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ON CENTRAL EXTENSION OF THREE DIMENSIONAL ASSOCIATIVE ALGEBRAS

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ABSTRACT. One of the methods to investigate the classification of algebras is the Skjelbred-Sund method. In order to use this method, some extension invariants are needed. In this paper, seven invariant classes of three dimensional nilpotent associative algebras are provided.

1. INTRODUCTION

The classification of associative algebra is categorized in an old problem which there are lot of researchers have been study about them before. Many other publications related to the problem have appeared. It was follows by others works such as Hazlett [1] studied on nilpotent algebras of dimension ≤ 4 over $\mathbb{C}$, Mazzola [2, 3] has investigated the associative unitary algebras of dimension 5 over algebraically closed fields of characteristic not 2 and the nilpotent commutative associative algebras of dimension ≤ 5, over algebraically closed fields of characteristic not 2, 3 and recently, respectively. Poonen [4] studied nilpotent commutative associative algebras of dimension ≤ 5, over algebraically closed fields.

Extension of Lie groups theory arise in several ways, for instance by using the central extension. A bijective correspondence between all central extensions of fixed Lie algebra and certain orbits in the set of all k-dimensional subspaces in the second cohomology group under the canonical action of automorphism group is

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established by Skjelbred and Sund [5]. A method for classifying nilpotent associative algebras is described by Graaf [6] in 2010. That is, analogous to Skjelbred-Sund method for classifying nilpotent Lie algebras.

In general, the central extension is used to enlarge from algebra dimension n to algebra dimension $n+1$. As an application, the set of all three-dimensional associative algebras are described.

2. PRELIMINARY

In this section some basic concepts regarding associative algebra are presented.

Definition 2.1. *[6] An associative algebra As is a vector space over a field K equipped with a bilinear map $f : As \times As \to As$ satisfying the associative identity, $f(f(x,y),z) = f(x, f(y,z))$, for all $x, y, z \in As$.*

An algebra A is said to be nilpotent, if there exist an integer $s \in N$, such that $A^s = 0$. The smallest integer s for that $A^s = 0$ is called the nilindex of A. [7]

Definition 2.2. *[6] Let As be an associative algebra and V be a vector space over K. Then the bilinear maps, $\theta : As \times As \to V \quad with \quad \theta((xy), z) = \theta(x, (yz))$ for all $x, y, z \in As$ are called associative cocycle.*

In [8], an associative algebra As over field K is called center of associative algebra if its binary map satisfies the following properties, $C(As) = \{a \in As | a \cdot As = As \cdot a = 0\}$.

Definition 2.3. *[6] Let A be an algebra over field K for $\theta \in Z^2(A, V)$. The set $\theta^\perp$ is called radical of algebra if its binary map satisfies the following properties:*

$$\theta^\perp = \{a \in A | \theta(a,b) = \theta(b,a) = 0\}, \text{ for all } b \in A.$$

An algebra A over field K is called maximum commutative subalgebra, $Com(A)$ and maximum abelian subalgebra, n_A if its binary map satisfies $xy = yx$ and $xy = 0$, respectively for all $x, y \in A$.

The following statements are stated in [8]. Let A be an arbitrary algebra over a field K. The centroid of A, $\Gamma(A)$ is defined by

$$\Gamma(A) = \{\phi \in End(A) | \phi(xy) = \phi(x)y = x\phi(y), \forall x, y \in A\}.$$

A derivation of an algebra A is a K-linear transformation $d : A \to A$ satisfying

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y), \forall x, y \in A.$$

The set of all derivations of an algebra A is denote as $Der(A)$.

Lemma 2.1. *Let As be n-dimensional associative algebra and let $\{e_1, e_2, \ldots, e_m\}$ be a basis of $As^{<2>}$. Then $B^2(As, K) = \langle \delta e_1^*, \delta e_2^*, \ldots, \delta e_m^* \rangle$ where $e_i^*(e_j) = \delta_{ij}$ and δ_{ij} is the Kronecker delta.*

This lemma is modified from [9] in Jordan algebra case to associative algebra, where $B^2(As, K)$ is coboundary of associative algebras.

