ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.8, 613–633
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.8.1

EXISTENCE RESULTS OF GENERALIZED PROPORTIONAL FRACTIONAL DIFFERENTIAL EQUATIONS AT RESONANCE CASE BY THE TOPOLOGICAL DEGREE THEORY

M. Azouzi[1], L. Guedda, and Z. Laadjal

ABSTRACT. In this paper we study the existence of solutions to multi-point boundary value problem of fractional differential equations at resonance, involving the Generalized Proportional Fractional derivative(GPF derivatives). the concerned results are obtained via extansion of Mawhin's continuation theorem. An illustrative example is presented.

1. INTRODUCTION

Recently, the theory of fractional differential equations (FDEs) is the subject of numerous research works. resulting from the modeling of various problems of physics, chemistry and biology,. . . . See, [8–11].

Indeed, many techniques are always used to prove the existance of solutions for ordinary and fractional equations. The readers can be referred to [6, 7, 13, 14], but here we interest on using the topological degree theory. This methode is an effective tool for the existance of solutions to boundary value problems (BVPs for short). See [15–17].

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 34E10, 34A45, 43A46, 34A50, 65CXX.
Key words and phrases. Generalized proportional fractional derivative, resonance, Mawhin's continuation theorem.
Submitted: 21.07.2022; *Accepted:* 05.08.2022; *Published:* 09.08.2022.

In [1], Fahd, *et al.*, proposed *the generalized proportional fractional derivatives* (GPF for short) and integrals. It have three advantages: the generated fracional integrals have a semi-group property, the kernal of the fractional operator include exponential function, and when the order ρ tends to 1 reduce to the Riemann-Liouville and Caputo fractional derivative and integral.

The aim of this paper is to study the existance of solutions for a class of fractional differential equations by using the extension of Mawhin's continuation theorem, More specifically, we consider the following generalized proportional fractional differential equation, with multi-point boundary conditions of the form:

$$ {}^{c}\mathfrak{D}_0^{\alpha,\rho}u(t) = f(t, u(t), {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(t)) \qquad 0 < t < 1 \tag{1.1} $$

$$ u(0) = 0, \tag{1.2} $$

$$ {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(1) = \sum_{i=1}^{i=m} \sigma_i \, {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(\eta_i), \tag{1.3} $$

where ${}^{c}\mathfrak{D}_0^{\alpha,\rho}$ denote the generalized proportional fractional derivative of Caputo type of order $\alpha \in (1,2], \rho \in (0,1]$, $0 < \eta_i < 1$, $\sigma_i \in \mathbb{R}$, $\sum\limits_{i=1}^{i=m} \sigma_i = 1,\ m \in \mathbb{N}^*$, and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

To investigate the problem, we use the condition

$$ \sum_{i=1}^{i=m} \sigma_i \eta_i^{2-\alpha} e^{-\delta(1-\eta_i)} = 1 \ , \tag{1.4} $$

where $\delta = \frac{\rho-1}{\rho}$.

The article is organized as following: Section 2, we give some definitions and lemmas, In Section 3, we will demonstrete a theorem of existence of solutions for the problem (1.1)-(1.2)-(1.3). Finally, we give an example to prove our results.

2. Preliminaries about the fractional calculus and coincidence degree theory

In this section we give some definitions and lemmas from the theory of fractional calculus.We start by defining Generalized Proportional fractional integrals and derivatives. these definitions are adopted from [12], [5].

Definition 2.1 ([1]). *For $\rho \in (0,1]$ and $\alpha > 0$. The left generalized proportional fractional integral of of f is defined by*

$$(\mathcal{J}_a^{\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{\frac{\rho-1}{\rho}(t-s)} f(s)\, ds, \tag{2.1}$$

where $t \in [a,b]$.

Definition 2.2 ([1]). *For $\rho \in (0,1]$ and $\alpha > 0$. The left generalized proportional fractional derivative of Caputo type of the function $f \in C^{(n)}[a,b]$ is given by*

$$\begin{aligned} {}^{\mathcal{C}}\mathfrak{D}_a^{\alpha,\rho} f(t) &= \mathcal{J}_a^{n-\alpha,\rho} (D^{n,\rho} f)(t) \\ &= \frac{1}{\rho^\alpha \Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} e^{\frac{\rho-1}{\rho}(t-s)} (D^{n,\rho} f)(s)\, ds, \end{aligned} \tag{2.2}$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $(D^{1,\rho} f)(t) = (D^\rho f)(t) = (1-\rho) f(t) + \rho f'(t)$, and

$$(D^{n,\rho} f)(t) = (\underbrace{D^\rho D^\rho \cdots D^\rho}_{n \text{ times}} f)(t), \quad \text{for } n \geq 1. \tag{2.3}$$

Remark 2.1. *In the case $\rho = 1$, the definitions 2.1 and 2.2 reduce to a left Riemann–Liouville fractional integral and left Caputo fractional derivative, respectively.*

Remark 2.2 ([5]). *We can writeen the formula (2.3) for $\rho \in (0,1]$, as follows*

$$(D^{n,\rho} f)(t) = \rho^n f^{(n)}(t) + \sum_{k=0}^{n-1} C_n^k \rho^k (1-\rho)^{n-k} f^{(k)}(t), \tag{2.4}$$

where $C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proposition 2.1 ([1]). *For $\rho \in (0,1]$, and $\alpha, \beta \in \mathbb{C}$ such that $\alpha > 0$, $\beta > 0$ and n is the integer part of α then for $f \in L^1[0,1]$ we have:*

$$\mathcal{J}_0^{\alpha,\rho} \mathcal{J}_0^{\beta,\rho} f(t) = \mathcal{J}_0^{\beta,\rho} \mathcal{J}_0^{\alpha,\rho} f(t) = \mathcal{J}_0^{\alpha+\beta,\rho} f(t), \tag{2.5}$$

$$(\mathcal{J}_0^{\alpha,\rho} t^{\beta-1} e^{\delta t})(x) = \frac{\Gamma(\beta)}{\rho^\alpha \Gamma(\alpha+\beta)} x^{\alpha+\beta-1} e^{\delta x}, \tag{2.6}$$

$$({}^{\mathcal{C}}\mathfrak{D}_0^{\alpha,\rho} t^{\beta-1} e^{\delta t})(x) = \frac{\rho^\alpha \Gamma(\beta)}{\Gamma(\beta-\alpha)} x^{\beta-\alpha-1} e^{\delta x}, \quad \Re(\beta) > n, \tag{2.7}$$

$${}^{\mathcal{C}}\mathfrak{D}_0^{\alpha,\rho} \mathcal{J}_a^{\alpha,\rho} f(t) = f(t), \tag{2.8}$$

$$(2.9)\qquad \mathcal{J}_0^{\alpha,\rho}\left({}^{\mathcal{C}}\mathfrak{D}_0^{\alpha,\rho} f\right)(t) = f(t) - \sum_{k=0}^{k=n-1} c_k t^k e^{\delta t},\ f \in C^{(n)}[0,1],$$

where $c_k = \frac{(D^{k,\rho} f)(a)}{\rho^k k!}$.

