Advances in Mathematics: Scientific Journal **11** (2022), no.8, 697–710
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.8.5

THE MATRIX LINEAR UNILATERAL AND BILATERAL EQUATIONS

G. Gomathi Eswari[1] and A. Rameshkumar

ABSTRACT. In this article the method of solving matrix linear equations over commutative Bezout domains by means of standard form of a pair of matrices with respect to generalized equivalence is found. The criterions of uniqueness of particular solutions of matrix linear equations are determined. The formulas of general solutions of matrix linear equations AX + BY = C and AX + YB = C are deduced.

1. INTRODUCTION

The matrix linear equations play a fundamental role in many talks in control and dynamical systems theory [1–4]. The such equations are the matrix linear bilateral equations with one and two variables

$$AX + XB = C \tag{1.1}$$

$$AX + YB = C \tag{1.2}$$

and the matrix linear unilateral equations

$$AX + BY = C, \tag{1.3}$$

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 15XX, 15A06.
Key words and phrases. Pair of matrices, Linear Unilateral Equations, Bilateral Equations.
Submitted: 31.07.2022; *Accepted:* 15.08.2022; *Published:* 24.08.2022.

where $A, B,$ and C are matrices of appropriate size over a certain field F or over a ring R. X, Y are unknown matrices. Equations (1.1), (1.2) are called Sylvester equations. The equation $AX + XA^T = C$, where matrix A^T is transpose of A, is called Lyapunov equation and it is special case of Sylvester equation. Equation (1.3) is called the matrix linear Diophantine equation [3, 4]. Roth [5] established the criterions of solvability of matrix equations (1.1), (1.2) whose coefficients $A, B,$ and C are the matrices over a field F.

2. Standard form of a pair of matrices

Let R be a commutative Bezout domain with diagonal reduction of matrices [9], that is, for every matrix A of the ring of matrices $M(n, R)$, there exist invertible matrices $U, V \in GL(n, R)$ such that

$$UAV_A = D^A = diag(\phi_1, \ldots, \phi_n), \phi_i | \phi_{i+1}, \quad i = 1, \ldots, n-1. \tag{2.1}$$

If $\phi_i \in R, i = 1, \ldots, n,$ then the matrix D^A is unique and is called the canonical diagonal form (Smith normal form) of the matrix A. Such rings are so-called adequate rings. The ring R is called an adequate if R is a commutative domain in which every finitely generated ideal is principal and for every $a, b \in R$ with $a \neq 0$; a can be represented as $a = cd$ where $(c, b) = 1$ and $(d_i, b) \neq 1$ for every nonunit factor d_i of d [10].

Definition 2.1. *The pairs* (A_1, A_2) *and* (B_1, B_2) *of matrices* $A_i, B_i \in M(n, R), i = 1, 2$ *are called generalized equivalent pairs if* $A_i = UB_iV_i, i = 1, 2$ *for some invertible matrices* U *and* V_i *over* R.

In [7,8], the forms of the pair of matrices with respect to generalized equivalence are established.

Theorem 2.1. *Let* R *be an adequate ring, and let* $A, B \in M(n, R)$ *be the nonsingular matrices and*

$$D^A = \Phi = diag(\phi_1, \ldots, \phi_n), D^B = \psi = diag(\psi_1, \ldots, \psi_n) \tag{2.2}$$

be their canonical diagonal forms then the pair of matrices (A, B) *is generalized equivalent to the pair* (D^A, T^B), *where* T^B *has the following form:*

$$T^B = \begin{Vmatrix} \psi_1 & 0 & \dots & 0 \\ t_{21}\psi_1 & \psi_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ t_{n1}\psi_1 & t_{n2}\psi_2 & \dots & \psi_n, \end{Vmatrix}$$

$t_{ij} \in R_{\delta_{ij}}$, *where* $\delta_{ij} = (\phi_i/\phi_j, \psi_i/\psi_j), i, j = 1, 2, \dots, n, i > j.$

The pair (D^A, D^B) defined in Theorem 2.2 is called the standard form of the pair of matrices (A, B) or the standard pair of matrices (A, B).

Definition 2.2. *The pair (A, B) is called diagonalizable if it is generalized equivalent to the pair of diagonal matrices(D^A, D^B) that is, its standard form is the pair of diagonal matrices (D^A, D^B).*

Example 1. *Let $A, B \in M(n, R)$. If $(\phi_n/\phi_1, \psi_n/\psi_1) = 1$, than the pair of matrices (A, B) is diagonalizable.*

It is clear taking into account by a Corollary that if $(\det A, \det B) = 1$, then the standard form of matrices (A, B) is the pair of diagonal matrices $(D^A,\ D^B)$. Let us formulate the criterion of diagonalizability of the pair of matrices [5].

