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SUBSEQUENCES OF CESÀRO CONVERGENT SEQUENCES

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ABSTRACT. Different types of convergence of sequences of real numbers and their properties have been studied by many authors in recent years. In these studies, frequently the relationship between the convergence of a sequence and its subsequences has been in focus. In this paper, we aim to revisit a classical type of convergence, Cesàro convergence and prove some new results on the related convergence of subsequences of a sequence.

1. INTRODUCTION

Summability of real valued sequences has been a subject of study for many mathematicians over the last century and in recent years. Many different types of convergence including statistical, Cesàro, almost and ideal convergence, and related properties have been researched. In some studies, the relationship of a sequence and its subsequences regarding some type of summability was investigated. For this purpose two different gauges of size were used: Lebesgue measure and Baire category, yielding many interesting results.

Buck [4] has initiated the study of the relationship between the convergence of a given sequence and the summability of its subsequences. Agnew [1], Buck [5], Buck and Pollard [6], Miller [11], Miller and Orhan [12], Zeager [18]

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have studied this relation with respect to some new types of convergence. Later on, in [2, 8, 9, 13, 14, 16, 17] additional kinds of convergence of a sequence and the related summability of its subsequences were studied, using Lebesgue measure as a gauge of the size of the set of convergent subsequences. Also, similar relations between sequences and their subsequences were studied, using category, by several authors, ([3, 10, 15]).

In this paper we will revisit a familiar type of convergence, known as Cesàro convergence. Our aim is to study the relationship of a Cesàro summable sequence and its subsequences, as it has been done for other types of convergence, using Lebesgue measure and Baire category as gauges of size. We will prove some results analogous to earlier results regarding statistical, uniform statistical and almost convergence ([11, 12, 16, 17]).

2. Preliminaries

Now let us recall some necessary notions. Let $x = \{x_n\}$ be a sequence of real numbers. The sequence x is said to be Cesàro convergent to a real number L if

$$\frac{\sum_{i=1}^{n} x_i}{n} \longrightarrow L.$$

It is well known that the class of Cesàro convergent sequences (strictly) contains the class of convergent sequences.

Let $K \subseteq \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers. If $m, n \in \mathbb{N}$, by $K(m,n)$ we denote the cardinality of the set of numbers i in K such that $m \leq i \leq n$. The numbers

$$\underline{d}(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n}, \quad \bar{d}(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n}$$

are called the lower and the upper asymptotic density of the set K, respectively. If $\underline{d}(K) = \bar{d}(K)$, then it is said that $d(K) = \underline{d}(K) = \bar{d}(K)$ is the asymptotic density of K.

The concept of statistical convergence has been introduced in [7] as follows: Let $x = \{x_n\}$ be a sequence of real numbers. The sequence x is said to be statistically convergent to a real number L provided that for every $\varepsilon > 0$ we have $d(K_\varepsilon) = 0$, where $K_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$. If $x = \{x_n\}$ converges statistically to L, then we write $st - \lim x = L$.

It is easy to prove that every bounded statistically convergent sequence is also Cesàro convergent to the same limit.

Uniform statistical convergence and almost convergence are types of summability that are similar, but more restrictive than statistical and Cesàro convergence, respectively. Their properties and the related summability of subsequences were extensively studied, using Lebesgue measure and category as gauges of the size of the set of convergent subsequences (see [12, 16, 17]).

Subsequences of a sequence x can be naturally identified with numbers $t \in (0, 1]$ written by a binary expansion with infinitely many 1's. Thus we can denote by $\{x(t)\}$ the subsequence of x corresponding to t.

3. Main results

If a sequence $x = \{x_n\}$ is convergent, then all of its subsequences converge to the same limit. The summability of subsequences of a sequence was studied with regards to many types of convergence. One of the classical results of this kind was proved by Miller in [11].

Theorem 3.1. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0, 1]$. Then x statistically converges to L if and only if the set of $t \in (0, 1]$ for which $x(t)$ statistically converges to L has Lebesgue measure 1.*

Concerning category, in place of measure, Miller proved the next theorem.

