(n_1+1)+n_2+\cdots+n_k}^{\Psi_{n_1,n_2,\ldots,n_k}});\ x] = C_{2r} \textit{ in } P[SD(P_{(n_1+1)+n_2+\cdots+n_{k-1}}^{\Psi_{n_1,n_2,\ldots,n_{k-1}}});\ x]\\
&\quad + \frac{{n_k}^2({n_k}^2-1)}{12}C_{2r-2} \textit{ in } P[SD(P_{(n_1+1)+n_2+\cdots+n_{k-1}}^{\Psi_{n_1,n_2,\ldots,n_{k-1}}});\ x]\\
&\quad + \frac{n_k(n_k+1(2n_k+1))}{6}C_{2r-2} \textit{ in } P[SD(P_{(n_1+1)+n_2+\cdots+n_{k-1}-1}^{\Psi_{n_1,n_2,\ldots,n_{k-1}-1}});\ x],
\end{aligned}$$

where $n = (n_1+1)+n_2+\cdots+n_k$.

Proof. Let $P\big[SD(P_{(n_1+1)+n_2+\cdots+n_k}^{\Psi_{n_1,n_2,\cdots,n_k}});\ x\big]$ be a path on $n = (n_1+1)+n_2+\cdots+n_k$ vertices with degenerate orientation $\Psi_{n_1,n_2,\cdots,n_k}$. Then,

$$\begin{aligned}
&C_{2r} \text{ in } P\big[SD(P_{(n_1+1)+n_2+\cdots+n_k}^{\Psi_{n_1,n_2,\cdots,n_k}});\ x\big]\\
&= \text{the coefficient of } x^{n-2r} \text{ in } P\big[SD(P_{(n_1+1)+n_2+\cdots+n_k}^{\Psi_{n_1,n_2,\cdots,n_k}});\ x\big]\\
&= \text{the coefficient of } x^{n-2r-1} \text{ in } P\big[SD(P_{(n_1+1)+n_2+\cdots+n_k-1}^{\Psi_{n_1,n_2,\cdots,n_k-1}});\ x\big]\\
&\quad + \frac{n_k(n_k-1)(2n_k-1)}{6} \text{ the coefficient of } x^{n-n_k-2r+2}\\
&\quad\quad \text{in } P\big[SD(P_{(n_1+1)+n_2+\cdots+n_{(k-1)}}^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}}});\ x\big]\\
&\quad + n_k^2 \text{ the coefficient of } x^{n-n_k-2r+1} \text{ in } P\big[SD(P_{(n_1+1)+n_2+\cdots+n_{(k-1)}-1}^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}-1}});\ x\big],\ [1]
\end{aligned}$$

$$= \text{the coefficient of } x^{n-2r-2} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_k-2}}_{(n_1+1)+n_2+\cdots+n_k-2});\ x\big]$$
$$+ \big[1^2+2^2+\cdots+(n_k-2)^2\big]\ C_{2r-2} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}});\ x\big]$$
$$+ n^2_{(k-1)}\ C_{2r-2} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}-1}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}-1});\ x\big]$$
$$+ \big[1^2+2^2+\cdots+(n_k-1)^2\big]\ C_{2r-2} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}});\ x\big]$$
$$+ n^2_k\ C_{2r-2} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}-1}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}-1});\ x\big]$$