We need a list of non-isomorphism algebra in two-dimensional before applying the Skjelbred-Sund method by using central extension to extend in three-dimensional algebras form.

Theorem 2.1. *[6] In two-dimensional associative algebras, there are the following non-isomorphism algebra.*

As_2^1 : *abelian;*	$As_2^2 : e_1e_1 = e_2$;
$As_2^3 : e_1e_1 = e_1, e_1e_2 = e_2$;	$As_2^4 : e_1e_1 = e_1, e_2e_1 = e_2$;
$As_2^5 : e_1e_1 = e_1, e_1e_2 = e_2e_1 = e_2$;	$As_2^6 : e_1e_1 = e_1, e_2e_2 = e_2$.

3. CLASSIFICATION OF THREE-DIMENSIONAL ASSOCIATIVE ALGEBRAS

In this section, by using algebraically approach, we give a list of classification of three-dimensional associative algebras. The group of automorphism for associative algebras in dimension two (see Theorem 2.1) are needed to obtain the following results:

Lemma 3.1. *Automorphism groups of two-dimensional associative algebras over $\mathbb{C}$ has in the following form:*

$Aut(As_2^1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, *where* $dim(Aut(As_2^1)) = 4$;

$Aut(As_2^2) = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{11}^2 \end{bmatrix}$, *where* $dim(Aut(As_2^2)) = 2$;

$Aut(As_2^3) = \begin{bmatrix} 1 & 0 \\ a_{21} & a_{22} \end{bmatrix}$, *where* $dim(Aut(As_2^3)) = 2$;

$Aut(As_2^4) = \begin{bmatrix} 1 & 0 \\ a_{21} & a_{22} \end{bmatrix}$, *where* $dim(Aut(As_2^4)) = 2;$

$Aut(As_2^5) = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix}$, *where* $dim(Aut(As_2^5)) = 1;$

$Aut(As_2^6) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *where* $dim(Aut(As_2^6)) = 0.$

Proof. Let $\{e_1, e_2\}$ be a basis of two-dimensional associative algebra, As_2 and $\phi = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a nonsingular matrix where $\phi \in Aut(As_2)$. Suppose $\{e_1', e_2'\}$ be a new basis that obtain by simply multiplying ϕ with the basis i.e.,

$$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Thus, $\{e_1', e_2'\}$ can be written as follows

(3.1) $$e_1' = a_{11}e_1 + a_{21}e_2, \quad \text{and} \quad e_2' = a_{12}e_1 + a_{22}e_2.$$

Now consider the algebra $As_2^2 : e_1e_1 = e_2$. By applying the new basis (3.1) to the algebra we get the following table of multiplications:

$$\begin{aligned} e_1'e_1' &= a_{11}^2e_2 = a_{12}e_1 + a_{22}e_2, \qquad & e_1'e_2' &= a_{11}a_{12}e_2 = 0, \\ e_2'e_1' &= a_{11}a_{12}e_2 = 0, & e_2'e_2' &= a_{12}^2e_2 = 0. \end{aligned}$$

Then we have the system $a_{11}^2 = a_{22}, a_{11}a_{12} = 0, a_{12} = 0$ and a_{21} is any. By solving the system, we obtain the group of automorphism for As_2^2 as follows

$$Aut(As_2^2) = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{11}^2 \end{bmatrix}, \text{where } a_{11}^3 \neq 0.$$

Since $\{a_{11}, a_{21}\}$ is a basis of Aut(As_2^2), therefore $dim(Aut(As_2^2)) = 2.$

By applying the similar method for other algebras in Theorem 2.1, we get all the list of automorphism groups as in Lemma 3.1. □

Now we want to classify three dimensional associative algebra by using Skjelbred Sund method. To apply this method, we need to find second cohomology of two dimensional associative algebras, $H^2(As_2, \mathbb{C})$ as a quotient group of cocycle, $Z^2(As_2, \mathbb{C})$ and coboundary, $B^2(As_2, \mathbb{C})$. The cocycles and coboundaries of As_2 (see in Table 1) can be found directly by applying Definition 2.2 and Lemma 2.1 to the list of algebras in Theorem 2.1.

