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HOMODERIVATIONS OF $\sigma-$PRIME $\Gamma-$RINGS

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ABSTRACT. Let M be a sigma-prime Gamma-ring with an additive mapping σ : $M \to M$ is called an involution on M. Let h be a homoderivation on M where h is also an additive mapping $h : M \to M$. In this paper, the commutativity properties of M admitting a homoderivation satisfying $h\sigma = \sigma h$ are proven.

1. INTRODUCTION

In this paper, M represents as a $\Gamma-$ring with center of $Z(M)$ and for any $a \in M$, if $2a = 0$ implies $a = 0$ then M is called 2-torsion free [10]. The commutator and anticommutator of M are defined as for any $x, y \in M$ and $\alpha \in \Gamma$ such that $[x, y]_\alpha = x\alpha y - y\alpha x$ and $\langle x, y\rangle_\alpha = x\alpha y + y\alpha x$, respectively. An additive mapping σ is said to be an involution on M if $\sigma : M \to M$ satisfies these two conditions: $\sigma(x\alpha y) = \sigma(y)\alpha\sigma(x)$ and $\sigma(\sigma(x)) = x$ [10]; and such M will be called as a $\sigma-\Gamma-$ring. The set $S_\alpha(M) = \{x \in M|\sigma(x) = \pm x\}$ is called the set of symmetric and skew symmetric elements of M and suppose $I \subseteq M$ such that $\sigma(I) = I$ then an ideal I of M is said to be a $\sigma-$ideal [5].

Definition 1.1. *Let M be a $\sigma - \Gamma-$ring. M is called a $\sigma-$prime if $x\Gamma M\Gamma y = 0 = x\Gamma M\Gamma\sigma(y)$ (or $x\Gamma M\Gamma y = 0 = \sigma(x)\Gamma M\Gamma y$) implies $x = 0$ or $y = 0$, for all $x, y \in M$.*

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In general, from Definition 1.1, we conclude that every prime $\sigma - \Gamma-$ring is $\sigma-$prime but the converse is not true. However, if M is a $\sigma-$prime $\Gamma-$ring such that for all $x \in M$ and $x\Gamma M\Gamma x = 0$, then $x\Gamma M\Gamma x\Gamma\sigma(x) = 0$.Since M is $\sigma-$prime, it implies that $x = 0$ or $x\Gamma M\Gamma\sigma(x) = 0$. Now, if $x\Gamma M\Gamma\sigma(x) = 0$, then $x\Gamma M\Gamma x = 0 = x\Gamma M\Gamma\sigma(x)$ implies $x = 0$, by M is $\sigma-$prime. Therefore, every $\sigma-$prime $\Gamma-$ring is a semiprime $\Gamma-$ring [6–8].

Works on homoderivations have been done by Melaibari et. al [2], Al-Kenani et. al [1] and Boua [3] in the cases of prime rings, 3-prime near-rings and $*-$prime rings, respectively. They used the concept of homoderivation on ring which was introduced by El Sofy Aly [4]. In this paper, we extend the work to commutativity of sigma-prime Gamma-ring.

Definition 1.2. *Let M be a $\sigma-$prime $\Gamma-$ring and h be an additive mapping $h : M \to M$. For all $x, y \in M$ and $\alpha \in \Gamma$, then h is called a homoderivation on M if $h(x\alpha y) = h(x)\alpha h(y) + h(x)\alpha y + x\alpha h(y)$.*

As example, let $h(x) = g(x) - x$ for all $x \in M$ where g is an endomorphism on M. A mapping $f : M \to M$ is centralizing on S where $S \subseteq M$, if $[x, f(x)]_\alpha \in Z(M)$ for all $x \in S$ and $\alpha \in \Gamma$. If $f(S) \subseteq S$ and for each $x \in S$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)}(x) = 0$, then f is called zero-power valued on S.

2. Sigma-prime Gamma-ring

This section begin with the following lemma.