Definition 2.3 ([5]). *Let* $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$, *and* $t > 0$. *The lower incomplete Gamma function is defined by*

$$(2.10)\qquad \gamma(\alpha, t) = \int_0^t y^{\alpha-1} e^{-y} dy.$$

Also, the lower regularized incomplete Gamma function is defined by

$$(2.11)\qquad \mathfrak{P}(\alpha, t) = \frac{\gamma(\alpha, t)}{\Gamma(\alpha)}.$$

Remark 2.3. *The function* $\mathfrak{P}$ *is also called "Incomplete Gamma function".*

Lemma 2.1 ([5]). *Let* $\alpha, \eta \in \mathbb{R}^+, \alpha \geq 0$ *It is clear that* $\mathfrak{P}(\alpha, t)$ *is a non-decreasing function with respect to* $t \in [0, 1]$. *And moreover*

$$(2.12)\qquad \int_{t_1}^{t_2} y^{\alpha-1} e^{-y} dy = \gamma(\alpha, t_2) - \gamma(\alpha, t_1), t_2 \geq t_1 > 0,$$

$$(2.13)\qquad \mathfrak{P}(\alpha, t) \in [0, 1] \text{ for all } t \in [0, 1],$$

$$(2.14)\qquad \max_{t\in[0,1]} \mathfrak{P}(\alpha, t)|_{t=1} = \mathfrak{P}(\alpha, 1),$$

$$(2.15)\qquad \min_{t\in[0,1]} \mathfrak{P}(\alpha, \eta(t-a)) = \mathfrak{P}(\alpha, t)|_{t=0} = 0.$$

Lemma 2.2 ([5]). *Let* $\rho \in (0, 1]$, $t_1, t_2 \in [0, 1]$ $(t_1 \leq t_2)$, *and* $\alpha > 0$. *Then*

$$(2.16)\qquad \int_{t_1}^{t_2} (1-s)^{\alpha-1} e^{\delta(b-s)}) ds = \frac{\rho^\alpha \Gamma(\alpha)}{(1-\rho)^\alpha} \left[\mathfrak{P}(\alpha, -\delta(1-t_1)) - \mathfrak{P}(\alpha, -\delta(1-t_2))\right].$$

Lemma 2.3 ([5]). *Let* $\rho \in (0, 1]$, *and* $0 \leq t_1 \leq t_2 \leq 1$. *For all* $0 < \alpha > 0$, *then*

$$(2.17)\qquad \lim_{t_2 \to t_1} \int_0^{t_1} \left|(t_2 - s)^{\alpha-1}\, e^{\delta(t_2-s)} - (t_1 - s)^{\alpha-1} e^{\delta(t_1-s)}\right| ds = 0.$$

Lemma 2.4 ([5]). *Let* $\rho \in (0,1]$*,* $\beta > 0$*, and* $g_\beta(t) = e^{\delta t}t^\beta, t \in [0,1]$*, then*

$$\max_{t\in[0,1]} g_\beta(t) = \begin{cases} (\frac{-\beta}{\delta e})^\beta, & \text{if} \quad -\frac{\beta}{\delta} \in [0,1], \\ \\ e^\delta, & \text{if} \quad -\frac{\beta}{\delta} \notin [0,1] \text{ or } \rho = 1. \end{cases} \tag{2.18}$$

Lemma 2.5. *For* $\theta > 0$*, the linear space*

$$C^\theta[0,1] = \left\{ u : u(t) = \mathcal{J}^{\theta,\rho}z(t) + \sum_{k=0}^{k=[\theta]} c_k t^k e^{\frac{\rho-1}{\rho}t},\ z \in C[0,1] \right\},$$

where $[\theta]$ *is the integer part of* θ *and* $c_k \in \mathbb{R}$ *with the norm*

$$||u||_{C^\theta} = ||u||_\infty + \sum_{k=1}^{k=[\theta]} \left\| {}^c\mathfrak{D}_0^{\alpha-1,\rho} u \right\|_\infty,$$

is a Banach space.

Lemma 2.6 ([12]). $M \subset C^\theta\,[0,1]$ *is relatively compact set if and only if*

(1) M *is uniformly bounded: there exists* $m > 0$*, such that* $\|u\|_{C^\theta} \leq m$*, for every* $u \in M$*.*

(2) M *is equicontinuous: for every* $\varepsilon > 0$*, there exists* $\delta > 0$*, such that for all* $t_1, t_2 \in [0,1]\,, |t_2 - t_1| < \delta$*, we have*

$$|u\,(t_2) - u\,(t_1)| < \varepsilon \text{ and } \left| {}^c\mathfrak{D}_0^{\theta-k,\rho} u\,(t_2) - {}^c\,\mathfrak{D}_0^{\theta-k,\rho} u\,(t_1) \right| < \varepsilon,$$

for all $u \in M$ *with* $k = 0, 1, 2, \ldots, [\theta]\,.$

Next, we present the notations and nomenclatures with regard to the coincidence degree, see ([12]).

Definition 2.4 ([12]). *Let* X, Y *be two real Banach spaces,* Ω *be an open bounded subset of* X*, and* $L : dom(L) \subset X \to Y$ *is a linear operator,* $N : X \to Y$ *is nonlinear mapping. If* $\operatorname{Im} L$ *is a closed set of* Y *and* $\dim\ker(L) = co\dim\ \operatorname{Im}(L) < +\infty$*, then* L *is called a Fredholm operator of index zero. In this case there exist two linear continuous projectors* $P : X \to X$*,* $Q : Y \to Y$ *such that* $\operatorname{Im} P = \ker L$*, and* $\ker Q = \operatorname{Im} L$ *and we can write* $X = \ker(L) \oplus \ker(P)$*,* $Y = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$*. It follows that* $L_P = L_{|dom(L)\cap\ker P} : dom(L) \cap \ker P \to \operatorname{Im} L$ *is invertible. We denote the inverse of* L_P *by* K_P*. If* $dom(L) \cap \bar{\Omega} \neq \emptyset$*, the mapping* N *will be called* L*-compact on* Ω *if* $QN(\bar{\Omega})$ *is bounded and* $K(I-Q)N : \bar{\Omega} \to X$ *is compact.*

Theorem 2.1. *Let X, Y be two real Banach spaces, $L : dom(L) \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be an L-compact mapping on Ω. Assume that the following conditions are satisfied:*

(1). $Lu \neq \lambda Nu$ *for all* $(u, \lambda) \in (dom(L) \backslash \ker L) \cap \partial\Omega \times (0,1)$*,*

(2). $QNu \neq 0$ *for all* $x \in \ker L \cap \partial\Omega$,

(3). $\deg(QN_{|\ker L}, \Omega \cap \ker L, 0) \neq 0$.

Then the equation $Lu = Nu$ *has at least one solution in* $dom\ L \cap \bar{\Omega}$.

In this paper,we will consider the Banach spaces

$$X = C^{\alpha-1}[0,1] = \left\{u : u(t) = \mathcal{J}^{\alpha-1,\rho} z(t); z(t) \in C[0,1]\right\},$$

with the norm

$$||u||_X = ||u||_\infty + \left\|{}^c\mathfrak{D}_0^{\alpha-1,\rho} u\right\|_\infty,$$

where $||u||_\infty = \max_{t\in[0,1]} |u(t)|$ and $Y = L^1[0,1]$ with the norm $||y||_Y = ||y||_1$.