Continuing in this way, we get

$$C_{2r} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_k}}_{(n_1+1)+n_2+\cdots+n_k});\ x\big]$$
$$= \text{the coefficient of } x^{n-2r} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_k}}_{(n_1+1)+n_2+\cdots+n_k});\ x\big]$$
$$= \text{the coefficient of } x^{n-2r-n_k} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}});\ x\big]$$
$$+ \big[1^2+(1^2+2^2)+\cdots+(1^2+2^2+\cdots+(n_k-1)^2)\big]\ C_{2r-2}$$
$$\text{in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}});\ x\big]$$
$$+ \big(1^2+2^2+\cdots+n_k^2\big)\ C_{2r-2} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}-1}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}-1});\ x\big]$$
$$= C_{2r} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}});\ x\big]$$
$$+ \frac{n_k^2(n_k^2-1)}{12}\ C_{2r-2} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}});\ x\big]$$
$$+ \frac{n_k(n_k+1)(2n_k+1)}{6}\ C_{2r-2} \text{ in } P\big[SD(P^{\Psi_{n_1,n_2,\cdots,n_{(k-1)}-1}}_{(n_1+1)+n_2+\cdots+n_{(k-1)}-1});\ x\big]$$

Hence the proof. □

Theorem 2.6. *Let* $P\Big[SD(P^{\Psi_{n_1,n_2,\cdots,n_k}}_{(n_1+1)+n_2+\cdots+n_k});\ x\Big] = C_0\ x^n + C_1\ x^{n-1} + C_2\ x^{n-2} + \cdots + C_n$*, where* $n = (n_1+1)+n_2+\cdots+n_k$*. Then*

(i) $C_2 = \sum_{i=1}^k \frac{n_i(n_i+1)^2(n_i+2)}{12} = \sum_{i=1}^k \frac{n_i^2(n_i^2-1)}{12} + \sum_{i=1}^k \frac{n_i(n_i+1)(2n_i+1)}{6}$
$= \sum_{1\le i<j\le k} Sd_{ij}^2$;

(ii) $C_4 = \sum_{1\le i<j\le k} \frac{n_i^2(n_i^2-1)}{12}\frac{n_j^2(n_j^2-1)}{12} + \sum_{\substack{i,j=1\\ i\neq j}}^k \frac{n_i^2(n_i^2-1)}{12}\frac{n_j(n_j+1)(2n_j+1)}{6}$
$+ \sum_{\substack{1\le i<j\le k\\ 2\le j-i\le k-1}} \frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_j(n_j+1)(2n_j+1)}{6}$;

(iii) $C_{2(k+1)} = 0$ *and hence* $C_{2i} = 0$ *for all* $i \ge k+1$.

Proof. (i) We use induction on *the number of segments* in P_n. When the number of segments in $P_n = 2$. Let n_1 and n_2 be the length of the first and second segments of P_n and $P^{\Psi_{n_1,n_2}}_{n_1+1+n_2}$. Then $n_1 + 1 + n_2 = n$. As

$$P\left[SD(P^{\Psi_{n_1,n_2}}_{n_1+1+n_2});\ x\right] = x\ P\left[SD(P^{\Psi_{n_1,n_2-1}}_{n_1+1+n_2-1});\ x\right]$$
$$+\frac{n_2(n_2-1)(2n_2-1)}{6}\ x^{n_2-2}\ P\left[SD(P^{\Psi_{n_1}}_{n_1+1});\ x\right]$$
$$+n_2^2\ x^{n_2-1}\ P\left[SD(P^{\Psi_{n_1-1}}_{n_1});\ x\right],$$

and $C_2 =$ the coefficient of x^{n-2} in $P[SD(P^{\Psi_{n_1,n_2}}_{n_1+1+n_2});\ x]$. By Theorem 2.5,

$$C_2 \text{ in } P[SD(P^{\Psi_{n_1,n_2}}_{n_1+1+n_2});\ x] = C_2 \text{ in } P[SD(P^{\Psi_{n_1}}_{n_1+1});\ x]$$
$$+\frac{n_2^2(n_2^2-1)}{12}\ C_0 \text{ in } P\left[SD(P^{\Psi_{n_1}}_{(n_1+1)});\ x\right]$$
$$+\frac{n_2(n_2+1)(2n_2+1)}{6}\ C_0 \text{ in } P\left[SD(P^{\Psi_{n_1-1}}_{n_1});\ x\right]$$
$$C_2 = \sum_{i=1}^{2}\frac{n_i(n_i+1)^2(n_i+2)}{12} = \sum_{i=1}^{2}\left[\frac{n_i^2(n_i^2-1)}{12}+\frac{n_i(n_i+1)(2n_i+1)}{6}\right].$$

Therefore the result (i) is true for the number of segments in $P_n = 2$.