Theorem 3.2. *Suppose $x = \{x_n\}$ is a divergent sequence of reals. The set of $t \in (0, 1]$ for which $x(t)$ is statistically convergent is meager.*

Here we prove an analogue of Theorem 3.1 concerning Cesàro convergence.

Theorem 3.3. *Suppose $x = \{x_n\}$ is a bounded sequence of reals in $(0, 1]$. Then x is Cesàro convergent if and only if the set of $t \in (0, 1]$ for which $x(t)$ is Cesàro convergent has Lebesgue measure 1. In this case, almost all subsequences of x (in the sense of Lebesque measure) have the same Cesàro limit as x.*

Proof. First, suppose that x Cesàro converges to L. Observe the independent random variables X_n, $n = 1, 2, \ldots$, where X_n takes on the values x_n, 0 with equal probability $\dfrac{1}{2}$. Clearly $E(X_n) = \dfrac{x_n}{2}$, and $Var(X_n) = \dfrac{x_n^2}{8}$ for $n \in \mathbb{N}$.

Since X_n are independent and $Var(X_n)$ are uniformly bounded, by the Kolmogorov strong law of large numbers:

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \longrightarrow \frac{x_1 + x_2 + \cdots + x_n}{2n}$$

almost surely, i.e.,

$$P\left(\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} - \frac{\sum_{i=1}^{n} x_i}{2n} = 0\right) = 1. \tag{3.1}$$

Since $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i}{n} = L$, from (3.1) we get

$$P\left(\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} - \frac{L}{2} = 0\right) = 1. \tag{3.2}$$

Let N denote the set normal numbers in $(0,1]$, i.e. the set of $t \in (0,1]$, $t = 0.t_1, t_2, \ldots, t_n, \ldots$ (binary representation with infinitely many $1'$s) for which the asymptotic density of $1'$ s ($0'$s) is exactly $\frac{1}{2}$. It is well known that $m(N) = 1$.

Now, suppose that $t \in N$. Let $m_n \leq n$ denote the number of $1'$s among the first n digits of t. Then

$$\left|\frac{\sum_{i=1}^{m_n}(x(t))_i}{2m_n} - \frac{\sum_{i=1}^{m_n}(x(t))_i}{n}\right| = \left|\frac{n}{2m_n} - 1\right| \cdot \left|\frac{\sum_{i=1}^{m_n}(x(t))_i}{n}\right|.$$

Since $\frac{\sum_{i=1}^{m_n}(x(t))_i}{n}$ is bounded and since $t \in N$ implies that $\left|\frac{n}{2m_n} - 1\right| \longrightarrow 0$ we have that

$$\left|\frac{\sum_{i=1}^{m_n}(x(t))_i}{2m_n} - \frac{\sum_{i=1}^{m_n}(x(t))_i}{n}\right| \longrightarrow 0$$

as $n \to \infty$. Since $t \in N$ was arbitrary and $m(N) = 1$, we conclude that

$$m\left(\{t \in (0,1] : \lim_{n\to\infty} \frac{\sum_{i=1}^{m_n}(x(t))_i}{2m_n} - \frac{\sum_{i=1}^{m_n}(x(t))_i}{n} = 0\}\right) = 1, \tag{3.3}$$

where m_n is defined as earlier (the number of $1'$s among the first n digits of t).

We can rewrite equation (3.2) as:

$$m\left(\{t \in (0,1] : \lim_{n\to\infty} \frac{\sum_{i=1}^{m_n}(x(t))_i}{n} - \frac{L}{2} = 0\}\right) = 1. \tag{3.4}$$

Combining equations (3.3) and (3.4) we conclude that

$$m\left(\{t \in (0,1] : \lim_{n\to\infty} \frac{\sum_{i=1}^{m_n}(x(t))_i}{2m_n} - \frac{L}{2} = 0\}\right) = 1,$$

i.e.,

$$m\left(\{t \in (0,1] : \lim_{n\to\infty} \frac{\sum_{i=1}^{m_n}(x(t))_i}{m_n} - L = 0\}\right) = 1.$$

This completes the proof in one direction.