Define the two operators $L, N : dom(L) \subset X \to Y$ as follows:

$$Lu(.) = {}^c\mathfrak{D}_0^{\alpha,\rho} u(t) \quad u \in dom(L), \tag{2.19}$$

and

$$Nu(t) = f(t, u(t), {}^c\mathfrak{D}_0^{\alpha-1,\rho} u(t)), \tag{2.20}$$

where

$$dom(L) = \Big\{u \in X \text{ s.t. } \mathfrak{D}_0^{\alpha,\rho} u(t) \in L^1[0,1], u(0) = 0,$$
$$\left. {}^c\mathfrak{D}_0^{\alpha-1,\rho} u(1) = \sum_{i=1}^{i=m} \sigma_i\ {}^c\mathfrak{D}_0^{\alpha-1,\rho} u(\eta_i)\right\}.$$

Notice that problem(1.1)-(1.2)-(1.3) can be converted to the abstract operator equation $Lu = Nu$, $u \in dom(L)$.

3. The existence of solutions to multi-point boundary value problem of fractional differential equations at resonance, involving the Generalized Proportional Fractional derivative

Firstly, we consider the following constants:

$$\Delta = \int_0^1 e^{\delta(1-s)}\, ds - \sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)} ds, \tag{3.1}$$

$$L_1 = \max_{0<t\leq 1} \left|te^{\delta t}\right| = \max\{\frac{-1}{\delta e}, e^{\delta}\}, \tag{3.2}$$

$$L_2 = \max_{0<t\leq 1} \left|t^{2-\alpha}e^{\delta t}\right| = \max\{(\frac{\alpha-2}{\delta e})^{2-\alpha}, e^{\delta}\}, \tag{3.3}$$

$$\kappa = \frac{\rho^{\alpha}\Gamma(\alpha)e^{2\delta}}{\rho\Gamma(\alpha)(L_1+L_2)+e^{\delta}(1+\rho^{\alpha-1}\Gamma(\alpha))} \tag{3.4}$$

and the function

$$\Lambda(t) = \frac{\rho^{\alpha-1}}{\Gamma(3-\alpha)}t^{2-\alpha}e^{\delta t}, t \in [0,1]. \tag{3.5}$$

Also, we define the two linear operators $I_1, I_2 : Y \to Y$ by

$$I_1 y = \int_0^1 e^{\delta(1-s)}y(s)ds, \tag{3.6}$$

and

$$I_2 y = \sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)} y\left(s\right) ds. \tag{3.7}$$

3.1. **Some auxiliary lemmas.**

Lemma 3.1. *Let L be the operator defined by (2.19), then*

$$\ker L = \left\{cte^{\delta t} : c \in \mathbb{R}\right\} \quad \textit{and} \quad \operatorname{Im} L = \{y \in Y : \ I_1 y - I_2 y = 0\}$$

Proof. For each $u \in \ker L$, we have $Lu\left(t\right) = {}^{c}\mathfrak{D}_0^{\alpha,\rho}u\left(t\right) = 0$. So, it's equivalent to

$$u(t) = c_0 e^{\delta t} + c_1 t e^{\delta t},\ t \in \left[0,1\right].$$

As condition (1.2), imply $c_0 = 0$. So, $u\left(t\right) = c_1 t e^{\delta t}$.

Now, for all $y \in \operatorname{Im} L$, there exist $u \in dom\left(L\right)$ such that

$${}^{c}\mathfrak{D}_0^{\alpha,\rho}u(t) = y\left(t\right).$$

By (2.9) we get

$$u\left(t\right) = \mathcal{J}_0^{\alpha,\rho}y\left(t\right) + c_0 e^{\delta t} + c_1 t e^{\delta t}.$$

From the condition (1.2), we find $c_0 = 0$, and in view of conditions (1.3)-(1.4), we obtain that

$$u(t) = c_1 t e^{\delta t} + \mathcal{J}_0^{\alpha,\rho} y(t).$$

Thus

$${}^c\mathfrak{D}_0^{\alpha-1,\rho} u(t) = c_1 \left({}^c\mathfrak{D}_0^{\alpha-1,\rho} t e^{\delta t}\right) + \frac{1}{\rho}\int_0^t e^{\delta(t-s)} y(s) ds.$$

Applying (2.7) we get

$${}^c\mathfrak{D}_0^{\alpha-1,\rho} u(t) = \frac{c_1 \rho^{\alpha-1}}{\Gamma(3-\alpha)} t^{2-\alpha} e^{\delta t} + \frac{1}{\rho}\int_0^t e^{\delta(t-s)} y(s) ds.$$

From ${}^c\mathfrak{D}_0^{\alpha-1,\rho} u(1) = \sum\limits_{i=1}^{i=m} \sigma_i \, {}^c\mathfrak{D}_0^{\alpha-1,\rho} u(\eta_i)$,we obtain

$$\begin{aligned}\frac{c_1 \rho^{\alpha-1}}{\Gamma(3-\alpha)} e^{\delta} + \frac{1}{\rho}\int_0^1 e^{\delta(1-s)} y(s) ds &= \sum_{i=1}^{i=m} \sigma_i \frac{c_1 \rho^{\alpha-1}}{\Gamma(3-\alpha)} \eta_i^{2-\alpha} e^{\delta \eta_i} \\ &\quad + \frac{1}{\rho}\sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)} y(s) ds,\end{aligned}$$

also

$$\int_0^1 e^{\delta(1-s)} y(s) ds = \sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)} y(s) ds.$$

We conclude that

$$I_1 y - I_2 y = 0. \tag{3.8}$$

On other hand, suppose that $y \in Y$ satisfies (3.8).

If $u(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\delta(t-s)} y(s) ds$, then $u \in dom(L)$, indeed $u(t) = \mathcal{J}_0^{\alpha-1,\rho} \mathcal{J}_0^{1,\rho} y(t)$ and we can easily show the boundary conditions(1.2)-(1.3) hold, which means that ${}^c\mathfrak{D}_0^{\alpha,\rho} u(t) = y(t)$ so $y \in \operatorname{Im}(L)$. □

Remark 3.1. *It easy to show that* $\Delta \neq 0$.