Assume that the result (i) is true for the number of segments in $P_n = k-1$, i.e, $C_2 = \sum_{i=1}^{k-1}\frac{n_i(n_i+1)^2(n_i+2)}{12}$. Then prove that the result (i) is true for the number of segments in $P_n = k$, i.e, to prove $C_2 = \sum_{i=1}^{k}\frac{n_i(n_i+1)^2(n_i+2)}{12}$.

Let $P^{\Psi_{n_1,n_2,\cdots,n_k}}_{n_1+1+n_2+\cdots+n_k}$ be the degenerate oriented path on $n_1+1+n_2+\cdots+n_k = n$ vertices. Let $n_i (i = 1, 2, \cdots, k)$ be the length of the i^{th} segment of the above path. Then, by Theorem 2.5

$$C_2 = C_2\ in\ P[SD(P^{\Psi_{n_1,n_2,\cdots,n_{k-1}}}_{n_1+1+n_2+\cdots+n_{k-1}});\ x]$$
$$+\frac{n_k^2(n_k^2-1)}{12}\ C_0\ in\ P[SD(P^{\Psi_{n_1,n_2,\cdots,n_{k-1}}}_{n_1+1+n_2+\cdots+n_{k-1}});\ x]$$
$$+\frac{n_k(n_k+1)(2n_k+1)}{6}\ C_0\ in\ P[SD(P^{\Psi_{n_1,n_2,\cdots,n_{k-1}-1}}_{n_1+1+n_2+\cdots+n_{k-1}-1});\ x]$$
$$C_2 = \sum_{i=1}^{k}\frac{n_i(n_i+1)^2(n_i+2)}{12} = \sum_{i=1}^{k}\left[\frac{n_i^2(n_i^2-1)}{12}+\frac{n_i(n_i+1)(2n_i+1)}{6}\right].$$

Thus, the result (i) is true for the number of segments in $P_n = k$, which completes the induction and hence the proof (i).

(iii) We use induction on *the number of segments in* P_n. When the number of segments in $P_n = 1$, all the edges are in one direction. Then by Theorem 2.2 [1], $P[SD(P_n^{\Psi_{n_1}});x] = P[SD(P_{n_1+1}^{\Psi_{n_1}});x] = x^{n_1+1} + \frac{n_1(n_1+1)^2(n_1+2)}{12}\, x^{n_1-1}$. Therefore,

$$C_4 = \text{the coefficient of } x^{n-4} \text{ in } P[SD(P_n^R);x] = 0$$

$$i.e,\ C_{2(1+1)} = 0 \text{ and hence by Theorem 2.5, } C_6 = C_8 = \cdots = 0.$$

Therefore $C_{2i} = 0$ for all $i \geq k+1$.

(2.1) Therefore the result (iii) is true for the number of segments in $P_n = 1$.

When the number of segments in $P_n = 2$, let $P_{n_1+1+n_2}^{\Psi_{n_1,n_2}}$ be the degenerate oriented path on n vertices with two segments.

Then by Theorem 2.5,

$$C_{2(2+1)} = C_6 = \text{the coefficient of } x^{n-6} \text{ in } P[SD(P_{n_1+1+n_2}^{\Psi_{n_1,n_2}});x] = 0 \text{ (as in (2.1))}.$$