Lemma 2.1. *Let M be a $\sigma-$prime $\Gamma-$ring and I be a nonzero $\sigma-$ideal of M. For all $x, y \in M$, if $x\Gamma I\Gamma y = 0 = x\Gamma I\Gamma\sigma(y)$ (or $x\Gamma I\Gamma y = 0 = \sigma(x)\Gamma I\Gamma y$), then $x = 0$ or $y = 0$.*

Proof. Let $a, b \in M$. Suppose $a \neq 0$, there exists some $x \in I$ and $\alpha \in \Gamma$ such that $a\alpha x \neq 0$. Indeed, otherwise $a\Gamma M\Gamma x = 0$ and $a\Gamma M\Gamma\sigma(x) = 0$ for all $x \in I$, therefore $a = 0$. Since $a\Gamma I\Gamma M\Gamma b = 0$ and $a\Gamma I\Gamma M\Gamma\sigma(b) = 0$, then $a\Gamma x\Gamma M\Gamma b = a\Gamma x\Gamma M\Gamma\sigma(b) = 0$ is obtained. In view of the $\sigma-$primeness of M this yields $b = 0$. □

Now the following lemmas need to be proven to launch in achieving our main results in the next section.

Lemma 2.2. *Let M be a $\sigma-$prime $\Gamma-$ring and I be a nonzero $\sigma-$ideal of M. Let h be a nonzero homoderivation on M such that $h\sigma = \sigma h$. For all $x \in I$ and $\alpha \in \Gamma$, if $[x, M]_\alpha \Gamma I \Gamma h(x) = 0$, then M is commutative.*

Proof. For all $x \in I$ and $\alpha, \beta, \gamma \in \Gamma$ we have $[x, M]_\alpha \beta I \gamma h(x) = 0$. Assume that $t = x - \sigma(x) \in I$, for any $x \in I$ and follows by the above expression, we get $[t, m]_\alpha \beta I \gamma h(t) = 0$ for all $m \in M$. Since $\sigma(t) = \sigma(x - \sigma(x)) = \sigma(x) - x = -t$, we find

$$\begin{aligned}\sigma([t, m]_\alpha)\beta I \gamma h(t) &= \sigma(t\alpha m - m\alpha t)\beta I \gamma h(t) = (\sigma(m)\alpha\sigma(t) - \sigma(t)\alpha\sigma(m))\beta I \gamma h(t) \\ &= (-\sigma(m)\alpha\sigma(t) + t\alpha\sigma(m))\beta I \gamma h(t) = [t, \sigma(m)]_\alpha \beta I \gamma h(t) = 0.\end{aligned}$$

Therefore, $[t, m]_\alpha \beta I \gamma h(t) = 0 = \sigma([t, m]_\alpha)\beta I \gamma h(t)$. According to Lemma 2.1, we have $[x, M]_\alpha = 0$ or $h(t) = 0$. Thus, for each $x \in I$ and $\alpha \in \Gamma$, we get either $[x, M]_\alpha = [\sigma(x), m]_\alpha$ or $h(x) = h(\sigma(x))$.

In case $[x, m]_\alpha = [\sigma(x), m]_\alpha$. For all $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$, we observe that

$$\begin{aligned}&\sigma([x, m]_\alpha)\beta I \gamma h(x) = \sigma(x\alpha m - m\alpha x)\beta I \gamma h(x) \\ &= (\sigma(m)\alpha\sigma(x) - \sigma(x)\alpha\sigma(m))\beta I \gamma h(x) = [\sigma(m), \sigma(x)]_\alpha \beta I \gamma h(x) \\ &= [\sigma(m), x]_\alpha \beta I \gamma h(x) = -[x, \sigma(m)]_\alpha \beta I \gamma h(x) = 0.\end{aligned}$$

Thus, $[x, m]_\alpha \beta I \gamma h(x) = 0 = \sigma([x, m]_\alpha)\beta I \gamma h(x)$ and by Lemma 2.1, we obtain $h(x) = 0$ or $[x, m]_\alpha = 0$. While in case $h(x) = h(\sigma(x))$, since h commutes with σ and $h(x) = \sigma(h(x))$. For all $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$, we find that $0 = [x, m]_\alpha \beta I \gamma h(x) = [x, m]_\alpha \beta I \gamma \sigma(h(x))$. Thus, by Lemma 2.1 we get $h(x) = 0$ or $[x, m]_\alpha = 0$.

Both cases above show that for each $x \in I$, either $h(x) = 0$ or $x \in Z(M)$. The sets of $x \in I$ in these two cases are additive subgroups of I whose union is I. Known that a group cannot be the union of two of its proper subgroups, therefore we obtain either $h(I) = 0$ or $I \subseteq (M)$.