TABLE 1. Cocycles and coboundaries of two dimensional associative algebras

IC	Cocycle, Z^2	$dim(Z^2)$	Coboundary, B^2	$dim(B^2)$
As_2^1	$\{\triangle_{11}, \triangle_{12}, \triangle_{21}, \triangle_{22}\}$	4	$\{\}$	0
As_2^2	$\{\triangle_{11}, \triangle_{12} + \triangle_{21}\}$	2	$\{\triangle_{11}\}$	1
As_2^3	$\{\triangle_{11}, \triangle_{12}\}$	2	$\{\triangle_{11}, \triangle_{12}\}$	2
As_2^4	$\{\triangle_{11}, \triangle_{21}\}$	2	$\{\triangle_{11}, \triangle_{21}\}$	2
As_2^5	$\{\triangle_{11}, \triangle_{12} + \triangle_{21}\}$	2	$\{\triangle_{11}, \triangle_{12} + \triangle_{21}\}$	2
As_2^6	$\{\triangle_{11}, \triangle_{22}\}$	2	$\{\triangle_{11}, \triangle_{22}\}$	2

The analogous of the Skjelbred Sund method is applied to classify three-dimensional associative algebras into three parts.

1. One dimensional Central Extension of As_2^1

From Table 1, we have $H^2(As_2^1, \mathbb{C}) = span\{\triangle_{11}, \triangle_{12}, \triangle_{21}, \triangle_{22}\}$. Furthermore, the center, $C(As_2^1) = span\{e_1, e_2\}$ and $\theta^\perp = \{\}$. Suppose that $\theta \in H^2(As_2^1, \mathbb{C})$:

$$\theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\triangle_{11} + b\triangle_{12} + c\triangle_{21} + d\triangle_{22} \text{ such that } \theta^\perp \cap \{e_1, e_2\} = 0.$$

Let $\phi = (a_{ij}) \in Aut(As_2^1)$. When ϕ act on θ, we get

$$\phi \cdot \theta = a^*\triangle_{11} + b^*\triangle_{12} + c^*\triangle_{21} + d^*\triangle_{22}.$$

We write $a = a^*, b = b^*, c = c^*, d = d^*$. Futhermore, $\phi = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Thus, the ϕ acts on θ as follows

$$\phi(\theta \cdot \phi^T) = \begin{bmatrix} a_{11}(aa_{11} + ba_{12}) + a_{12}(ca_{11} + da_{12}) & a_{11}(aa_{21} + ba_{22}) + a_{12}(ca_{21} + da_{22}) \\ a_{21}(aa_{11} + ba_{12}) + a_{22}(ca_{11} + da_{12}) & a_{21}(aa_{21} + ba_{22}) + a_{22}(ca_{21} + da_{22}) \end{bmatrix}.$$

Based on the action above, we describe the following relations :

$$a^* = a_{11}(aa_{11} + ba_{12}) + a_{12}(ca_{11} + da_{12}), \quad b^* = a_{11}(aa_{21} + ba_{22}) + a_{12}(ca_{21} + da_{22}),$$
$$c^* = a_{21}(aa_{11} + ba_{12}) + a_{22}(ca_{11} + da_{12}), \quad d^* = a_{21}(aa_{21} + ba_{22}) + a_{22}(ca_{21} + da_{22}).$$

We consider two cases : $a \neq 0$ and $a = 0$.