Therefore the result (iii) is true for the number of segments in $P_n = 2$. Assume that the result (iii) is true for the number of segments in $P_n = r$, i.e, $C_{2(r+1)} = 0$ and $C_{2i} = 0$ for all $i \geq r+1$. Then prove that the result (iii) is true for the number of segments in $P_n = r+1$, i.e, to prove $C_{2(r+2)} = 0$ and $C_{2i} = 0$ for all $i \geq r+2$, $C_{2((r+1)+1)} = C_{2r+4}$ = the coefficient of $x^{n-(2r+4)}$ in $P[SD(P_n^{\Psi_{n_1,n_2,\cdots,n_r,n_{r+1}}});x]$. By Theorem 2.5, $C_{2(r+2)}$ in $P[SD(P_{(n_1+1)+n_2+\cdots+n_r+n_{r+1}}^{\Psi_{n_1,n_2,\cdots,n_r,n_{r+1}}});x] = 0$, and hence $C_{2i} = 0$ for all $i \geq r+2$, (since by induction hypothesis).

This completes the induction and hence the proof.

Proof of (ii)

$$\begin{aligned} C_4 &= \text{the coefficient of } x^{n-4} \text{ in } P[SD(P_n^{\Psi_{n_1,n_2}});x] \\ &= C_4 \text{ in } P[SD(P_{n_1+1}^{\Psi_{n_1}});x] + \frac{n_2^2(n_2^2-1)}{12}\, C_2 \text{ in } P[SD(P_{n_1+1}^{\Psi_{n_1}});x] \\ &\quad + \frac{n_2(n_2+1)(2n_2+1)}{6}\, C_2 \text{ in } P[SD(P_{n_1}^{\Psi_{n_1-1}});x]. \end{aligned}$$

Therefore,

$$\begin{aligned} C_4 = {} & \frac{n_1^2(n_1^2-1)}{12}\,\frac{n_2^2(n_2^2-1)}{12} + \frac{n_1^2(n_1^2-1)}{12}\,\frac{n_2(n_2+1)(2n_2+1)}{6} \\ & + \frac{n_2^2(n_2^2-1)}{12}\,\frac{n_1(n_1+1)(2n_1+1)}{6}. \end{aligned}$$

Hence the claim.

To prove the result (ii) of Theorem 2.6, we use induction on *the number of segments in* P_n. When the number of segments in $P_n = 3$, let $P_n^{\Psi_{n_1,n_2,n_3}} = P_{n_1+1+n_2+n_3}^{\Psi_{n_1,n_2,n_3}}$ be,

$$\begin{aligned}
&C_4 \text{ in } P[SD(P_{n_1+1+n_2+n_3}^{\Psi_{n_1,n_2,n_3}});x] \\
=&C_4 \text{ in } P[SD(P_{n_1+1+n_2}^{\Psi_{n_1,n_2}});x] + \frac{n_3^2(n_3^2-1)}{12}\, C_2 \text{ in } P[SD(P_{n_1+1+n_2}^{\Psi_{n_1,n_2}});x] \\
&+ \frac{n_3(n_3+1)(2n_3+1)}{6}\, C_2 \text{ in } P[SD(P_{n_1+1+n_2-1}^{\Psi_{n_1,n_2-1}});x] \\
=&\frac{n_1^2(n_1^2-1)}{12}\,\frac{n_2^2(n_2^2-1)}{12} + \frac{n_2^2(n_2^2-1)}{12}\,\frac{n_3^2(n_3^2-1)}{12} + \frac{n_3^2(n_3^2-1)}{12}\,\frac{n_1^2(n_1^2-1)}{12} \\
&+ \left(\frac{n_1^2(n_1^2-1)}{12} + \frac{n_2^2(n_2^2-1)}{12}\right)\frac{n_3(n_3+1)(2n_3+1)}{6} \\
&+ \left(\frac{n_2^2(n_2^2-1)}{12} + \frac{n_3^2(n_3^2-1)}{12}\right)\frac{n_1(n_1+1)(2n_1+1)}{6} \\
&+ \left(\frac{n_3^2(n_3^2-1)}{12} + \frac{n_1^2(n_1^2-1)}{12}\right)\frac{n_2(n_2+1)(2n_2+1)}{6} \\
&+ \frac{n_1(n_1+1)(2n_1+1)}{6}\,\frac{n_3(n_3+1)(2n_3+1)}{6}
\end{aligned}$$