Consider the case $h(I) = 0$. Then for all $x \in I$, we have $h(x) = 0$ and it implies that $0 = h(x\alpha m) = h(x)\alpha h(m) + h(x)\alpha m + x\alpha h(m) = x\alpha h(m)$, for all $m \in M$ and $\alpha \in \Gamma$. It follows by $I\Gamma h(m) = 0$ implies $I\Gamma M \Gamma h(m) = 0 = \sigma(I)\Gamma M \Gamma h(m)$. By $\sigma-$primeness of M, $h = 0$ which is a contradiction. Now, consider the case $I \subseteq Z(M)$. Let $m, n \in M$, $x \in I$ and $\alpha, \beta \in \Gamma$, then we obtain $m\alpha n\beta x = m\alpha x\beta n = n\alpha m\beta x$ and $[m, n]_\alpha \beta x = 0$. Thus, $[m, n]_\alpha \Gamma I = 0$ and $[m, n]_\alpha \Gamma M \Gamma I = 0 = [m, n]_\alpha \Gamma M \Gamma \sigma(I)$. By $\sigma-$primeness of M, $[m, n]_\alpha = 0$. Hence, M is commutative. $\square$

Lemma 2.3. *Let M be a $\sigma-$prime $\Gamma-$ring and I be a nonzero $\sigma-$ideal of M. Let h be a nonzero homoderivation on M such that $h\sigma = \sigma h$. For all $x \in I$ and $\alpha \in \Gamma$, if h is a zero-power valued on I and $[h(x), x]_\alpha = 0$, then M is commutative.*

Proof. Given that $[h(x), x]_\alpha = 0$, for all $x \in I$ and $\alpha \in \Gamma$. Now for all $x, y \in I$ and $\alpha \in \Gamma$, by linearizing the given expression, we get $[h(x), y]_\alpha + [h(y), x]_\alpha = 0$. Take $\beta \in \Gamma$ and replaces $y = y\beta x$, gives $[h(x), y\beta x]_\alpha + [h(y\beta x), x]_\alpha = 0$ and it can be extended as $[h(x), y]_\alpha \beta x + [h(y), x]_\alpha \beta h(x) + [h(y), x]_\alpha \beta x + [y, x]_\alpha \beta h(x) = 0$. Thus we have $[h(y) + y, x]_\alpha \beta h(x) = 0$. Since h is a zero-power valued on I, we can replace $y = y - h(y) + h^2(y) + \cdots + (-1)^{(n(y)-1)} h^{(n(y)-1)}(y)$ to get $[x, y]_\alpha \beta h(x) = 0$.

Now, for arbitrary $m \in M$ and take $\gamma \in \Gamma$ and by replacing $y = m\gamma y$, we find $0 = [x, m\gamma y]_\alpha \beta h(x) = [x, m]_\alpha \gamma y \beta h(x)$, which can imply $[x, m]_\alpha \gamma y \beta h(x) = 0$, for all $x \in I$ and $\alpha, \beta, \gamma \in \Gamma$. It is prove that by Lemma 2.2, M is commutative. □

Lemma 2.4. *Let M be a $\sigma-$prime $\Gamma-$ring and let I be a nonzero $\sigma-$ideal of M. If $x \in M$ and x centralizes I, then $x \in Z(M)$.*

Proof. Let $x \in M$. For all $u \in I$ and $\alpha \in \Gamma$ such that $[x, u]_\alpha = 0$. Then, for arbitrary $m \in M$ and $\beta \in \Gamma$, we obtain $0 = [x, m\beta u]_\alpha = [x, m]_\alpha \beta u$ which can implies $[x, M]_\alpha \Gamma I = 0$. Therefore, we get $[x, M]_\alpha \Gamma M \Gamma I = 0 = [x, M]_\alpha \Gamma M \Gamma \alpha(I)$. Since M is $\alpha-$prime, then $[x, M]_\alpha = 0$. Thus $x \in Z(M)$. □

3. The commutativity of M admitting centralizing homoderivations

Motivated by the work in [1], the concept of homoderivations on $\sigma-$prime $\Gamma-$rings are presented in the following theorems.