Proof. As $\frac{1}{\delta} < 0$, suffices to proof that $\sum\limits_{i=1}^{i=m} \sigma_i \left(1 - e^{\delta \eta_i}\right) + e^{\delta} - 1 < 0$, or $\sum\limits_{i=1}^{i=m} \sigma_i \left(1 - e^{\delta \eta_i}\right) + e^{\delta} - 1 > 0$, By the resonance condition (1.4), we have

$$\sum_{i=1}^{i=m} \sigma_i \left(1 - e^{\delta \eta_i}\right) + e^{\delta} - 1 = \sum_{i=1}^{i=m} \sigma_i \, e^{\delta \eta_i} \left(\eta_i^{2-\alpha} - 1\right).$$

In fact, for all $i = 1, 2, \ldots, m$, $\sigma_i \geq 0, \eta_i \in]0;1[$, $0 < e^{\delta\eta_i} < 1$, $\sigma_i e^{\delta\eta_i} \geq 0$, and $\left(\eta_i^{2-\alpha} - 1\right) < 0$, we have $\sigma_i\, e^{\delta\eta_i}\left(\eta_i^{2-\alpha} - 1\right) \leq 0$, by the condition $\sum_{i=1}^{i=m} \sigma_i = 1$, there exists at least $i_0 \in \{1, 2, \ldots, m\}$ such that $\sigma_{i_0} \neq 0$ and hence $\sigma_{i_0}\, e^{\delta\eta_{i_0}}\left(\eta_{i_0}^{2-\alpha} - 1\right) < 0$ which prove that

$$\sum_{i=1}^{i=m} \sigma_i\, e^{\delta\eta_i}\left(\eta_i^{2-\alpha} - 1\right) < 0.$$

Therefore $\Delta \neq 0$. □

Lemma 3.2. *We can define two linear continuous projectors P and Q as follow*

$$P : X \to X \text{ such that } Pu\,(t) = \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} te^{\delta t}\, {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u(1),$$

and

$$Q : Y \to Y \text{ such that } Qy(t) = \frac{1}{\Delta}(I_1 y - I_2 y),$$

where $\Delta \neq 0$.

The inverse of the operator $L_P = L_{|dom(L)\cap \ker P}$ is the operator $K_P : \operatorname{Im} L \to dom\,(L) \cap \ker P$ defined by

$$K_P y(t) = \mathcal{J}_0^{\alpha,\rho} y\,(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\delta(t-s)} y(s) ds,$$

and checking

$$\|K_P y\|_X \leq C' \|y\|_1 , \tag{3.9}$$

where $C' = \frac{1+\rho^{\alpha-1}\Gamma(\alpha)}{\rho^{\alpha}\Gamma(\alpha)}$.

Proof. For all $u \in X$, we get

$${}^{c}\mathfrak{D}_0^{\alpha-1,\rho} Pu\,(1) = {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u\,(1)\,.$$

We have

$${}^{c}\mathfrak{D}_0^{\alpha-1,\rho} Pu\,(t) = \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} \frac{\rho^{\alpha-1}}{\Gamma(3-\alpha)}\, {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u\,(1)\, e^{\delta} = {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u\,(1)\,,$$

then

$$P\left(Pu(t)\right) = \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} te^{\delta t}\, {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} Pu(1) = Pu(t),$$

for $t \in [0, 1]$.

We note that $\operatorname{Im} P = \ker L$, $\ker P = \left\{u \in X; {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u\,(1) = 0\right\}$.

From Lemma 2.6 we have

$$\begin{aligned} |Pu(t)| &= \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} \left| te^{\delta t} \, {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u(1) \right| \\ &\leq \frac{L_1 \Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} \left| {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u(1) \right| \\ &\leq \frac{L_1}{\rho^{\alpha-1}e^{\delta}} \|u\|_X , \end{aligned}$$

on other hand

$$\left| {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} Pu(t) \right| \leq \frac{L_2}{\rho^{\alpha-1}e^{\delta}} \left| {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u(1) \right| \leq \frac{L_2}{\rho^{\alpha-1}e^{\delta}} \|u\|_X . \tag{3.10}$$

Then

$$\|Pu\|_X = \|Pu\|_\infty + \left\| {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} Pu \right\|_\infty \leq \left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right) \|u\|_X . \tag{3.11}$$

For all $y \in Y$, taking $\frac{1}{\Delta}(\;\; I_1 y - I_2 y) = \upsilon$, thus

$$\begin{aligned} Q^2 y &= Q\,(Qy) \\ &= \frac{1}{\Delta} \left(\int_0^1 e^{\delta(1-s)} \; \upsilon ds - \sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)} \upsilon ds \right) \\ &= \frac{\upsilon}{\Delta} \Delta = \upsilon, \end{aligned}$$

so $Q^2 = Q$. Furthermore, we have

$$\|Qy\|_1 \leq C \|y\|_1 . \tag{3.12}$$

where $C = \frac{1+\Delta}{\Delta}$

For any $u \in dom\,(L) \cap \ker P$, by proposition 2.1 - 2.9 we can write

$$K_P L u\,(t) = \mathcal{J}_0^{\alpha,\rho} \; {}^{c}\mathfrak{D}_0^{\alpha,\rho} u\,(t) = u(t) + c_0 e^{\delta t} + c_1 t e^{\delta t},$$

with $t \in (0,1]$ and c_0, c_1 are two real constants. As $K_P L x \in dom\,(L) \cap \ker P$, then $c_0 = 0$ and

$$\begin{aligned} {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} \left(u\,(t) + c_1 t e^{\delta t} \right)\Big|_{t=1} &= {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} u\,(1) + c_1 \frac{\rho^{\alpha-1}}{\Gamma(3-\alpha)} e^{\delta} \\ &= c_1 \frac{\rho^{\alpha-1}}{\Gamma(3-\alpha)} e^{\delta} = 0, \end{aligned}$$

which imply that $c_1 = 0,$ therefore $K_P L u = u$.

If $y \in \operatorname{Im} L$,we get $LK_P y(t) =^c \mathfrak{D}_0^{\alpha,\rho} \mathcal{J}_0^{\alpha,\rho} y(t) = y(t)$ which show that $K_P = (L_P)^{-1}$., and the other hand

$$\begin{aligned} \|K_P y\|_X &= \|\mathcal{J}_0^{\alpha,\rho} y\|_\infty + \left\|\mathcal{J}_0^{1,\rho} y\right\|_\infty \\ &\leq C' \|y\|_1 . \end{aligned}$$

This complete the proof. □

Lemma 3.3. *L is a Fredholm operator of index 0.*

Proof. For any $y \in Y$, we can write $y = (I - Q) y + Qy$. $(I - Q) y \in \ker Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$ then $y \in \operatorname{Im} L + \operatorname{Im} Q$. Assume that $y \in \operatorname{Im} L \cap \operatorname{Im} Q$ thus $y = c \in \mathbb{R}$ such that

$$I_1 c - I_2 c = 0 = c\left(\int_0^1 e^{\delta(1-s)} ds - \sum_{i=1}^{i=m} \sigma_i \int_0^{\eta_i} e^{\delta(\eta_i - s)} ds\right) = \Delta c = 0,$$

i.e $c = 0$, which imply that $\operatorname{Im} L \cap \operatorname{Im} Q = \phi$ an hence $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. As $\dim \operatorname{Im} Q = \dim \ker L = 1$, then L is a Fredholm operator of index 0. □

Lemma 3.4. *Assume that M is an open bounded subset in X such that $dom(L) \cap M \neq \phi$.The operator N L-compact on $\overline{M}$.*