Therefore,

$$\begin{aligned}
C_4 =& \sum_{1\le i<j\le 3} \frac{n_i^2(n_i^2-1)}{12}\,\frac{n_j^2(n_j^2-1)}{12} + \sum_{\substack{i,j=1\\ i\ne j}}^{3} \frac{n_i^2(n_i^2-1)}{12}\,\frac{n_j(n_j+1)(2n_j+1)}{6} \\
&+ \frac{n_1(n_1+1)(2n_1+1)}{6}\,\frac{n_3(n_3+1)(2n_3+1)}{6}.
\end{aligned}$$

Therefore, the result (ii) is true for *the number of segments in* P_n *= 3*. Assume that the result (ii) is true for *the number of segments in* P_n *= r*, i.e,

$$\begin{aligned}
C_4 =& \sum_{1\le i<j\le r} \frac{n_i^2(n_i^2-1)}{12}\,\frac{n_j^2(n_j^2-1)}{12} + \sum_{\substack{i,j=1\\ i\ne j}}^{r} \frac{n_i^2(n_i^2-1)}{12}\,\frac{n_j(n_j+1)(2n_j+1)}{6} \\
&+ \sum_{\substack{1\le i<j\le r\\ 2\le j-i\le r-1}} \frac{n_i(n_i+1)(2n_i+1)}{6}\,\frac{n_j(n_j+1)(2n_j+1)}{6}.
\end{aligned}$$

Then prove that the result (ii) is true for *the number of segments in* P_n = *r+1*. Now,

$$\begin{aligned}
C_4 = &\text{ the coefficient of } x^{n-4} \text{ in } P[SD(P^{\Psi_{n_1,n_2,\cdots,n_r,n_{r+1}}}_{n_1+1+n_2+\cdots+n_r+n_{r+1}});x] \\
&= C_4 \text{ in } P[SD(P^{\Psi_{n_1,n_2,\cdots,n_r}}_{n_1+1+n_2+\cdots+n_r});x] \\
&\quad + \frac{n_{r+1}^2(n_{r+1}^2-1)}{12} C_2 \text{ in } P[SD(P^{\Psi_{n_1,n_2,\cdots,n_r}}_{n_1+1+n_2+\cdots+n_r});x] \\
&\quad + \frac{n_{r+1}(n_{r+1}+1)(2n_{r+1}+1)}{6}\, C_2 \text{ in } P[SD(P^{\Psi_{n_1,n_2,\cdots,n_{r-1}}}_{n_1+1+n_2+\cdots+n_r-1});x] \\
&= \sum_{1\le i<j\le r+1} \frac{n_i^2(n_i^2-1)}{12}\,\frac{n_j^2(n_j^2-1)}{12} + \sum_{\substack{i,j=1\\ i\ne j}}^{r+1} \frac{n_i^2(n_i^2-1)}{12}\,\frac{n_j(n_j+1)(2n_j+1)}{6} \\
&\quad + \sum_{\substack{1\,\le\, i\,<\,j\,\le\, r+1\\ 2\,\le\, j-i\,\le\, r}} \frac{n_i(n_i+1)(2n_i+1)}{6}\,\frac{n_j(n_j+1)(2n_j+1)}{6}.
\end{aligned}$$

Therefore the result (ii) is true when *the number of segments in* $P_n = r+1$, which completes the induction and hence the proof. □

