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VULNERABILITY PARAMETERS IN HONEYCOMB NETWORKS

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ABSTRACT. The study of the vulnerability of a network is the most important task for the network engineers. There are many stability measures are available to construct or reconstruct any communication network. The honeycomb meshes are having better topological properties than meshes. In this paper, we find the vulnerability parameters of honeycomb mesh.

1. INTRODUCTION

Tessellations is a repeating pattern of polygons that covers a plane with no gaps or overlaps. Square mesh (Figure 1 a)), triangular mesh (Figure 1 b)), and hexagonal mesh (Figure 1 c)) are three fundamental tessellations of a plane with regular polygons of the same kind. They are all isogonal and monohedra. Out of these, the square tessellation is the basis for mesh-connected computers.

The mesh-connected computers are made by the square tessellation. The hexagon mesh is developed by the triangular tessellation. The honeycomb networks are defined by hexagonal tessellation. Honeycomb networks are the most

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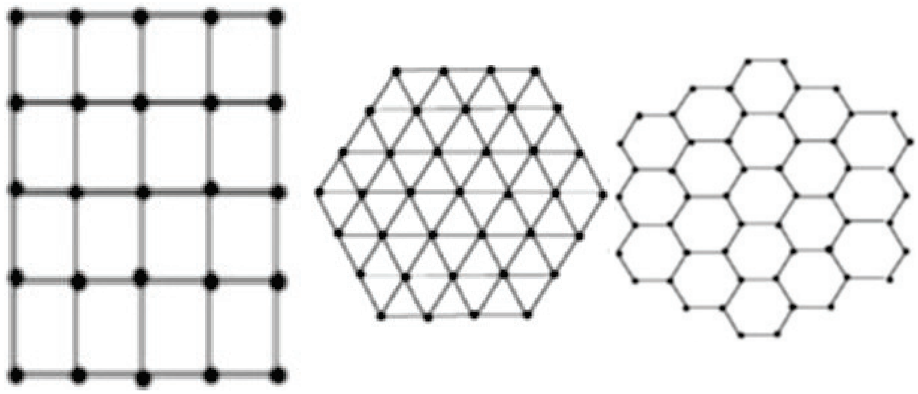

FIGURE 1. a)Square mesh b) Triangular mesh c) Hexagonal mesh

reliable networks in terms of network parameters like degree, number of links, cost, etc., when we are comparing with other mesh connected planar graphs. Stojmenovic has given a detail study of the topological properties of honeycomb networks in [5, 6]. The honeycomb network $HM(n)$ is built from the hexagon tessellations through the recursive process, see Figure 2. The number of vertices and edges of $HM(n)$ are $6t^2$ and $9t^2-3t$ respectively. The diameter is $4t-1$ [5,6].

The stability of any communication network is nothing but the resistance of the network to the disruption if the network begins to lose its links or processors. Graphs are the most reliable mathematical model of any communication network, which are composed of processors (nodes) and communication links (edges). The loss in network effectiveness depends links cuts, transmission failures at various points can interrupt service in a period of time and hardware failures. Network engineers design the network that they do not easily disrupted with external attack. To measure the performance of a network, designers use the following major performance metrics after the external attack in the network, which are the number of nodes that are not functioning, the total number of connected subnetworks and the size of a largest remaining subnetwork within which mutual communication can still occur. Based on these metrics, there are many vulnerability parameters are available in the literature. There are toughness, integrity, scattering number, tenacity and rupture degree.