Theorem 3.1. *Let M be a 2 torsion-free $\sigma-$prime $\Gamma-$ring and I be a nonzero $\sigma-$ideal of M. Suppose that h is a nonzero homoderivation on M such that $h\sigma = \sigma h$. If h is centralizing and a zero-power valued on I, then M is commutative.*

Proof. Given for all $x \in I$ and $\alpha \in \Gamma$, we have $[h(x), x]_\alpha \in Z(M)$. Now, for all $x, y \in I$, $\alpha \in \Gamma$ and by linearizing the given expression above, we find $[h(x), y]_\alpha + [h(y), x]_\alpha \in Z(M)$. Take $\beta \in \Gamma$ and replaces $y = x\beta x$ to obtain $[h(x), x\beta x]_\alpha + [h(x\beta x), x]_\alpha \in Z(M)$. By extending this expression, we get

$$x\beta[h(x), x]_\alpha + [h(x), x]_\alpha \beta x + h(x)\beta[h(x), x]_\alpha + [h(x), x]_\alpha \beta h(x)$$
$$+ [h(x), x]_\alpha \beta x + x\beta[h(x), x]_\alpha \in Z(M).$$

The above expression can be simplified as $(4x + 2h(x))\beta[h(x), x]_\alpha \in Z(M)$. Thus, it becomes $(2x+h(x))\beta[h(x), x]_\alpha \in Z(M)$, since M is 2 torsion-free. Therefore, for arbitrary $m \in M$, we have

$$[(2x + h(x))\beta[h(x), x]_\alpha, m]_\alpha = [2x + h(x), m]_\alpha\beta[h(x), x]_\alpha = 0.$$

In particular, for all $x \in I$ and $\alpha, \beta \in \Gamma$, we find

$$[2x + h(x), x]_\alpha\beta[h(x), x]_\alpha = [h(x), x]_\alpha\beta[h(x), x]_\alpha = 0.$$

Since every $\sigma-$prime $\Gamma-$ring is semiprime and since the center of semiprime $\Gamma-$ring contains a no nonzero nilpotent elements [9]. Then for all $x \in I$ and $\alpha \in \Gamma$, we obtain that $[h(x), x]_\alpha = 0$. Hence by Lemma 2.3, M is commutative. □

Theorem 3.2. *Let M be a 2 torsion-free $\sigma-$prime $\Gamma-$ring and I be a nonzero $\sigma-$ideal of M. Suppose that h is a nonzero homoderivation on M such that $h\sigma = \sigma h$. If h is a zero-power valued on I and $a \in S_\sigma(M)$ such that $[a\beta h(x), x]_\alpha = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$. Then $a = 0$ or M is commutative.*

Proof. Given for all $x \in I$ and $\alpha, \beta \in \Gamma$, we have

$$[a\beta h(x), x]_\alpha = 0. \tag{3.1}$$

Now, for all $x, y \in I$ and $\alpha, \beta \in \Gamma$, we get $[a\beta h(x), y]_\alpha + [a\beta h(y), x]_\alpha = 0$. Take $\gamma \in \Gamma$ and replaces $y = y\gamma x$, we obtain

$$[a\beta h(x), y\gamma x]_\alpha + [a\beta h(y)\gamma h(x), x]_\alpha + [a\beta h(y)\gamma x, x]_\alpha + [a\beta y\gamma h(x), x]_\alpha = 0,$$

which is equivalent to

$$y\gamma[a\beta h(x), x]_\alpha + [a\beta h(x), y]_\alpha\gamma x + a\beta h(y)\gamma[h(x), x]_\alpha + [a\beta h(y), x]_\alpha\gamma h(x)$$

$$+[a\beta h(y), x]_\alpha\gamma x + a\beta y\gamma[h(x), x]_\alpha + a\beta[y, x]_\alpha\gamma h(x) + [a, x]_\alpha\beta y\gamma h(x) = 0$$

or

$$\begin{aligned} &a\beta h(y)\gamma[h(x), x]_\alpha + [a\beta h(y), x]_\alpha\gamma h(x) + a\beta y\gamma[h(x), x]_\alpha \\ &+ a\beta[y, x]_\alpha\gamma h(x) + [a, x]_\alpha\beta y\gamma h(x) = 0. \end{aligned}$$

The last expression above can be written as

$$a\beta(h(y) + y)\gamma[h(x), x]_\alpha + [a, x]_\alpha\beta(h(y) + y)\gamma h(x) + a\beta[h(y) + y, x]_\alpha\gamma h(x) = 0.$$