Theorem 2.7. *Let* $P[SD(P^{\chi_{n_1,n_2,\ldots,n_t}}_{(n_1+1)+n_2+\ldots+n_t});x] = C_0x^n + C_1x^{n-1} + \ldots + C_rx^{n-r} + \ldots + C_n$*, where* $n = (n_1+1)+n_2+\ldots+n_t$*. Then*

$$\begin{aligned}
C_6 = C_{2.3} = &\sum_{1\le i<j<k\le t} \frac{n_i^2(n_i^2-1)}{12}\frac{n_j^2(n_j^2-1)}{12}\frac{n_k^2(n_k^2-1)}{12} \\
&+ \sum_{\substack{i\ne j\ne k=1\\ 1\le i<j<k\le t}}^{t} \frac{n_i^2(n_i^2-1)}{12}\frac{n_j^2(n_j^2-1)}{12}\frac{n_k(n_k+1)(2n_k+1)}{6} \\
&+ \sum_{\substack{1\le i<j<k\le t\\ i,j,k \text{ consecutive integers}}} \frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_j(n_j-1)^2(n_j-2)}{12}\frac{n_k(n_k+1)(2n_k+1)}{6} \\
&+ \sum_{\substack{i\ne k\ne j=1\\ 2\le(k-i)\le(t-1)}} \frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_j^2(n_j^2-1)}{12}\frac{n_k(n_k+1)(2n_k+1)}{6} \\
&+ \sum_{\substack{1\le i<j\le k\le t\\ 2\le(j-i),\ k-j\le(t-3)}} \frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_j(n_j+1)(2n_j+1)}{6}\frac{n_k(n_k+1)(2n_k+1)}{6},
\end{aligned}$$

for all $t \ge 5$.

Proof. To prove the theorem, we use induction on the number of segments in P_n. Taking the number of segments in $P_n = 5$, and using Theorem 2.5 and Theorem 2.6 we arrive at the required result. □

Theorem 2.8. *Let* $P[SD(P^{\psi_{n_1,n_2,\cdots,n_t}}_{(n_1+1)+n_2+\cdots+n_t});x] = C_0x^n + C_1x^{n-1} + \cdots + C_{2r}x^{n-2r} + \cdots + C_n$*, where* $(n_1+1)+n_2+\cdots+n_t = n$ *and* $1 \le t \le n-2$*. Then,*

C_{2r} *in* $P[SD(P^{\psi_{n_1,n_2,\cdots,n_t}}_{(n_1+1)+n_2+\cdots+n_t});x]$

$$= \sum_{1\le i<j<k\le t} \frac{n_i^2(n_i^2-1)}{12} \cdots \frac{n_j^2(n_j^2-1)}{12} \cdots \frac{n_k^2(n_k^2-1)}{12}$$

$$+ \sum_{i\ne j\ne\cdots\ne k=1}^{t} \frac{n_i(n_i+1)(2n_i+1)}{6} \frac{n_j^2(n_j^2-1)}{12} \cdots \frac{n_k^2(n_k^2-1)}{12}$$

$$+ \Bigg\{ \sum_{\substack{1\le i\le t-2\\ 1\le j,\cdots,k\le t\\ i\ne i+1\ne i+2\ne j\ne\cdots\ne k}} \frac{n_i(n_i+1)(2n_i+1)}{6} \frac{n_{i+1}(n_{i+1}-1)^2(n_{i+1}-2)}{12}$$

$$\frac{n_{i+2}(n_{i+2}+1)(2n_{i+2}+1)}{6} \frac{n_j^2(n_j^2-1)}{12} \cdots \frac{n_k^2(n_k^2-1)}{12}$$

$$+ \sum_{\substack{1\le i\le t-2\\ 1\le j,\cdots,k\le t\\ j,\cdots,k\ne i+1}} \frac{n_i(n_i+1)(2n_i+1)}{6} \frac{n_{i+2}(n_{i+2}+1)(2n_{i+2}+1)}{6} \frac{n_j^2(n_j^2-1)}{12} \cdots \frac{n_k^2(n_k^2-1)}{12} \Bigg\}$$

