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# A CUBIC SELF-CENTERED DISTANCE DEGREE INJECTIVE (DDI) GRAPH

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ABSTRACT. A graph is *distance degree injective (DDI) graph* if no two vertices have the same distance degree sequence. In this short note we show the existence of a regular(cubic) self-centered distance degree injective (DDI) graph, which was an open problem.

## 1. INTRODUCTION

Distance degree sequence of a vertex in a graph gives a distribution of other vertices depending on their distance to this vertex. When these sequences are listed for all vertices of a graph, we get an overall picture of distance distribution along with the number of vertices. There were two major class of graphs defined which depended on these sequences, namely, the distance degree regular (DDR) and distance degree injective (DDI) graphs by Bloom et al. [2]. DDR graphs are highly symmetric in nature by having equal distance degree sequences for all vertices and DDI graphs are the ones in which no two distance degree sequences are equal. Hence this sets up a dual kind of study pattern. Many researchers have worked in both these classes of graphs viz., [1–3, 6–10] to cite a few. But characterizations for both classes of graphs are still elusive. Many open problems still persist.

In this short communication we settle one such open problem on existence of a DDI graph.

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## 2. Preliminaries

For all undefined terms we refer Buckley and Harary [4]. Let $G = (V, E)$ denote a graph with set of *vertices* $V$, whose cardinality is the order $p$ and two element subsets of $V$, known as the *edges* forming $E$, whose cardinality is size $q$.

Unless mentioned otherwise, in this article, by a *graph* we mean an undirected, finite graph without multiple edges and self-loops.

The *distance* $d(u, v)$ from a vertex $u$ of $G$ to a vertex $v$ is the length of a shortest $u$ to $v$ path. The *degree* of the vertex $u$ is the number of vertices at distance one. A graph is said to be regular if all the vertices have the same degree. If the regularity is three, then the graph is called a cubic graph. The *eccentricity* $e(v)$ of $v$ is the distance of a farthest vertex from $v$. The minimum of the eccentricities is the *radius*, $rad(G)$ and the maximum is the *diameter*, $diam(G)$ of $G$. A graph is said to be *self centered* if all the vertices have the same eccentricity. The *distance degree sequence (dds)* of a vertex $v$ in a graph $G$ is a list of the number of vertices at distance $1, 2, \ldots, e(v)$ in that order, where $e(v)$ denotes the eccentricity of $v$ in $G$. Thus the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$ is the distance degree sequence of a vertex $v_i$ in $G$ where $d_{i_j}$ denotes the number of vertices at distance $j$ from $v_i$. The $p-$tuple of distance degree sequences of the vertices of $G$ with entries arranged in lexicographic order is the *distance degree sequence (DDS) of G.* A graph is called a *Distance Degree Injective (DDI) graph* if no two vertices have the same distance degree sequence (dds). These were defined by Bloom et al. [2].
Since then there have been open problems being posed on DDI graphs on characterizations and even existence all these years. To cite a few we list the following. First one was on $k$-regular DDI graphs as posed by Bloom et al. [2].

**Problem 1.** *[2] Does there exist a non-trivial $k$-regular DDI graph?*

This problem got resolved by the existence of a cubic DDI graph on $24$ vertices and having diameter $10$ as found in [8]. The general existence problem was resolved by Bollobas in [3] where he showed the following.

**Theorem 2.1.** *[3]: Let $k \geq 3$ and $\epsilon > 0$ be fixed. Set $k = \left\lfloor \frac{(1/2+\epsilon)(\log p)}{\log(k-1)} \right\rfloor$. Then, as $p$ goes to infinity, the probability tends to one that every vertex $v_i$ in a $k$-regular labeled graph of order $p$ is uniquely determined by the initial segment $d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}$ of its distance degree sequence.*

Since the distance degree sequence of a graph is independent of a labeling, this result shows that almost all $k$-regular graphs of order $p$ are DDI provided $p$ is large enough. The problem that remained unresolved is that of finding minimal DDI regular graphs.

**Problem 2.** *[8] For $k \geq 3$ what is the smallest order and/or diameter for which there exists a $k$-regular DDI graph?*

Martinez and Quintas [7] found a cubic DDI graph having diameter $8$ and order $22$, as in Figure 1. They also showed that, if in the graph of Figure 1 the edges *ab* and *cd* are replaced by the graph shown in Figure 2, one can obtain a cubic DDI graph with $22 + 2m$ vertices and diameter $8 + m$.