Now, for the other direction, suppose that $x(t)$ is Cesàro convergent for almost all $t \in (0,1]$.

Let X denote the set of $t \in (0,1]$ for which $x(t)$ is Cesàro convergent. Then X has measure 1. Since $m(X) = 1$ implies that $m(1-X) = 1$ where $1 - X = \{1 - t : t \in (0,1]\}$, and $m(N) = 1$, we can fix some $t \in X \cap (1-X) \cap N$.

Now, t and $1-t$ are both normal and $x(t)$ and $x(1-t)$ are both Cesàro convergent. Let $L_1 = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(x(t))_i}{n}$ and $L_2 = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(x(1-t))_i}{n}$

Now suppose n is arbitrarily fixed. Let n_1 denote the number of $1'$s among the first n indices of t, and n_2 the number of $0'$s among the first n indices of t.

Then,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i}{n} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n_1}(x(t))_i}{n} + \lim_{n\to\infty} \frac{\sum_{i=1}^{n_2}(x(1-t))_i}{n} =$$

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n_1}(x(t))_i}{n_1} \cdot \frac{n_1}{n} + \lim_{n\to\infty} \frac{\sum_{i=1}^{n_2}(x(1-t))_i}{n_2} \cdot \frac{n_2}{n}.$$

Now, if we let $n \to \infty$, we have that $n_1 \to \infty$, $n_2 \to \infty$, and that $\frac{n_1}{n} \to \frac{1}{2}$, $\frac{n_2}{n} \to \frac{1}{2}$.

From the above we can conclude that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i}{n} = \frac{L_1 + L_2}{2}.$$

Hence x is Cesàro convergent. From the first part of the proof, consequently almost all subsequences have the same Cesàro limit as x, so the proof is complete. □

Next we prove an analogue of Theorem 3.2 for Cesàro convergence.

Theorem 3.4. *Suppose $x = \{x_n\}$ is a divergent sequence of reals. The set of $t \in (0,1]$ for which $x(t)$ is Cesàro convergent is meager.*

Proof. If $\lim_{n\to\infty} x_n = \infty$ or $-\infty$, then all of its subsequences have the same limit and hence are not Cesàro convergent so the theorem holds in this case. Now suppose that x has at least two distinct limit points. First consider the case when

x has two distinct finite limit points, a and b. For $t \in (0,1]$ we define $\overline{x(t)}$ to be the sequence of means of $x(t)$, i.e. $\overline{x(t)}_n = \dfrac{\sum_{i=1}^{n}(x(t))_i}{n}$.

Let

$$X_{a,\varepsilon,m} = \{t \in (0,1] : \text{ there exists } n > m, \overline{x(t)}_n \in (a-\varepsilon, a+\varepsilon)\}.$$

Define $X_{b,\varepsilon,m}$ analogously. We will show that $X_{a,\varepsilon,m}$, $X_{b,\varepsilon,m}$ are comeager for m, ε.

Let m and ε be fixed. Let x_{n_j}, $j = 1, 2 \ldots$, denote a subsequence of x with limit a.

Let $\bar{t} = (t_1, t_2, \ldots, t_d)$ be an arbitrary fixed finite sequence of $0's$ and $1's$. Suppose there are exactly k $1's$ among them, we can assume WLOG that $k \geq 1$. It is sufficient to show that there exists a finite extension t^* of $\bar{t}$ such that every $t \in [0,1)$ starting with t^* is in $X_{a,\varepsilon,m}$. Let $M = |y_1 + y_2 + \cdots + y_d|$ where $y_i = x_i$ if $t_i = 1$ and $y_i = 0$ if $t_i = 0$.

Let g denote a fixed positive integer large enough that $\frac{k}{k+g}|a| < \frac{\varepsilon}{4}$, $k + g > m$ and $\frac{M}{k+g} < \frac{\varepsilon}{2}$.