Proof. The boundness of M imply that there exists $R > 0$ such that for all $u \in M$, we have $\|u\|_X = \|u\|_\infty + \left\|{}^c\mathfrak{D}_0^{\alpha-1,\rho} u\right\|_\infty \leq R$. By the continuity of f there exists $A > 0$ such that $|f(s, u(s))| \leq A$ for all $u \in M$. So, we get

$$\|QNu\|_1 \leq C \|Nu\|_1 \leq CA,$$

and

$$\|(I - Q) Nu\|_1 \leq \|Nu\|_1 + \|QNu\|_1 \leq (C + 1)A, \tag{3.13}$$

we conclude that

$$\|K_P (I - Q) Nu\|_X \leq C' \|(I - Q) Nu\|_\infty \leq C'(C + 1)A,$$

then $QN(M)$ and $K_{P,Q}N(M)$ are bounded, we only need to prove that $K_{P,Q}N(M)$ is equicontinuous. Putting $0 \leq t_1 \leq t_2 \leq 1$,

$$\begin{aligned}
&\left|K_P\left(I-Q\right)Nu\left(t_2\right)-K_P\left(I-Q\right)Nu\left(t_1\right)\right|\\
&=\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\left(I-Q\right)Nu(s)ds\right.\\
&\quad\left.-\int_0^{t_1}(t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\left(I-Q\right)Nu(s)ds\right|\\
&=\frac{1}{\rho^\alpha\Gamma(\alpha)}\left(\left|\int_0^{t_1}(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\left(I-Q\right)Nu(s)ds\right.\right.\\
&\quad-\int_0^{t_1}(t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\left(I-Q\right)Nu(s)ds\\
&\quad\left.\left.+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\left(I-Q\right)Nu(s)ds\right|\right)\\
&\le\frac{1}{\rho^\alpha\Gamma(\alpha)}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}-(t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\right|\left|\left(I-Q\right)Nu(s)\right|ds\right.\\
&\quad\left.+\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\right|\left|\left(I-Q\right)Nu(s)\right|ds\right)\\
&\le\frac{\left\|\left(I-Q\right)Nu\right\|_1}{\rho^\alpha\Gamma(\alpha)}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}-(t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\right|ds\right.\\
&\quad\left.+\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}\right|ds\right).
\end{aligned}$$

Using Lemma 2.2 and the inequality (3.13) we get

$$\begin{aligned}
&\left|K_P\left(I-Q\right)Nu\left(t_2\right)-K_P\left(I-Q\right)Nu\left(t_1\right)\right|\\
\le\ &\frac{(C+1)A}{\rho^\alpha\Gamma(\alpha)}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}e^{\delta(t_2-s)}-(t_1-s)^{\alpha-1}e^{\delta(t_1-s)}\right|ds\right.\\
+\ &\frac{(C+1)A}{(1-\rho)^\alpha}\left[\mathfrak{P}(\alpha,-\delta(t_2-t_1))-0\right].
\end{aligned}$$

From Lemma 2.3 we obtain

$$\left|K_P\left(I-Q\right)Nu\left(t_2\right)-K_P\left(I-Q\right)Nu\left(t_1\right)\right|\to 0 \text{ as } t_1\to t_2.$$

On other hand, we have

$$\begin{aligned}{}^{c}\mathfrak{D}_0^{\alpha-1,\rho} K_P\left(I-Q\right) Nu\left(t\right) &= \mathcal{J}_0^{1,\rho}\left(I-Q\right)Nu(t)\\ &= \frac{1}{\rho}\int_0^t e^{\delta(t-s)}\left(I-Q\right)Nu\left(s\right)ds.\end{aligned}$$

Similarity,

$$\begin{aligned} &\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho} K_P\left(I-Q\right) Nu\left(t_2\right) - {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} K_P\left(I-Q\right) Nu\left(t_1\right)\right|\\ = &\frac{1}{\rho}\left|\int_0^{t_2} e^{\delta(t_2-s)}\left(I-Q\right)Nu(s)ds - \int_0^{t_1} e^{\delta(t_1-s)}\left(I-Q\right)Nu(s)ds\right|\\ \leq &\frac{(C+1)A}{\rho}\left(\int_0^{t_1}\left|e^{\delta(t_2-s)} - e^{\delta(t_1-s)}\right|ds + \int_{t_1}^{t_2} e^{\delta(t_2-s)}ds\right),\end{aligned}$$

from Lemma 2.3 (with $\alpha = 1$) we get

$$\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho} K_P\left(I-Q\right) Nu\left(t_2\right) - {}^{c}\mathfrak{D}_0^{\alpha-1,\rho} K_P\left(I-Q\right) Nu\left(t_1\right)\right| \to 0 \text{ as } t_1 \to t_2.$$

According to the Lemma 2.6, $K_P\left(I-Q\right)N\left(M\right)$ is compact, which show that N is L-compact on M. □

3.2. An existence theorem for the Generalized Proportional fractional differential equations.

Theorem 3.1. *Suppose that there exists:*

(C_1) *There exists a L^1-Carathéodory function $\Phi : [0,1]\times\mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ which is non decreasing with respect to the last two variables such that*

$$\left|f\left(t,x,y\right)\right| \leq \Phi\left(t,\left|x\right|,\left|y\right|\right),$$

for all $(x;y)\in\mathbb{R}^2$ and $t\in[0,1]$.

(C_2) *A real $M_0 > 0$, such that if we have $\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u\left(t\right)\right| > M_0$ for all $t\in[0,1]$, then*

$$I_1 f(t,u(t),{}^{c}\mathfrak{D}_0^{\frac{1}{4},\frac{1}{2}}u(t)) - I_2 f(t,u(t),{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(t)) \neq 0.$$

(C_3) *A real $M_1 > 0$, such that for $|c| > M_1$, then either*

$$c\left(I_1 N(cte^{\delta t}) - I_2 N(cte^{\delta t}\right) > 0, \tag{3.14}$$

or

$$c(I_1N(cte^{\delta t}) - I_2N(cte^{\delta t})) < 0, \tag{3.15}$$

then the fractional BVPs (1.1)-(1.2)-(1.3) has at least one solution in $dom\,(L) \subset X$*, provided that*

$$\int_0^1 \Phi\,(t,r,r)\,dt \le \frac{\rho^{\alpha}\Gamma(\alpha)e^{2\delta}}{\rho\Gamma(\alpha)(L_1+L_2)+e^{\delta}(1+\rho^{\alpha-1}\Gamma(\alpha))}r+\beta. \tag{3.16}$$

Where β *is a positive constant.*

Proof. **Step1**: Let

$$\Omega_1 = \{u \in dom\,(L) \setminus \ker L : Lu = \lambda Nu, \lambda \in [0,1]\}$$

We will show that it is a bounded set. Notice that if $u \in \Omega_1$ then $\lambda \in (0,1]$, because $\Omega_1 \cap \ker L = \phi$,which allows us to write $Nu = L\frac{1}{\lambda}u \in \operatorname{Im} L = \ker Q$, then

$$\begin{aligned} QNu = &\int_0^1 e^{\delta(1-s)} f(s,u(s),{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(s))ds \\ &-\sum_{i=1}^{i=m}\sigma_i\int_0^{\eta_i} e^{\delta(\eta_i-s)} f(s,u(s),{}^c\mathfrak{D}_0^{\alpha-1,\rho}u(s))ds = 0. \end{aligned}$$