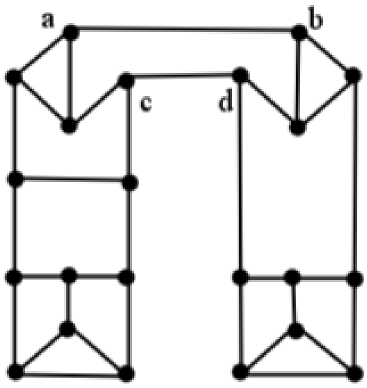


FIGURE 1. Cubic DDI graph

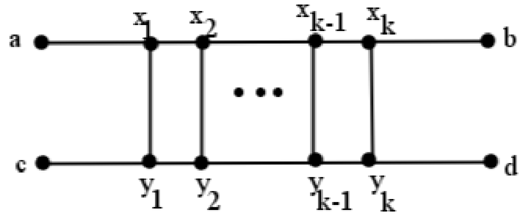


FIGURE 2. Bigger cubic DDI graph

This was further reduced to order $18$ and diameter $7$ cubic graph by Jiri Volf [11] by constructing the graph of Figure 3.

From [8], [7], [11], [5] it follows that

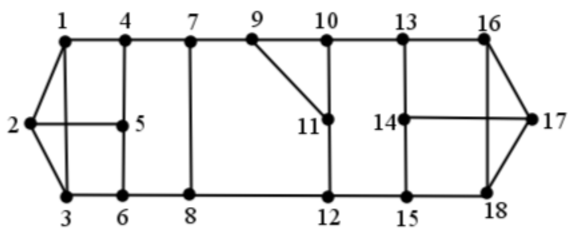


FIGURE 3. Smallest cubic DDI graph

(i) if there is a cubic *DDI* graph having less than $18$ vertices, then its order must be $16$; and

(ii) if there is a cubic *DDI* graph having diameter less than $7$, then its diameter must be $4, 5$,or $6$.

So all these cases are considered and proved by Itagi Huilgol et al. [10] that there does not exist a cubic DDI graph of order 16 with diameter 4, 5, 6.

**Theorem 2.2.** *[10]: There does not exist a cubic DDI graph of order 16 with diameter 4,5,6.*

So the graph of order $18$ as in [11] shown in Figure 3 is the smallest order cubic DDI graph. One more problem was posed by Itagi Huilgol [9] as follows:

**Problem 3.** *Does there exist a self-centered, $k$-regular DDI graph?*

Addressing the above problem we have constructed a $3-$regular self-centered DDI graph hence by proving it true.

## 3. Main Results

**Theorem 3.1.** *There exists a cubic self-centered distance degree injective (DDI) graph.*

*Proof.* Follows from the example below. Let $H$ be the graph with vertex set

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38\}$$

and edge set

$$\{\{1, 9\}, \{1, 24\}, \{1, 33\}, \{2, 7\}, \{2, 25\}, \{2, 34\}, \{3, 5\}, \{3, 11\}, \{3, 23\}, \{4, 6\}, \{4, 14\}, \{4, 18\}, \{5, 28\}, \{5, 37\}, \{6, 9\}, \{6, 23\}, \{7, 10\}, \{7, 37\}, \{8, 12\}, \{8, 30\}, \{8, 38\}, \{9, 17\}, \{10, 17\}, \{10, 22\}, \{11, 15\}, \{11, 16\}, \{12, 21\}, \{12, 37\}, \{13, 18\}, \{13, 20\}, \{13, 35\}, \{14, 20\}, \{14, 35\}, \{15, 29\}, \{15, 34\}, \{16, 32\}, \{16, 36\}, \{17, 22\}, \{18, 31\}, \{19, 24\}, \{19, 36\}, \{19, 38\}, \{20, 26\}, \{21, 28\}, \{21, 32\}, \{22, 33\}, \{23, 31\}, \{24, 29\}, \{25, 27\}, \{25, 30\}, \{26, 30\}, \{26, 38\}, \{27, 28\}, \{27, 32\}, \{29, 36\}, \{31, 35\}, \{33, 34\}\}.$$