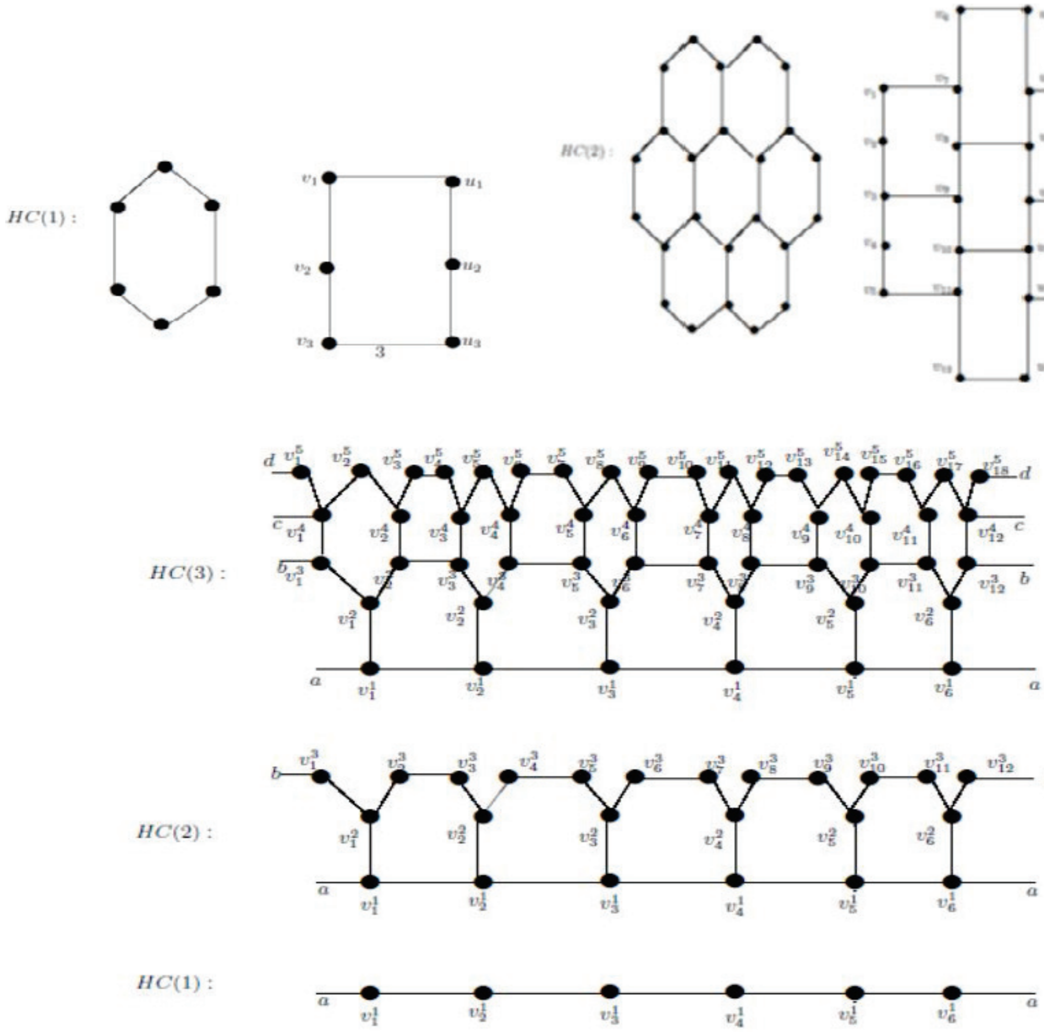

FIGURE 2. Recursive construction of HM(n)

2. VULNERABILITY PARAMETERS

Let $H = (V, E)$ be a simple graph. We denote $\kappa(H), \alpha(H), \beta(H)$ the connectivity of H, vertex covering and independence number of H respectively. The following are the definitions of some vulnerability parameters in the literature. Graphs are very useful mathematical model to represent anycommunication network with processors and communication links.The vulnerability of communication networks can be measured by their links cuts, node interruptions, software errors or hardware failures, and transmission failures at various points can interrupt service for long periods of time. In order to measure the performance, the following quantites are used by the network engineers.

(1) the number of elements that are not functioning.
(2) the number of remaining connected subnetworks.
(3) the size of a largest remaining group within which mutual communication can still occour.

Based on the above quantities, the following measures were introduced in the literature.

The connectivity is a parameter defined based on quantity (1).

Vertex connectivity is $\tau(H) = \min\{|T| : T \subset V(H)$ is a cut set of $H\}$.

Both toughness and scattering number take into account quantities (1) and (2). The toughness (Chvtal, 1973, [3]) is defined as

$$t(H) = \min\{|T|/\omega(H-T) : T \subset V(H) \text{ is a vertex cut of } H\}.$$

The scattering number (see [11, 12]) is

$$sc(H) = \max\{\omega(H-T) - |T| : T \subset V(H) \text{ is a vertex cut of } H\}.$$

The integrity is defined based on quantities (1) and (3). The integrity (Barefoot, et al., 1987, [2]):

$$I(H) = \min\{|T| + m(H-T) : T \subset V(H) \text{ is a vertex cut of } H\}.$$

Further results in integrity, the readers refer [1, 4].