$$+ \Bigg\{ \sum_{\substack{1\le i\le t-3\\ 1\le j,\cdots,k\le t\\ j,\cdots,k\ne i,i+1,i+2,i+3}} \frac{n_i(n_i+1)(2n_i+1)}{6} \frac{n_{i+1}^2(n_{i+1}^2-1)}{12} \frac{n_{i+2}(n_{i+2}-1)^2(n_{i+2}-2)}{12}$$

$$\frac{n_{i+3}(n_{i+3}+1)(2n_{i+3}+1)}{6} \frac{n_j^2(n_j^2-1)}{12} \cdots \frac{n_k^2(n_k^2-1)}{12}$$

$$+ \sum_{\substack{1\le i\le t-3\\ 1\le j,\cdots,k\le t\\ i\ne i+1\ne\cdots\ne j\ne k}} \frac{n_i(n_i+1)(2n_i+1)}{6} \frac{n_{i+1}(n_{i+1}-1)^2(n_{i+1}-2)}{12} \frac{n_{i+2}(n_{i+2}-1)(2n_{i+2}-1)}{6}$$

$$\frac{n_{i+3}(n_{i+3}+1)(2n_{i+3}+1)}{6} \frac{n_j^2(n_j^2-1)}{12} \cdots \frac{n_k^2(n_k^2-1)}{12}$$

$$+ \sum_{\substack{1\le i\le t-3\\ 1\le j,\cdots,k\le t\\ j,k=i+1\ (or)\ i+2}} \frac{n_i(n_i+1)(2n_i+1)}{6} \frac{n_{i+3}(n_{i+3}+1)(2n_{i+3}+1)}{6} \frac{n_j^2(n_j^2-1)}{12} \cdots \frac{n_k^2(n_k^2-1)}{12} \Bigg\}$$

$$+ \Bigg\{ \sum_{\substack{1\le i\le t-4\\ 1\le j,\cdots,k\le t}} \frac{n_i(n_i+1)(2n_i+1)}{6} \frac{n_{i+1}^2(n_{i+1}^2-1)}{12} \frac{n_{i+2}^2(n_{i+2}^2-1)}{12} \frac{n_{i+3}(n_{i+3}-1)^2(n_{i+3}-2)}{12}$$

$$\frac{n_{i+4}(n_{i+4}+1)(2n_{i+4}+1)}{6} \frac{n_j^2(n_j^2-1)}{12} \cdots \frac{n_k^2(n_k^2-1)}{12}$$

$$+ \sum_{\substack{1\le i\le t-4\\ 1\le j,\cdots,k\le t}} \frac{n_i(n_i+1)(2n_i+1)}{6} \frac{n_{i+1}^2(n_{i+1}^2-1)}{12} \frac{n_{i+2}(n_{i+2}-1)^2(n_{i+2}-2)}{12}$$

$$\frac{n_{i+3}(n_{i+3}-1)(2n_{i+3}-1)}{6}\frac{n_{i+4}(n_{i+4}+1)(2n_{i+4}+1)}{6}\frac{n_j^2(n_j^2-1)}{12}\cdots\frac{n_k^2(n_k^2-1)}{12}$$

$$+\sum_{\substack{1\le i\le t-4\\1\le j,\cdots,k\le t}}\frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_{i+1}(n_{i+1}-1)^2(n_{i+1}-2)}{12}\frac{n_{i+2}(n_{i+2}-1)(2n_{i+2}-1)}{6}$$

$$\frac{n_{i+3}(n_{i+3}-1)(2n_{i+3}-1)}{6}\frac{n_{i+4}(n_{i+4}+1)(2n_{i+4}+1)}{6}\frac{n_j^2(n_j^2-1)}{12}\cdots\frac{n_k^2(n_k^2-1)}{12}$$

$$+\sum_{\substack{1\le i\le t-4\\1\le j,\cdots,k\le t\\ j,\cdots,k \text{ takes any}\\ \text{two values of}\\ i+1,i+2,i+3}}\frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_{i+4}(n_{i+4}+1)(2n_{i+4}+1)}{6}\frac{n_j^2(n_j^2-1)}{12}\cdots\frac{n_k^2(n_k^2-1)}{12}\Bigg\}+\cdots$$