Since h is a zero-power valued on I, for all $x, y \in I$ and $\alpha, \beta, \gamma \in \Gamma$, we have $a\beta y\gamma[h(x), x]_\alpha + [a, x]_\alpha \beta y\gamma h(x) + a\beta[y, x]_\alpha \gamma h(x) = 0$. Now, take $\lambda \in \Gamma$. By replacing $y = a\lambda y$ we obtain

$$a\beta a\lambda y\gamma[h(x), x]_\alpha + [a, x]_\alpha \beta a\lambda y\gamma h(x) + a\beta a\lambda[y, x]_\alpha \gamma h(x) + a\beta[a, x]_\alpha \lambda y\gamma h(x) = 0.$$

Thus, we get $[a, x]_\alpha \beta a\lambda y\gamma h(x) = 0$, which implies

$$[a, x]_\alpha \beta a\Gamma I\Gamma h(x) = 0. \tag{3.2}$$

Clear that for $x \in I \cap S_\sigma(M)$ we have $\sigma(x) = x$. Therefore, since h commutes with σ it implies that $\sigma(h(x)) = h(\sigma(x)) = h(x)$. Then we find

$$[a, x]_\alpha \beta a\Gamma I\Gamma h(x) = 0 = [a, x]_\alpha \beta a\Gamma I\Gamma \sigma(h(x))$$

and by Lemma 2.1, it follows that $[a, x]_\alpha \beta a = 0$.

Now consider $y \in I$. Since $(y + \sigma(y)) \in I \cap S_\alpha(M)$, we have $[a, y + \sigma(y)]_\alpha \beta a = 0$ or $h(y + \sigma(y)) = 0$. We need to consider two cases:

Case 1: Let $[a, y + \sigma(y)]_\alpha \beta a = 0$. Since $(y - \sigma(y)) \in I \cap S_\sigma(M)$, we have either $h(y - \sigma(y)) = 0$ or $[a, y - \sigma(y)]_\alpha \beta a = 0$. If $h(y - \sigma(y)) = 0$, then by a similar approach from above, we get $[a, y]_\alpha \beta a = 0$ or $h(y) = 0$. If $[a, y - \sigma(y)]_\alpha \beta a = 0$, then $[a, y - \sigma(y)]_\alpha \beta a + [a, y + \sigma(y)]_\alpha \beta a = 0$ which can be reduced to $2[a, y]_\alpha \beta a = 0$. Since M is 2-torsion free, we obtain $[a, y]_\alpha \beta a = 0$.

Case 2: Let $h(y + \sigma(y)) = 0$. Then $h(y) = -h(\sigma(y)) = -\sigma(h(y))$. Thus, by (3.2), gives $0 = [a, y]_\alpha \beta a\Gamma I\Gamma h(y) = [a, y]_\alpha \beta a\Gamma I\Gamma \sigma(h(y))$ and by Lemma 2.1, it shows that $[a, y]_\alpha \beta a = 0$ or $h(y) = 0$.

Clearly, both cases show that for each $y \in I$ then $[a, y]_\alpha \beta a = 0$ or $h(y) = 0$. Similar approach as in the proving of Lemma 2.2, we have either $[a, I]_\alpha \beta a = 0$ or $h(I) = 0$. Now we consider two cases again.

First case: Let $h(I) = 0$, then $h(x) = 0$ for all $x \in I$. For arbitrary $m \in M$ and $\mu \in \Gamma$, we obtain $0 = h(m\mu x) = h(m)\mu h(x) + h(m)\mu x + m\mu h(x) = h(m)\mu x$. Therefore, we have $h(m)\Gamma I = 0$ and $h(m)\Gamma M\Gamma I = 0 = h(m)\Gamma M\Gamma \sigma(I)$ that implies $h = 0$, since $\sigma-$primeness of M. This is contradictory.

Second case: Let $[a, I]_\alpha \beta a = 0$. Then we have $[a, x]_\alpha \beta a = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$. Now, take $\mu \in \Gamma$ and by replacing $x = x\mu y$ yields $[a, x]_\alpha \mu y\beta a = 0$. Thus, $[a, x]_\alpha \mu I\beta a = 0$. As $a \in S_\sigma(M)$, then $0 = [a, x]_\alpha \Gamma I\beta a = [a, x]_\alpha \Gamma I\beta \sigma(a)$ for all $x \in I$ and $\alpha, \beta \in \Gamma$. By Lemma 2.1, a centralizes I or $a = 0$ and by Lemma 2.4, $a \in Z(M)$

or $a = 0$. If $0 \neq a \in Z(M)$, then by (3.1) we find

$$0 = [a\mu h(x), x]_\alpha = a\mu[h(x), x]_\alpha + [a, x]_\alpha \mu h(x) = a\mu[h(x), x]_\alpha.$$