By the condition (C_2), there exists $t_0 \in [0,1]$ such that $\left|\mathfrak{D}_0^{\alpha-1,\rho}u\,(t_0)\right| \le M_0$. On the other hand, we have

$$\begin{aligned} {}^c\mathfrak{D}_0^{\alpha-1,\rho}u\,(t) &= {}^c\mathfrak{D}_0^{\alpha-1,\rho}u\,(t_0) + \int_{t_0}^t e^{\delta(t-s)}\ {}^c\mathfrak{D}_0^{\alpha,\rho}u\,(s)\,ds \\ &= {}^c\mathfrak{D}_0^{\alpha-1,\rho}u\,(t_0) + \int_{t_0}^t e^{\delta(t-s)} f\left(s,u\,(s)\,,{}^c\mathfrak{D}_0^{\alpha-1,\rho}u\,(s)\right)ds, \end{aligned}$$

then

$$\begin{aligned} \left|{}^c\mathfrak{D}_0^{\alpha-1,\rho}u\,(t)\right| &\le \left|{}^c\mathfrak{D}_0^{\alpha-1,\rho}u\,(t_0)\right| + \left|\int_{t_0}^t e^{\delta(t-s)} f\left(s,u\,(s)\,,{}^c\mathfrak{D}_0^{\alpha-1,\rho}u\,(s)\right)ds\right| \\ &\le M_0 + \|Nu\|_1\,. \end{aligned} \tag{3.17}$$

Furthermore, we can write

$$\begin{aligned} u = (I-P)\,u + Pu &= K_PL\,(I-P)\,u + Pu \\ &= K_PLu + Pu, \end{aligned}$$

then

$$\|u\|_X \leq \|K_P Lu\|_X + \|Pu\|_X .$$

By using (3.17), we obtain

$$\begin{aligned} |Pu(t)| &= \frac{\Gamma(3-\alpha)}{\rho^{\alpha-1}e^{\delta}} \left|te^{\delta t}\, {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(1)\right| \\ &\leq \frac{L_1}{\rho^{\alpha-1}e^{\delta}} \left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(1)\right| \\ &\leq \frac{L_1}{\rho^{\alpha-1}e^{\delta}} \left(M_0 + \|Nu\|_1\right) \end{aligned}$$

and

$$\begin{aligned} \left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}Pu(t)\right| &= \frac{L_2}{e^{\delta}} \left|{}^{c}\mathfrak{D}_0^{\alpha-1}u(1)\right| \\ &\leq \frac{L_2}{\rho^{\alpha-1}e^{\delta}}(M_0 + \|Nx\|_1). \end{aligned}$$

Then

$$\|Pu\|_X \leq \frac{L_1 + L_2}{\rho^{\alpha-1}e^{\delta}} \left(M_0 + \|Nu\|_1\right).$$

By simple calculations, we have

$$\|K_P Lu\|_X \leq \frac{1+\rho^{\alpha-1}\Gamma(\alpha)}{\rho^{\alpha}\Gamma(\alpha)} \|Nu\|_1 ,$$

which gives

$$\|u\|_X \leq \left(\frac{L_1 + L_2}{\rho^{\alpha-1}e^{\delta}}\right) M_0 + \left(\frac{\rho\Gamma(\alpha)\left(L_1+L_2\right) + e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right) \|Nu\|_1. \tag{3.18}$$

It is easy to see that

$$|Nu(t)| = \left|f\left(s, u(s), {}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(s)\right) ds\right|. \tag{3.19}$$

According to conditions (C_1) and (3.16), we obtain

$$\begin{aligned} \int_0^1 \left|f\left(s, u(s), {}^{c}\mathfrak{D}_0^{\alpha-1}u(s)\right)\right| ds &\leq \int_0^1 \Phi\left(s, |u(s)|, \left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u(s)\right|\right) ds \\ &\leq \int_0^1 \Phi\left(s, \|u\|_X, \|u\|_X\right) ds \\ &\leq k\|u\|_X + \beta. \end{aligned} \tag{3.20}$$

Then $\|Nu\|_1 \leq \kappa.\|u\|_X + \beta$ substitude this result in (3.20) we get

$$\begin{aligned}\|u\|_X &\leq \left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right)M_0+\left(\frac{\rho\Gamma(\alpha)\left(L_1+L_2\right)+e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)\|Nu\|_1\\ &\leq \left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right)M_0+\left(\frac{\rho\Gamma(\alpha)\left(L_1+L_2\right)+e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)(\kappa.\|u\|_X+\beta)\\ &\left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right)M_0+e^{\delta}\|u\|_X+\left(\frac{\rho\Gamma(\alpha)\left(L_1+L_2\right)+e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)\beta.\end{aligned}$$

We conclude that

$$\|u\|_X \leq \frac{\left(\frac{L_1+L_2}{\rho^{\alpha-1}e^{\delta}}\right)M_0+\left(\frac{\rho\Gamma(\alpha)(L_1+L_2)+e^{\delta}\left(1+\rho^{\alpha-1}\Gamma(\alpha)\right)}{\rho^{\alpha}\Gamma(\alpha)e^{\delta}}\right)\beta}{1-e^{\delta}}.$$

Thus Ω_1 is bounded.

Step 2: Let

$$\Omega_2=\left\{u\in\ker L: Nu\in \operatorname{Im} L\right\}.$$

For all $u\in\Omega_2$ there exists a real constant c such that $u\left(t\right)=c\,te^{\delta t}, t\in[0,1]$ and as $Nu\in \operatorname{Im} L$ then

$$QNu=0.$$

In view of $\left(C_3\right)$, there exists $t_1\in[0,1]$ satisfying $\left|{}^{c}\mathfrak{D}_0^{\alpha-1,\rho}u\left(t_1\right)\right|=\left|\frac{c\rho^{\alpha-1}}{\Gamma(3-\alpha)}t_1^{2-\alpha}e^{\delta t_1}\right|\leq M_0$ i.e $|c|\leq\frac{M_0}{L_2\rho^{\alpha-1}}$, which yields that

$$\begin{aligned}\|u\|_X &= |cL_1|+\left|c\frac{L_2\rho^{\alpha-1}}{\Gamma(3-\alpha)}\right|\\ &= |c|\left(L_1+\frac{L_2\rho^{\alpha-1}}{\Gamma(3-\alpha)}\right)\\ &\leq \frac{M_0}{\rho^{\alpha-1}L_2}\left(L_1+1\right),\end{aligned}$$

then Ω_2 is bounded.

Step 3: Assume that condition $\left(C_3\right)-(3.14)$ holds. Let

$$\Omega_3=\left\{u\in\ker L:\lambda Ju+\left(1-\lambda\right)QNu=0,\lambda\in[0,1]\right\},$$

where J is the isomorphism defined by $J:\ker L\to \operatorname{Im} Q$; $J\left(c\,te^{\delta t}\right)=c.$

For $u = c\, te^{\delta t} \in \Omega_3$, we have

$$\lambda Ju + (1-\lambda)\, QNu = \lambda c + \frac{1-\lambda}{\Delta}\big(I_1 f(s, cse^{\delta s}, \Lambda(s)) \\ - I_2 f(s, cse^{\delta s}, \Lambda(s))\big) = 0. \tag{3.21}$$

If $\lambda = 0$, we get $QNu = 0$ so by the condition (C_3) there exists $t_2 \in [0,1]$ such that

$$\left|{}^c\mathfrak{D}_0^{\alpha-1,\rho} u\,(t_2)\right| = \left|\frac{c\rho^{\alpha-1}}{\Gamma(3-\alpha)} t_2^{2-\alpha} e^{\delta t_2}\right| \leq M_0.$$

So, $|c| \leq \frac{M_0}{L_2\rho^{\alpha-1}}$, and hence

$$\|u\|_X = |cL_1| + \left|c\frac{L_2\rho^{\alpha-1}}{\Gamma(3-\alpha)}\right| = |c|\left(L_1 + \frac{L_2\rho^{\alpha-1}}{\Gamma(3-\alpha)}\right) \leq \frac{M_0}{\rho^{\alpha-1}L_2}\,(L_1+1)\,.$$

In the case $\lambda \neq 0$, in view of the condition $(C_3) - (3.14)$ we get

$$-\lambda c^2 = \frac{(1-\lambda)c}{\Delta}(I_1 f(s, cse^{\delta s}, \Lambda(s)) - I_2 f(s, cse^{\delta s}, \Lambda(s))) > 0,$$

which contradict (3.14). Then $|c| \leq M_1$ which show that Ω_3 is bounded.