The distance degree sequences of all the vertices is given by

$dds(1) = (1, 3, 6, 7, 10, 9, 2)$; $dds(2) = (1, 3, 6, 11, 8, 5, 4)$;

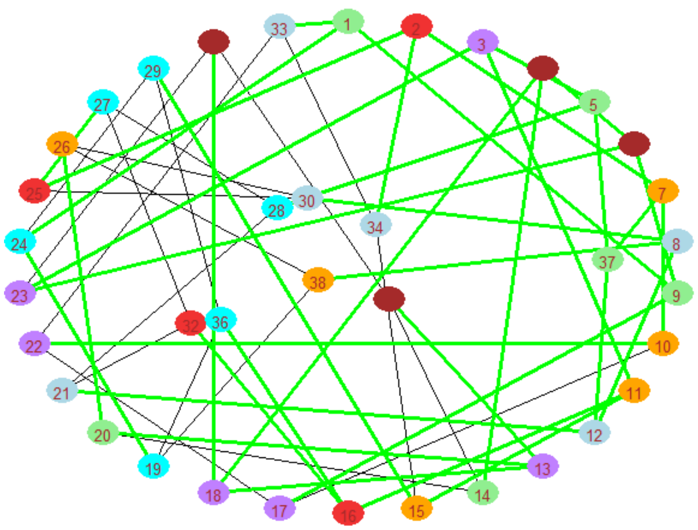


FIGURE 4. H: Cubic self-centered DDI graph

$dds(3) = (1, 3, 6, 12, 11, 4, 1)$; $dds(4) = (1, 3, 6, 4, 8, 10, 6)$;
$dds(5) = (1, 3, 6, 9, 11, 7, 1)$; $dds(6) = (1, 3, 6, 9, 9, 8, 2)$;
$dds(7) = (1, 3, 6, 9, 8, 7, 4)$; $dds(8) = (1, 3, 5, 9, 8, 10, 2)$;
$dds(9) = (1, 3, 6, 8, 10, 6, 4)$; $dds(10) = (1, 3, 4, 6, 10, 9, 5)$;
$dds(11) = (1, 3, 6, 10, 10, 7, 1)$; $dds(12) = (1, 3, 6, 8, 8, 9, 3)$;
$dds(13) = (1, 3, 4, 4, 5, 9, 12)$; $dds(14) = (1, 3, 5, 4, 6, 10, 9)$;
$dds(15) = (1, 3, 6, 8, 11, 6, 3)$; $dds(16) = (1, 3, 6, 8, 9, 7, 4)$;
$dds(17) = (1, 3, 4, 6, 10, 11, 3)$; $dds(18) = (1, 3, 5, 3, 6, 11, 9)$;
$dds(19) = (1, 3, 5, 8, 11, 9, 1)$; $dds(20) = (1, 3, 5, 5, 7, 11, 6)$;
$dds(21) = (1, 3, 5, 7, 7, 7, 8)$; $dds(22) = (1, 3, 4, 5, 8, 12, 5)$;
$dds(23) = (1, 3, 6, 8, 13, 5, 2)$; $dds(24) = (1, 3, 5, 8, 9, 10, 2)$;
$dds(25) = (1, 3, 6, 10, 10, 6, 2)$; $dds(26) = (1, 3, 5, 8, 11, 7, 3)$;
$dds(27) = (1, 3, 5, 9, 8, 8, 4)$; $dds(28) = (1, 3, 5, 7, 8, 10, 4)$;
$dds(29) = (1, 3, 5, 6, 11, 8, 4)$; $dds(30) = (1, 3, 5, 9, 10, 8, 2)$;
$dds(31) = (1, 3, 5, 4, 7, 14, 4)$; $dds(32) = (1, 3, 5, 9, 6, 6, 8)$;
$dds(33) = (1, 3, 6, 6, 9, 9, 4)$; $dds(34) = (1, 3, 6, 10, 9, 5, 4)$;
$dds(35) = (1, 3, 4, 3, 5, 9, 13)$; $dds(36) = (1, 3, 5, 7, 10, 8, 4)$;
$dds(37) = (1, 3, 6, 10, 8, 8, 2)$; $dds(38) = (1, 3, 5, 8, 13, 7, 1)$.

From the above sequences it is clear that $H$ is a $3-$regular self-centered DDI graph on $38$ vertices and $57$ edges. Hence Proved. □

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