Since $a \in Z(M)$, $a\Gamma M\Gamma[h(x), x]_\alpha = 0$ and $a \in S_\sigma(M)$ then for all $x \in I$ and $\alpha \in \Gamma$, we obtain that $0 = a\Gamma M\Gamma[h(x), x]_\alpha = \sigma(a)\Gamma M\Gamma[h(x), x]_\alpha$. As $a \neq 0$, then $\sigma-$primeness of M implies $[h(x), x]_\alpha = 0$ for all $x \in I$ and $\alpha \in \Gamma$. It follows from Lemma 2.3 that M is commutative. □

The following theorems investigate the identities on homoderivations.

Theorem 3.3. *Let M be a $\sigma-$prime $\Gamma-$ring and I be a nonzero $\sigma-$ideal of M. Suppose that h is a nonzero homoderivation on M such that $h\sigma = \sigma h$. For all $x, y \in I$ and $\alpha \in \Gamma$, if h satisfies either $h([x, y]_\alpha) = 0$ or $h(\langle x, y\rangle_\alpha) = 0$ then M is commutative.*

Proof. We start with the first condition. For all $x, y \in I$ and $\alpha \in \Gamma$, we have $h([x, y]_\alpha) = 0$. Take $\beta \in \Gamma$ and by replacing $y = y\beta x$, we get

$$0 = h([x, y\beta x]_\alpha) = h([x, y]_\alpha \beta x) = h([x, y]_\alpha)\beta h(x) + h([x, y]_\alpha)\beta x + [x, y]_\alpha \beta h(x),$$

which implies

$$[x, y]_\alpha \beta h(x) = 0. \tag{3.3}$$

Now for arbitrary $m \in M$ and $\gamma \in \Gamma$, replaces $y = m\gamma y$, the expression $[x, m\gamma y]_\alpha \beta h(x) = [x, m]_\alpha \gamma y \beta h(x) = 0$ is obtained.

Next in second condition. For all $x, y \in I$ and $\alpha \in \Gamma$, we have $h(\langle x, y\rangle_\alpha) = 0$. Again, take $\beta \in \Gamma$ and replaces $y = y\beta x$, gives

$$0 = h(\langle x, y\beta x\rangle_\alpha) = h(\langle x, y\rangle_\alpha \beta x) = h(\langle x, y\rangle_\alpha)\beta h(x) + h(\langle x, y\rangle_\alpha)\beta x + \langle x, y\rangle_\alpha \beta h(x).$$

Thus $\langle x, y\rangle_\alpha \beta h(x) = 0$ which is equivalent to

$$x\alpha y\beta h(x) = -y\alpha x\beta h(x). \tag{3.4}$$

For arbitrary $m \in M$ and $\gamma \in \Gamma$, we replace $y = m\gamma y$ in $x\alpha y\beta h(x) = -y\alpha x\beta h(x)$ to obtain $x\alpha m\gamma y\beta h(x) = -m\gamma y\alpha x\beta h(x)$ which can implies $x\alpha m\gamma y\beta h(x) = m\alpha x\gamma y\beta h(x)$. Therefore $[x, m]_\alpha \gamma y\beta h(x) = 0$.

From the both conditions, we can conclude that for all $x \in I$, $\alpha \in \Gamma$ and by Lemma 2.2, the expression $[x, M]_\alpha \Gamma I\Gamma h(x) = 0$ implies M is commutative. □

Theorem 3.4. *Let M be a 2 torsion-free $\sigma-$prime $\Gamma-$ring and I be a nonzero $\sigma-$ideal of M. Suppose that h is a nonzero homoderivation on M such that $h\sigma = \sigma h$. For all $x, y \in I$ and $\alpha \in \Gamma$, if h satisfies these two conditions either $h([x, y]_\alpha) = [x, y]_\alpha$ or $h(\langle x, y\rangle_\alpha) = \langle x, y\rangle_\alpha$, then M is commutative.*

Proof.