$$+\Bigg\{\sum_{1\le i\le t-(r-1)}\frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_{i+1}^2(n_{i+1}^2-1)}{12}\cdots\frac{n_{i+r-3}^2(n_{i+r-3}^2-1)}{12}$$

$$\frac{n_{i+r-2}(n_{i+r-2}-1)^2(n_{i+r-2}-2)}{12}\frac{n_{i+r-1}(n_{i+r-1}+1)(2n_{i+r-1}+1)}{6}$$

$$+\sum_{1\le i\le t-(r-1)}\frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_{i+1}^2(n_{i+1}^2-1)}{12}\cdots\frac{n_{i+r-3}(n_{i+r-3}-1)^2(n_{i+r-3}-2)}{12}$$

$$\frac{n_{i+r-2}(n_{i+r-2}-1)(2n_{i+r-2}-1)}{6}\frac{n_{i+r-1}(n_{i+r-1}+1)(2n_{i+r-1}+1)}{6}+\cdots$$

$$+\sum_{1\le i\le t-(r-1)}\frac{n_i(n_i+1)(2n_i+1)}{6}\frac{n_{i+1}(n_{i+1}-1)^2(n_{i+1}-2)}{12}\frac{n_{i+2}(n_{i+2}-1)(2n_{i+2}-1)}{6}\cdots$$

$$\frac{n_{i+r-2}(n_{i+r-2}-1)(2n_{i+r-2}-1)}{6}\frac{n_{i+r-1}(n_{i+r-1}+1)(2n_{i+r-1}+1)}{6}$$

$$+\sum_{1\le i\le t-(r-1)}\frac{n_i(n_i+1)(2n_i+1)}{6}\cdots\frac{n_{i+r-1}(n_{i+r-1}+1)(2n_{i+r-1}+1)}{6}$$

$$\frac{n_j^2(n_j^2-1)}{12}\cdots\frac{n_k^2(n_k^2-1)}{12}$$

where j, k *takes any r - 3 values from* $i+1,\cdots,i+r-2\Bigg\}$

$+\Bigg\{$*Similar terms involving three* $\frac{n_i(n_i+1)(2n_i+1)}{6}$*'s* $(1\le i\le t)$,

namely $\frac{n_i(n_i+1)(2n_i+1)}{6}, \frac{n_j(n_j+1)(2n_j+1)}{6}, \frac{n_k(n_k+1)(2n_k+1)}{6}$,

such that $1\le i<j<k\le t$ *and* $2\le j-i, k-j\le t-3$,

and the remaining (r - 3) elements from $\frac{n_i^2(n_i^2-1)}{12}(1\le i\le t)$, $\frac{n_j(n_j-1)^2(n_j-2)}{12}$

$(2\le j\le t-2)$ *and* $\frac{n_k(n_k-1)(2n_k-1)}{6}(3\le k\le t-1)\Bigg\}+\cdots$

$$+\left\{\textit{Terms involving (r - 1) } \frac{n_i(n_i+1)(2n_i+1)}{6}\text{'s}(1 \leq i \leq t) \textit{and one element from}\right.$$

$$\left.\frac{n_j(n_j-1)^2(n_j-2)}{12}(2 \leq j \leq t-1) \textit{ or } \frac{n_k^2(n_k^2-1)}{12}(1 \leq k \leq t)\right\}$$

$$+\sum_{\substack{1\leq i<j<k\leq t\\ 2\leq j-i,\cdots,k-j\leq t-3}} \frac{n_i(n_i+1)(2n_i+1)}{6}\cdots\frac{n_j(n_j+1)(2n_j+1)}{6}\cdots\frac{n_k(n_k+1)(2n_k+1)}{6}$$

each summation contains product of r *terms.*

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