Both the tenacity and rupture degree take into account all the three quantities The tenacity (Cozzens, et al., 1995, [9]),

$$T(H) = \min\left\{\frac{|T| + m(H-T)}{\omega(H-T)} : T \subset V(H) \text{ is a vertex cut of } H\right\}.$$

The rupture degree (Li. Zhang and Li [10]) is

$$r(H) = \max\{\omega(H-T) - |T| - m(H-T) : T \subset V(H) \text{ is a vertex cut of } H\}.$$

Edge analogues of these parameters are defined similarly.

We refer the following theorems to obtain the bounds of the vulnerabiltiy parameters in honeycomb meshs.

Theorem 2.1. *[3] Let H be a graph with n vertices. Then $t(H) \leq \frac{n-\beta(H)}{\beta(H)}$.*

Theorem 2.2. *[8] Let H be a graph of order n. Then $2\beta(H) - n \leq sc(H) \leq \beta(H) - \kappa(H)$.*

Theorem 2.3. *[7] Let H be a graph of order n. Then $t(H) \geq \frac{\kappa(H)}{\kappa(H)+sc(H)}$.*

Theorem 2.4. *[6] If X is a spanning subgraph of a graph H, $r(H) \leq r(X)$.*

Theorem 2.5. *[6] If $1 \leq m \leq n$, then $r(K_(m,n)) = n - m - 1$.*

Theorem 2.6. *Let $G = HM(t), t \geq 3$. Then*

$$I(G) = \begin{cases} \frac{3t^2}{2} + 6 & ,t \text{ is even} \\ \frac{3(t^2-1)}{2} + 6 & ,t \text{ is odd} \end{cases}.$$

Proof. It is easy to see that, if $t = 1$, then $HM(1) = C_6$ and $I(C_6) = 4$ and if $t = 2$, then, $I(HM(2)) = 10$.

Let $G = HM(t), t \geq 3$.

Case(i): When t is even. Let $S = \{v_1, v_5, \cdots, v_{3t^2-4}\} \cup \{u_1, u_5, \cdots, u_{3t^2-4}\}$ be a cut set of G and $|S| = \frac{3t^2}{2}$. Then $m(G - S) = 6$. Therefore, $|S| + m(G - S) \leq \frac{3t^2}{2} + 6$. Let S be a vertex cut of G. Then clearly, $m(G - S) \geq \frac{n-|S|}{\omega(G-S)}$ and $m(G - S) \geq 6$, since $\omega(G - S) = \frac{3t^2+2t}{4}$. Therefore, we get $m(G - S) \geq \frac{6}{(1+\frac{2}{3}t)}$. Since $t \geq 4, m(G - S) \geq \lceil \frac{6}{(1+\frac{2}{3}t)} \rceil \geq 6$. $|S| + m(G - S) \geq \frac{3t^2}{2} + 6$. Hence $I(HM(t)) = \frac{3t^2}{2} + 6$.

Case(ii): When t is odd. Let $S = \{v_1, v_5, \cdots, v_{3t^2-3}\} \cup \{u_1, u_5, \ldots, u_{3t^2-3}\}$ be a cut set of G and $|S| = \frac{3(t^2-1)}{2}$. Then $m(G - S) = 6$. Therefore, $|S| + m(G - S) \leq \frac{3(t^2-1)}{2} + 6$. Let S be a vertex cut of G. Then clearly, $m(G - S) \geq \frac{n-|S|}{\omega(G-S)}$ and $m(G - S) \geq 6$, since $\omega(G - S) = \frac{3(t^2+1)}{4}$. Therefore, we get $m(G - S) \geq 6$. Also, $|S| + m(G - S) \geq \frac{3(t^2-1)}{2} + 6$. Hence $I(HM(t)) = \frac{3(t^2-1)}{2} + 6$. □

Theorem 2.7. *Let $G = HM(t)$. Then*

(1) $t(G) = \frac{3(t^2-t)}{3(t^2-t)+1)}$
(2) $sc(G) = 0$
(3) $r(G) = -1$
(4) $T(G) = \frac{(3t^2+1)}{3t^2}$

Proof.