Now fix j_0, such that $n_{j_0} > d$ and $\left|x_{n_j} - a\right| < \frac{\varepsilon}{4}$ for $j \geq j_0$.

Consider the following extension of $\bar{t}$

$$t^* = (t_1, t_2, \ldots, t_d, \ldots, t_{n_{j_0}}, \ldots, t_{n_{j_0+g-1}})$$

where for $i > d$: $t_i = 1 \; for \; i = n_j, j_0 \leq j \leq j_0 + g - 1$ and $x_i = 0$, otherwise. Then for every $t \in [0,1)$ that extends t^* we have that:

$$\begin{aligned}\left|\overline{x(t)}_n - a\right| &= \left|\frac{\sum_{i=1}^{n}(x(t))_i}{n} - a\right| \leq \left|\frac{\sum_{i=1}^{d} y_i}{n}\right| + \left|\frac{\sum_{i=0}^{g-1} x_{n_{j_0+i}} - ga}{n}\right| + \left|\frac{g}{n} - 1\right| \cdot |a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon\end{aligned}$$

for $n = k + g$. Hence every $t \in [0,1)$ that extends t^* is in $X_{a,\varepsilon,m}$.

This proves that $X_{a,\varepsilon,m}$, and likewise $X_{b,\varepsilon,m}$ is comeager for any ε, m. Let $X_a = \bigcap_{j,m} X_{a,\frac{1}{j},m}$, $X_b = \bigcap_{j,m} X_{b,\frac{1}{j},m}$. Then X_a represents the set of all $t \in (0,1]$ for which a is a limit point of $\overline{x(t)}_n$ and X_b represents the set of all $t \in (0,1]$ for which b is a limit point of $\overline{x(t)}_n$. Clearly X_a and X_b are both comeager and hence $X_a \cap X_b$ is also comeager. But this means that the set of all $t \in (0,1]$ for which $\overline{x(t)}_n$ has both a and b as limit points and hence is not convergent is comeager, so the theorem is proved in this case.

In the case of x having two distinct limit points of which one or both are infinite, an analogous construction can be made, proving the theorem in that case. □

4. Sequences of $0'$s and $1'$s

Finally we give our attention to sequences of $0'$s and $1'$s. Here we are able to prove a few simple but revealing facts connecting statistical convergence and Cesàro convergence.

Theorem 4.1. *Suppose $x = \{x_n\}$ is a sequence of $0'$ s and $1'$s. Then x Cesàro converges to l, $l \in [0,1]$, if and only if $d(\{n : x_n = 1\}) = l$.*

Proof. Suppose x is Cesàro convergent to $l \in [0,1]$. Then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i}{n} = l,$$

which means that

$$\lim_{n\to\infty} \frac{|\{i : 1 \leq i \leq n, x_i = 1\}|}{n} = l.$$

Hence $d(\{n : x_n = 1\}) = l$. The converse also follows, considering the above equations. □

Theorem 4.2. *Suppose $x = \{x_n\}$ is a sequence of $0'$s and $1'$s. Then x statistically converges to 0 (1) if and only if x Cesàro converges to 0 (1).*

Proof. We treat the case when the limit is 0. Suppose x statistically converges to 0. From the definition, then clearly $d(\{n : x_n = 1\}) = 0$. Hence by Theorem 4.1, x Cesàro converges to 0. Conversely if x Cesàro converges to 0, by Theorem 4.1, $d(\{n : x_n = 1\}) = 0$. Consequently since x contains only $0'$s and $1'$s, $d(\{n : |x_n| > \varepsilon\}) = 0$ for each $\varepsilon > 0$, so x statistically converges to 0. The case when the limit is 1 works analogously, so the proof is complete. □

We also mention that from Theorem 4.1 it is clear that all normal sequences of $0'$s and $1'$s Cesàro converge to $\frac{1}{2}$, which implies that almost all (in the sense of Lebesgue measure) sequences of $0'$s and $1'$s have Cesàro limit $\frac{1}{2}$.

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