(a) For case $a \neq 0$. Choosing $a^* = 1$, by taking $a_{12} = 0$, then we have $a_{11} = \frac{1}{\sqrt{a}}$.

$$a^* = 1, \quad b^* = a_{21} + ba_{22},$$
$$c^* = a_{21} + ca_{22}, \quad d^* = a_{21}^2 + ba_{21}a_{22} + ca_{21}a_{22} + da_{22}^2.$$

Choosing $c^* = 0$, by taking $a_{21} = 0$, then $a^* = 1$, $b^* = ba_{22}$, $c^* = 0$, and $d^* = da_{22}^2$.

For case $b = 0$. By depending $d = 0$ or not, we would have two representatives $[1, 0, 0, 0]$ and $[1, 0, 0, 1]$. Our algebras are:

$$As_3^1 : e_1e_1 = e_3, \qquad As_3^2 : e_1e_1 = e_3, e_2e_2 = e_3.$$

For case $b \neq 0$. We would have a representatives $[1, 1, 0, \alpha]$. Then the algebra is: $As_3^3 : e_1e_1 = e_3, e_1e_2 = e_3, e_2e_2 = \alpha e_3$.

(b) For case $a = 0$. By taking $a_{12} = 0$, we get $a^* = 0, b^* = ba_{11}a_{22}$, $c^* = ca_{11}a_{22}$, and $d^* = ba_{21}a_{22} + ca_{21}a_{22} + da_{22}^2$.

For case $b \neq 0$. By taking $b^* = 1$, we choose $a_{22} = 1, a_{11} = \frac{1}{b}$. Then $a^* = 0, b^* = 1$, $c^* = c$, and $d^* = a_{21}(1 + c) + d$.

If $c = -1$, by depending $d = 0$ or not, we would have two representatives $[0, 1, -1, 0]$ and $[0, 1, -1, \alpha]$. Hence, we obtain :

$$As_3^4 : e_1e_2 = e_3, e_2e_1 = -e_3, \qquad As_3^6 : e_1e_2 = e_3, e_2e_1 = -e_3, e_2e_2 = \alpha e_3.$$

If $c \neq -1$, by depending $d = 0$ or not, we would have two representatives $[0, 1, 1, 0]$ and $[0, 1, 1, \alpha]$. Our algebras are :

$$As_3^8 : e_1e_2 = e_3, e_2e_1 = e_3, \qquad As_3^7 : e_1e_2 = e_3, e_2e_1 = e_3, e_2e_2 = \alpha e_3.$$

2. One dimensional Central Extension of As_2^2

In Table 1 gives $H^2(As_2^2, \mathbb{C}) = span\{\triangle_{12} + \triangle_{21}\}$. Furthermore, the center, $C(As_2^2) = span\{e_2\}$ and $\theta^{\perp} = \{\}$. Suppose that $\theta \in H^2(As_2^2, \mathbb{C})$:

$$\theta = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} = a\triangle_{12} + a\triangle_{21} \text{ such that } \theta^{\perp} \cap \{e_2\} = 0.$$

Let $\phi = (a_{ij}) \in Aut(As_2^2)$. When ϕ act on θ, we obtain $\phi \cdot \theta = a^*\triangle_{12} + a^*\triangle_{21}$.

We write $a = a^*$. Futhermore we have $\phi = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{11}^2 \end{bmatrix}$. Thus, ϕ acts on θ as follows:

$$\phi(\theta \cdot \phi^T) = \begin{bmatrix} 0 & aa_{11}^3 \\ aa_{11}^3 & 2aa_{11}^2a_{21} \end{bmatrix}.$$

Based on the the action above, we describe the following relations: $a^* = aa_{11}^3$. We consider the case $a \neq 0$ and $a_{11} \neq 0$, by taking $a_{11} = \dfrac{1}{a^3}$. Thus, $a^* = 1$. Therefore we obtain the algebra $As_3^5 : e_1e_1 = e_2, e_2e_1 = e_3$.