If $(C_3) - (3.14)$ holds, we prove by the same method that

$$\Omega_3 = \{u \in \ker L : -\lambda Ju + (1-\lambda)\, QNu = 0, \lambda \in [0,1]\}$$

is bounded set. It remains to check that all conditions of Theorem (2.1) are fulfilled. Let $\Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \Omega$. As Ω_1, Ω_2 and Ω_3 are bounded sets, then

(1) $Lu \neq \lambda Nu$ for all $(u, \lambda) \in (dom(L)\backslash \ker L) \cap \partial\Omega \times (0,1)$,

(2) $QNu \neq 0$ for all $x \in \ker L \cap \partial\Omega$,

(3) Without loss of generality, assume that $(C_3) - (3.14)$ holds and define the operator

$$F(u, \lambda) = \lambda Ju + (1-\lambda)QNu;$$

as Ω_3 is bounded then, $F(\lambda, u) \neq 0$ for all $(u, \lambda) \in (\ker L \cap \partial\Omega) \times (0,1)$ Thus, by the homotopy property of degree, we have

$$\begin{aligned} \deg(QN_{|\ker L}, \Omega \cap \ker L, 0) &= \deg(F(.,0), \Omega \cap \ker L, 0) = \deg(F(.,1), \Omega \cap \ker L, 0) \\ &= \deg(J, \Omega \cap \ker L, 0) \neq 0. \end{aligned}$$

Consequently, the equation $Lu = Nu$ has at least one solution in $dom(L) \subset X$. Namely, BVPs (1.1)-(1.2)-(1.3) has at least one solution in the space X. $\square$

4. A NUMERICAL EXAMPLE

Consider the boudary value problem

$$\begin{cases} {}^cD_0^{\frac{3}{2},\frac{1}{2}}u(t) = f\left(t, u(t), {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(t)\right), t \in [0;1] \\ u(0) = 0 \\ {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(1) = \sigma_1 {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(\eta_1) + \sigma_2 {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(\eta_2) \end{cases},$$

where

$$\alpha = \frac{3}{2}, \rho = \frac{1}{2}, \delta = \frac{\rho - 1}{\rho} = -1, \sigma_1 = \frac{6 - 2e^{\frac{8}{9}}}{3e^{\frac{3}{4}} - 2e^{\frac{8}{9}}} \simeq 0.7638..,$$

$$\sigma_2 = \frac{3e^{\frac{3}{4}} - 6}{3e^{\frac{3}{4}} - 2e^{\frac{8}{9}}} \simeq 0.23.., \eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{9},$$

and

$$f\left(t, u(t), {}^cD_0^{\frac{1}{2},\frac{1}{2}}u(t)\right) = \begin{cases} 0 & \text{if } t \in \left[0; \frac{1}{4}\right] \\ \frac{\kappa}{4e^{\frac{3}{4}}-7}\left[\sin u(t) + {}^c D_0^{\frac{1}{2},\frac{1}{2}}u(t) - 1\right]\left(t - \frac{1}{4}\right)e^{1-t} & \text{if } t \in \left[\frac{1}{4}; 1\right] \end{cases}$$

Notice that obviously,condition (A_1) holds with

$$\Phi(t, x, y) = \begin{cases} 0 & \text{if } t \in \left[0; \frac{1}{4}\right] \\ \frac{\kappa}{4e^{\frac{3}{4}}-7}[x + y + 1]\left(t - \frac{1}{4}\right)e^{1-t} & \text{if } t \in \left[\frac{1}{4}; 1\right] \end{cases} \tag{4.1}$$

and

$$\begin{aligned} \int_0^1 \Phi(t, r, r)dt &= \int_{\frac{1}{4}}^1 \Phi(t, r, r)dt = \frac{(2r+1)\kappa}{4e^{\frac{3}{4}} - 7} e \int_{\frac{1}{4}}^1 \left(t - \frac{1}{4}\right) e^{-t} dt \\ &= \frac{(2r+1)\kappa}{4e^{\frac{3}{4}} - 7} \cdot \left(\frac{4e^{\frac{3}{4}} - 7}{4}\right) = \frac{\kappa}{2} r + \frac{\kappa}{4} \leq \kappa r + \frac{\kappa}{4}. \end{aligned} \tag{4.2}$$

Choosing $M_0 = 3$ Assume that $|{}^c D_0^{\frac{1}{2},\frac{1}{2}}u(t)| > M_0$ for each $t \in [0;1]$. if $|{}^c D_0^{\frac{1}{2},\frac{1}{2}}u(t)| > M_0$ for all $t \in [0;1]$ we have

$$f\left(t, u(t), {}^c D_0^{\frac{1}{2},\frac{1}{2}}u(t)\right) > \frac{\kappa}{4e^{\frac{3}{4}} - 7}\left[-1 + M_0 - 1\right]\left(t - \frac{1}{4}\right)e^{1-t},$$

then

$$
\begin{aligned}
I_1 N u &= \int_{\frac{1}{4}}^{1} e^{-(1-s)} f\left(s, u(s),{}^{c} D_0^{\frac{1}{2}, \frac{1}{2}} u(s)\right) ds > \frac{\kappa}{4 e^{\frac{3}{4}}-7}\left[-2+M_0\right] \int_{\frac{1}{4}}^{1}\left(s-\frac{1}{4}\right) ds \\
&= \frac{9 \kappa}{32\left(4 e^{\frac{3}{4}}-7\right)}\left(M_0-2\right)>0 \\
I_2 N u &= \sigma_1 \cdot \int_0^{\frac{1}{4}} e^{-\left(\frac{1}{4}-s\right)} f\left(s, u(s),{}^{c} D_0^{\frac{1}{2}, \frac{1}{2}} u(s)\right) ds+\sigma_2 \\
&\quad \cdot \int_0^{\frac{1}{9}} e^{-\left(\frac{1}{9}-s\right)} f\left(s, u(s),{}^{c} D_0^{\frac{1}{2}, \frac{1}{2}} u(s)\right) ds=0 .
\end{aligned}
$$

On the other hand, if ${}^{c}D_0^{\frac{1}{2},\frac{1}{2}} u(t) \leq -M_0$ for all $t \in \left[\frac{1}{4}; 1\right]$ we get

$$
f\left(t, u(t),{}^{c} D_0^{\frac{1}{2}, \frac{1}{2}} u(t)\right)<\frac{\kappa}{4 e^{\frac{3}{4}}-7}\left[1-M_0-1\right]\left(t-\frac{1}{4}\right) e^{1-t},
$$

so,

$$
\begin{aligned}
I_1 N u &= \int_{\frac{1}{4}}^{1} e^{-(1-s)} f\left(s, u(s),{}^{c} D_0^{\frac{1}{2}, \frac{1}{2}} u(s)\right) ds<\frac{\kappa}{4 e^{\frac{k}{4}}-7}\left(-M_0\right) \int_{\frac{1}{4}}^{1}\left(s-\frac{1}{4}\right) ds \\
&= -\frac{9 \kappa}{32\left(4 e^{\frac{2}{4}}-7\right)} M_0<0 \\
I_2 N u &= 0 .
\end{aligned}
$$

This assure that the condition (A_2) is satisfied.