Condition 1: Given for all $x, y \in I$, $\alpha \in \Gamma$ and $h([x, y]_\alpha) = [x, y]_\alpha$. By taking $\beta \in \Gamma$ and replaces $y = y\beta x$ gives $h([x, y]_\alpha\beta x) = [x, y]_\alpha\beta x$. Then we have $h([x, y]_\alpha)\beta h(x)+h([x, y]_\alpha)\beta x+[x, y]_\alpha\beta h(x) = [x, y]_\alpha\beta x$ which implies $2[x, y]_\alpha\beta h(x) = 0$. Since M is 2 torsion-free, we get (3.3). By a similar approach as the proving in Theorem 3.3 for first condition, we have $[x, m]_\alpha\gamma y\beta h(x) = 0$.

Condition 2: Given for all $x, y \in I$, $\alpha \in \Gamma$ and $h\langle x, y\rangle_\alpha) = \langle x, y\rangle_\alpha$. Again, take $\beta \in \Gamma$ and replaces $y = y\beta x$, gives $h(\langle x, y\rangle_\alpha\beta x) = \langle x, y\rangle_\alpha\beta x$. Then we get $h(\langle x, y\rangle_\alpha)\beta h(x)+h(\langle x, y\rangle_\alpha)\beta x+\langle x, y\rangle_\alpha\beta h(x) = \langle x, y\rangle_\alpha\beta x$ which implies $2\langle x, y\rangle_\alpha\beta h(x) = 0$. Since M is 2 torsion-free, the expression $\langle x, y\rangle_\alpha\beta h(x) = 0$ is equivalent to (3.4). By a similar approach as the proving in Theorem 3.3 for second condition, we can show that $[x, m]_\alpha\gamma y\beta h(x) = 0$.

From the conditions above, clearly that for all $x \in I$, $\alpha \in \Gamma$ and by Lemma 2.2, the expression $[x, M]_\alpha\Gamma I\Gamma h(x) = 0$ implies M is commutative. □

4. Conclusion

From Theorems 3.1 and 3.2, we prove that sigma-prime Gamma-ring is commutative if a homoderivation is centralizing and a zero-power valued on sigma-ideal. While Theorems 3.3 and 3.4 show that the commutativity of sigma-prime Gamma-ring admitting a homoderivation satisfies some conditions of commutator dan anticommutator of sigma-prime Gamma-ring.

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REFERENCES

[1] A. AL-KENANI, A. MELAIBARI, N. MUTHANA: *Homoderivations and Commutativity of $*-$Prime Rings*, East-West J. of Mathematics, **17**(2) (2015), 117–126.

[2] A. MELAIBARI, N. MUTHANA, A. AL-KENANI: *Homoderivations on Rings*, General Mathematics Notes, **35**(1) (2016), 1–8.

[3] A. BOUA: *Homoderivations and Jordan Right Ideals in 3-Prime Near-Rings*, AIP Conference Proceedings, **2074** (2019), 020010.

[4] M.M. EL SOFY ALY: *Rings with Some Kinds of Mappings*, M.Sc. Thesis, Cairo University, Branch of Fayoum, Egypt, 2000.

[5] I.N. HERSTEIN: *Rings with Involution*, The University of Chicago Press, 1976.

[6] K.K. DEY, A.C. PAUL: *On Centralizing Automorphisms and Jordan Left Derivations on Sigma-Prime Gamma Rings*, Journal of Scientific Research, **4**(1) (2012), 33–37.

[7] K.K. DEY, A.C. PAUL: *Commutativity of Prime and Semiprime Gamma Rings*, GANIT J. Bangladesh Math. Soc., **34** (2014), 27–33.

[8] K.K. DEY, A.C. PAUL, B. DAVVAZ: *On Centralizing Automorphisms and Jordan Left Derivations on sigma-Prime Gamma Rings*, Heceppte Journal of Mathematics and Statistics, **44**(1) (2015), 51–57.

[9] K.K. DEY, A.C. PAUL, B. DAVVAZ: *Derivations with Nilpotent Values on Gamma-Rings*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., **21**(4) (2014), 237–246.

[10] K.K. DEY, A.C. PAUL, I.S. RAKHIMOV: *Commutativity of $\sigma-$Prime $\Gamma-$Rings with Semiderivations*, International Journal of Pure and Applied Math., **113**(5) (2017), 541–552.
N. NABUSAWA: *On a Generalization of The Ring Theory*, Osaka J. Math., **1** (1964), 65–75.

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