3. One dimensional Central Extension of $As_2^3, As_2^4, As_2^5, As_2^6$.

We consider $H^2(As_2^3, \mathbb{C}) = H^2(As_2^4, \mathbb{C}) = H^2(As_2^5, \mathbb{C}) = H^2(As_2^6, \mathbb{C}) = span\{\}$. Furthermore, the center, $C(As_2^3) = C(As_2^4) = C(As_2^5) = C(As_2^6) = span\{\}$ and

$\theta^{\perp} = \{\}$. Suppose that $\theta \in H^2(As_2^3, \mathbb{C}) = H^2(As_2^4, \mathbb{C}) = H^2(As_2^5, \mathbb{C}) = H^2(As_2^6, \mathbb{C})$: $\theta = 0$ such that $\theta^{\perp} \cap \{\} = 0$.

Since $\theta = 0$, the automorphism group, ϕ does not act on $\theta \in H^2(As_2^3, \mathbb{C}) = H^2(As_2^4, \mathbb{C}) = H^2(As_2^5, \mathbb{C}) = H^2(As_2^6, \mathbb{C})$. Thus, there is no central extension of one dimensional associative algebra for $As_2^3, As_2^4, As_2^5, As_2^6$.

From the calculation above, we have the following algebras.

$As_3^1 : e_1e_1 = e_3;$ $As_3^2 : e_1e_1 = e_3, e_2e_2 = e_3;$
$As_3^3 : e_1e_1 = e_3, e_1e_2 = e_3, e_2e_2 = \alpha e_3;$ $As_3^4 : e_1e_2 = e_3, e_2e_1 = -e_3;$
$As_3^5 : e_1e_1 = e_2, e_2e_1 = e_3;$ $As_3^6 : e_1e_2 = e_3, e_2e_1 = -e_3, e_2e_2 = \alpha e_3;$
$As_3^7 : e_1e_2 = e_3, e_2e_1 = e_3, e_2e_2 = \alpha e_3;$ $As_3^8 : e_1e_2 = e_3, e_2e_1 = e_3.$

Now, some isomorphism invariants such as cocycle, coboundary, center, radical, maximum commutative subalgebra, maximum albelian subalgebra, centroid, derivation and automorphism are applied to investigate the isomorphishm between these algebras. The dimension of isomorphism invariants for three dimensional associative algebras presented in Table 2.

TABLE 2. Dimension of isomorphism Invariants for three dimensional associative algebras

IC	Z^2	B^2	$\theta^{\perp}$	C	ϕ	n_{As}	Com	d	Γ
As_3^1	5	1	0	2	5	2	3	6	6
As_3^2	4	2	1	1	4	1	3	4	3
As_3^3	4	3	1	1	4	1	1	5	3
As_3^4	4	1	1	1	6	1	1	5	3
As_3^5	2	2	2	1	3	1	1	3	4
As_3^6	4	1	1	1	4	1	1	4	3
As_3^7	4	1	1	1	4	1	3	4	3
As_3^8	4	1	1	1	4	1	3	4	3

From Table 2, there exists two algebras which isomorphic each other which is $As_3^7 \cong As_3^8$. By using Maple Programme (see [10]), we obtain the following matrix:

$$\begin{bmatrix} a_{11} & 0 & a_{13} \\ -\frac{1}{2}\alpha a_{22} & a_{22} & a_{23} \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}.$$

Summarizing all the observations above, we have the following list of algebra:

Proposition 3.1. *There exist seven explicit isomorphism representatives of three dimensional associative algebras over complex given as follows:*

$As_3^1 : e_1e_1 = e_3$; $As_3^2 : e_1e_1 = e_3, e_2e_2 = e_3$;
$As_3^3 : e_1e_1 = e_3, e_1e_2 = e_3, e_2e_2 = \alpha e_3$; $As_3^4 : e_1e_2 = e_3, e_2e_1 = -e_3$;
$As_3^5 : e_1e_1 = e_2, e_2e_1 = e_3$; $As_3^6 : e_1e_2 = e_3, e_2e_1 = -e_3, e_2e_2 = \alpha e_3$;
$As_3^7 : e_1e_2 = e_3, e_2e_1 = e_3$.