Taking $M_1 = 7$,

$$
\begin{aligned}
c\left(I_1 N\left(c t e^{-t}\right)-I_2 N\left(c t e^{-t}\right)\right)=c I_1 N\left(c t e^{-t}\right)=c & \int_{\frac{1}{4}}^{1} \frac{\kappa}{4 e^{\frac{3}{4}}-7} \\
& \cdot\left[\sin \left(\mathrm{cte}^{-t}\right)+\frac{c}{\sqrt{2}} \frac{\sqrt{s} e^{-s}}{\Gamma\left(\frac{3}{2}\right)}-1\right]\left(s-\frac{1}{4}\right) ds \\
= & \frac{\kappa}{4 e^{\frac{3}{4}}-7} c[A c+B]
\end{aligned}
$$

where

$$
A=\frac{1}{\sqrt{2} \cdot \Gamma\left(\frac{3}{2}\right)} \int_{\frac{1}{4}}^{1}\left(s-\frac{1}{4}\right) \sqrt{s} e^{-s} ds, B=\int_{\frac{1}{4}}^{1} \sin \left(\text{ cte }^{-t}-1\right)\left(s-\frac{1}{4}\right) ds .
$$

We have $A = \sqrt{\frac{2}{\pi}} \int_{\frac{1}{4}}^{1} \left(s - \frac{1}{4}\right) \sqrt{s} e^{-s} ds \simeq 9.0401 \times 10^{-2} > 0, |B| \leq \int_{\frac{1}{4}}^{1} 2\left(s - \frac{1}{4}\right) ds = \frac{9}{16}$ thus $\left|\frac{B}{A}\right| \leq \frac{\frac{9}{16}}{0.09} = \frac{100}{16} = 6.25$. If $c > 7$, we have $-c < -7 < -6.25 \leq \frac{B}{A}$ then $Ac + B > 0$ which imply that $c\left(I_1 N\left(cte^{-t}\right) + I_2 N\left(cte^{-t}\right)\right) > 0$ If $c < -7$, we get $\frac{B}{A} \leq 6.25 < 7 < -c$ thus $Ac + B < 0$, therefore $c\left(I_1 N\left(cte^{-t}\right) + I_2 N\left(cte^{-t}\right)\right) > 0$ Then condition is fufilled and the problem 1 has at least one solution in $C^{\frac{1}{2}}[0;1]$

References

[1] F. Jarad, T. Abdeljawad, J. Alzabut: *Generalized fractional derivatives generated by a class of local proportional derivatives*, Eur. Phys. J. Spec. Top. **226** (2017), 3457-3471.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi: *Higher Transcendental Functions*, Vol. II. McGraw-Hill, New York, (1953).

[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo: *Theory and Applications of Fractional Differential Equations*, Elsevier B.V., Netherlands, 2006.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi: *Higher Transcendental Functions*, Vol. II. McGraw-Hill, New York, (1953).

[5] Z. Laadjal, T. Abdeljawad, F. Jarad: *On existence–uniqueness results for proportional fractional differential equations and incomplete gamma functions*, Adv. Differ. Equ. **2020** (2020), art. no. 641.

[6] Z. Laadjal, T. Abdeljawad, F. Jarad: *Sharp estimates of the unique solution for two-point fractional boundary value problems with conformable derivative*, Numer. Methods Partial Differential Eq., (2021), 1-10,

[7] Z. Laadjal, Q.H. Ma: *Existence and uniqueness of solutions for nonlinear Volterra-Fredholm integro-differential equation of fractional order with boundary conditions*, Math. Meth. Appl. Sci., **44**(10) (2019), 8215-8227.

[8] Y. Chen, X. Tang: *Positive solutions of fractional di erential equations at resonance on the half-line*, Bound. Value Probl. **2012**(64) (2012), 13 pp.

[9] H.E. Zhang: *Nonlocal boundary value problems of fractional order at resonance with integral conditions*, Adv. Difference Equ. (2017), paper no. 326, 12 pp.

[10] M. Shaoib, T. Abdeljawad, M. Sarwar, F. Jarad: *Fixed point theorems for multi-valued contractions in metric spaces with applications to fractional differential and integral equations*, IEEE Access, 7, (2019), 127373-127383.

[11] B. Alqahtani, A. Fulga, F. Jarad, E. Karapinar: *Nonlinear F-contractions on b-metric spaces and differential equations in the frame of fractional derivatives with Mittag-Leffler kernel*, Chaos Soliton Fract., **128** (2019), 349-354.

[12] J. Mawin: *Topological degree and boundary value problems for nonlinear differential equations in topological methods for ordinary differential equations*, Lecture Notes Math., **1537** (1993), 74-142.

[13] J. WANG: *The existence of solutions to boundary value problems of fractional differential equations at resonance*, Nonlinear Analysis, **74** (2011), 1987-1994.

[14] S.DJEBALI AND L. GUEDDA: *Nonlocal p-Laplacian boundary value problems of fractional order at resonance*, Panamerican Mathematical Journal. **29** (2019), 45-63.

[15] A. ALI, B. SAMET, K. SHAH, ET AL: *Existence and stability of solution to a toppled systems of differential equations of non-integer order*, Bound. Value Probl., **2017** (2017), art. no. 16.

[16] R.A. KHAN AND K. SHAH: *Existence and uniqueness of solutions to fractional order multi-point boundary value problems*, Communications in Applied Analysis, **19** (2015), 515–520.

[17] M.B. ZADA, K. SHAH AND R. A. KHAN: *Existence theory to a coupled system of higher orderfractional hybrid di erential equations by topological degree theory*, Int. J. Appl. Comput. Math., **4** (2018), art.no. 102.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HAMMA LAKHDER EL-OUED
FACULTY OF EXACT SCIENCES, OPERATOR THEORY, EDP AND APPLICATIONS LABORATORY
ALGERIA.
Email address: azouzi-mounira@univ-eloued.dz

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HAMMA LAKHDER EL-OUED
FACULTY OF EXACT SCIENCES, OPERATOR THEORY, EDP AND APPLICATIONS LABORATORY
ALGERIA.
Email address: guedda-lamine@univ-eloued.dz

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY CENTER OF ILLIZI
33000 ILLIZI
ALGERIA.
Email address: z.laadjal@cuillizi.dz