(1) From the structure of Honeycomb meshes (see Figure 2), if the vertex cut S of $HM(t)$ is composed from the vertices belonging to $2, 4, \cdots, 2t-2$ level, then $\omega(G-S) < |S|+1$. This is the only vertex cut which has the property $|S|+1 = \omega(G-S)$. For any other vertex cut $\omega(G-S) \leq |S|$. Hence $min\{\frac{|S|}{\omega(G-S)}\}$ is attained when $\omega(G-S) = |S|+1$.

Therefore, $t(G) \geq \frac{|S|}{|S|+1} = \frac{(3t^2-t)}{(3t^2-t)+1)}$.

On the other hand, let S_1 denote the vertex cover of G. Then $|S_1| = 3(t^2-t)$ and $\omega(G-S_1) = 3(t^2-t)+1$.

Therefore, $t(G) \leq \frac{3(t^2-t))}{3(t^2-t)+1}$.

(2) Since $t(G) < 1$, $\frac{\kappa(G)}{\kappa(G)+sc(G)} < 1$ by Theorem 2.3. It implies that $sc(G) \geq 0$.

On the other hand, let S be a cut set of G with $|S| = 3t^2$. If we remove S vertices from G, then $\omega(G-S) = 3t^2$.

Therefore, $sc(G) \leq 0$. Hence $sc(G) = 0$.

(3) By Theorem 2.4 and Theorem 2.5, $HM(t) \subseteq K_{3t^2,3t^2}, r(HM(t) \geq r(K_{3t^2,3t^2}) = -1$. Let S be a cut set of G with $|S| = k$. If $k \leq 3t^2 \Rightarrow \frac{-1}{k} \leq \frac{-1}{3t^2}$, then $\omega(G-S) \leq k$.

Therefore, $m(G-S) \leq \lceil \frac{|V(G)|-k}{k} \rceil$.

Also, $\omega(G-S) - |S| - m(G-S) \leq k - k - \lceil \frac{|V(G)|-k)}{k} \rceil = -\lceil \frac{6t^2-k}{k} \rceil \leq -1$. If $k \geq 3t^2$, then $\omega(G-S) \leq |V(G)| - k$.

Therefore, $-m(G-S) \geq \lceil \frac{|V(G)|-k}{\omega(G-S)} \rceil = \frac{(|V(G)|-k)}{\omega(G-S)} - 1, = -\frac{(6t^2-k)}{\omega(G-S)} - 1 = \frac{(6t^2-k)}{-3t^2} - 1 = 1 - \frac{k}{3t^2}$, and $\omega(G-S) - |S| - m(G-S) \leq 3t^2 - 3t^2 + 1 - \frac{k}{3t^2} \leq -1$.

Therefore, $r(G) \leq -1$. Hence $r(G) = -1$.

(4) Let S be a cut set of G with $|S| = x$. Then $\omega(G-S) \leq x$ and so $m(G-S) \geq \frac{(6t^2-x)}{x}$ (such a set is guaranteed, since $\alpha(G) = \beta(G) = 3t^2$). Since $\frac{(6t^2-x)}{x} \geq 1, x$ must be at most $3t^2$. Thus, we get $T(G) \geq \{\frac{x+\frac{6t^2-x}{x}}{x}\}$, where $x \leq 3t^2$.

Now we consider the function $f(x) \geq min\frac{(x^2-x+6t^2)}{x^2}$ and $f'(x) = \frac{(x-12t^2)}{x^3}$. Since $x \leq 3t^2 < 12t^2$, we have $f'(x) < 0$, and so $f(x)$ is a decreasing function. So $f(x)$ takes its minimum value at $x = 3t^2$ and $f_{min}(x) = \frac{(3t^2+1)}{3t^2}$.

Hence $T(G) \geq \frac{(3t^2+1)}{3t^2}$.

On the other hand, let S_1 denote the vertex cover of G. Then $|S_1| = 3t^2$, $m(G-S_1) = 1$ and $\omega(G-S_1) = 3t^2$. Therefore, $T(G) \leq \frac{(3t^2+1)}{3t^2}$.

Hence $T(G) = \frac{(3t^2+1)}{3t^2}$. □

3. Conclusion

The resistance of a communication network can be measured by the vulnerability parameters after the failure of certain processors or communication links. The network designers want to design a network with less vulnerability or more reliability. They consider not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network while the construction of any communication networks. There are many vulnerability parameterS can be used to describe the vulnerability of communication networks. In this paper, we found the exact values of such vulnerability parameters of honeycomb networks.

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