The following lemma shows centroids and derivations of three dimensional associative algebras:

Lemma 3.2. *Centroids and derivations of three dimensional associative algebras over* $\mathbb{C}$ *has the form as in Table 3.*

Table 3: Centroid and derivations of three dimensional associative algebras

IC	Centroid, Γ	dim	Derivation, d	dim
As_3^1	$\begin{bmatrix} a_{33} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	5	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 2a_{11} \end{bmatrix}$	5
As_3^2	$\begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{33} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	3	$\begin{bmatrix} a_{22} & -a_{21} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 2a_{22} \end{bmatrix}$	4
As_3^3	$\begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{33} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	3	$\begin{bmatrix} a_{33} - a_{22} & -a(2a_{22} - a_{33}) & 0 \\ 2a_{22} - a_{33} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	4
As_3^4	$\begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{33} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	3	$\begin{bmatrix} a_{33} - a_{22} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	6
As_3^5	$\begin{bmatrix} a_{22} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}$	3	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{32} & 2a_{11} & 0 \\ a_{31} & a_{32} & 3a_{11} \end{bmatrix}$	4

As_3^6	$\begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{33} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	3	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{32} & 2a_{11} & 0 \\ a_{31} & a_{32} & 3a_{11} \end{bmatrix}$	4
As_3^7	$\begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{33} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	3	$\begin{bmatrix} a_{33}-a_{22} & -\alpha a_{22}+\frac{1}{2}aa_{33} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 3a_{11} \end{bmatrix}$	4

Proof. Let $\{e_1, e_2, e_3\}$ be a basis of three dimensional associative algebras, As_3. For centroid case. By definition of centroid, ϕ and algorithm in [8], we have $\phi \in \Gamma(As_3)$ as follows:

$$\phi = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ where } |\phi| \neq 0.$$

Then, we get the centroid for each basis:

$$\phi(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3, \quad \phi(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3,$$

$$\phi(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3.$$

Similar for derivation. By using the definition of derivation, d as stated in Preliminary section and algorithm in [8], then we have

$$d = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ where } |d| \neq 0.$$

Thus, the derivation of the basis can be written as follows:

$$d(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3, \quad d(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3,$$

$$d(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3.$$

Now, since in dimension three we have e_ie_j, where $i, j = 1, 2, 3$, thus nine cases are needed to consider.

Here we show detail calculation to find centroid and derivation of As_3^2 only. By using the algebra $As_3^2 : e_1e_1 = e_3, e_2e_2 = e_3$ (from Proposition 3.1) in nine cases into definition of centroid, we obtain $a_{33} = a_{22} = a_{11}, a_{12} = a_{13} = a_{21} = a_{23} = 0$. Thus the centroid of As_3^2 is:

$$\Gamma(As_3^2) = \begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{33} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

For derivation case, element d of the $d(As_3^2)$ are substitute into equation $d(e_ie_j) = d(e_i)e_j + e_id(e_j)$. We build the system of equation that we obtain from nine cases, then we get: $a_{13} = a_{23} = 0, a_{33} = 2a_{22} = 2a_{11}, a_{12} = -a_{21}$. As a result:

$$d(As_3^2) = \begin{bmatrix} a_{22} & -a_{21} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 2a_{22} \end{bmatrix}.$$

By using both similar methods for all algebras in Proposition 3.1, we obtain all centroids and derivations of three dimensional algebras as shown in table above (see Lemma 3.2). □

From Lemma 3.2 we can conclude the result in the following corollary.

Corollary 3.1. 1. *The dimension of centroids for two dimensional complex algebras are three and five.*
2. *2. The dimension of derivation for two dimensional complex algebras are four, five and six.*

4. Conclusion

In this paper, seven invariant classes of three dimensional associative algebra are obtained. We also provide centroids and derivations of algebras.

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