Definition 2.3. *Diophantine equation is a polynomial equation usually involving two (or) more unknown variables, Such that the only solutions of interest are the integer ones. $ax + by = c$ where x, y are unknowns and a, b, c are integers.*

3. THE MATRIX LINEAR UNILATERAL EQUATIONS $AX + BY = C$

3.1. **The Construction of the Solutions of the Matrix Linear Unilateral Equations with Two Variables.**

Suppose that the matrix linear unilateral equation (1.3) is solvable, and let (D^A, T^B) be a standard form of a pair of matrices (A, B) from (1.3) with respect to generalized equivalence, that is,

$$D^A = \Phi = UAV_A = diag(\phi_1, \dots, \phi_n),$$

$$T^B = UBV_B = \begin{Vmatrix} \psi_1 & 0 & \dots & 0 \\ t_{21}\psi_1 & \psi_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ t_{n1}\psi_1 & t_{n2}\psi_2 & \dots & \psi_n \end{Vmatrix} \tag{3.1}$$

is a lower triangular matrix of the form (2.3) with the principal diagonal

$$D^B = \Psi = diag(\psi_1, \dots, \psi_n), \tag{3.2}$$

where $U, V_A, V_B \in GL(n, R)$.

Then (1.3) is equivalent to the equation

$$D^A \tilde{x} + T^B \tilde{y} = \tilde{c}, \tag{3.3}$$

where $\tilde{x} = v_A^{-1} x, \tilde{y} = v_B^{-1} y$ and $\tilde{c} = uc$.

The pair of matrices $\tilde{x_0}, \tilde{y_0}$ satisfying (3.3) is called the solution of this equation. Then

$$x_0 = V_A \tilde{x_0}, \qquad y_0 = V_B \tilde{y_0} \tag{3.4}$$

is the solution of (1.3). The matrix equation (3.3) is equivalent to the system of linear equation:

$$\begin{aligned}
\phi_1 \tilde{x}_{11} + \phi_1 \tilde{y}_{11} &= \tilde{c}_{11}, \\
\phi_1 \tilde{x}_{12} + \phi_1 \tilde{y}_{12} &= \tilde{c}_{12}, \\
&\vdots \\
\phi_1 \tilde{x}_{1n} + \phi_1 \tilde{y}_{1n} &= \tilde{c}_{1n}, \\
\phi_2 \tilde{x}_{21} + \phi_1 t_{21} \tilde{y}_{11} + \phi_2 \tilde{y}_{21} &= \tilde{c}_{21}, \\
&\vdots \\
\phi_n \tilde{x}_{nn} + \phi_1 t_{n1} \tilde{y}_{1n} + \ldots + \psi_{n-1} t_{n,n-1} \tilde{y}_{n-1,n} + \psi_n \tilde{y}_{nn} &= \tilde{c}_{nn},
\end{aligned} \tag{3.5}$$

with the variables $\tilde{x}_{ij}, \tilde{y}_{ij},\ i, j = 1, \dots, n$, where $t_{ij},\ i, j = 1, \dots, n$, from (3.3), or

$$\phi_i \tilde{x}_{ij} + \sum_{i=1}^{i=n} \psi_i t_{ij} \tilde{y}_{ij} + \phi_i \tilde{y}_{ij} = \tilde{c}_{ij},\ \ i, j = 1, \dots, n, \tag{3.6}$$

where $\tilde{x} = \|\tilde{x}_{ij}\|_1^n$, $\tilde{y} = \|\tilde{y}_{ij}\|_1^n$ and $\tilde{c} = \|\tilde{c}_{ij}\|_1^n$.

The solving of this system reduces to the successive solving of linear Diophantine equations of the form

$$\phi_i \tilde{x}_{ij} + \phi_i \tilde{y}_{ij} = \tilde{c}_{ij}. \tag{3.7}$$

Using solutions of system (3.6), we construct the solutions $\tilde{x}, \tilde{y}$ of matrix equation (3.3). Then $X = V_A \tilde{x}$ and $Y = V_B \tilde{y}$ are the solutions of matrix equation (1.3).

3.2. **The General solutions of the Matrix Equation $AX + BY = C$ with the Diagonalizable Pair of Matrices (A, B).**

Suppose that the pair of matrices (A, B) is diagonalizable, that is,

$$UAV_A = D^A = \Phi = diag(\phi_1, \dots, \phi_n),$$
$$UBV_B = D^B = \Psi = diag(\psi_1, \dots, \psi_n). \tag{3.8}$$

For some matrices $U, V_A, V_B \in GL(n, R)$. Then (1.3) is equivalent to the equation

$$\Phi\tilde{X} = \Psi\tilde{Y} = \tilde{C}, \tag{3.9}$$

where $\tilde{X} = V_A^{-1}X, \tilde{Y} = V_B^{-1}Y$ and $\tilde{C} = UC$.

From matrix equation (3.9) we get the system of linear Diophantine equation:

$$\phi_i\tilde{x}_{ij} + \phi_i\tilde{y}_{ij} = \tilde{c}_{ij} \quad i, j = 1, \dots, n. \tag{3.10}$$

Let $\tilde{x}_{ij}^{(0)}, \tilde{y}_{ij}^{(0)},\ i, j = 1, \dots, n$ be a particular solution of corresponding equation of system (3.10), that is, $\tilde{x}_{ij}^{(0)}$ is the solution of congruence $\phi_i\tilde{x}_{ij} \equiv \tilde{c}_{ij}(\mod \psi_i)$, $\tilde{x}_{ij}^{(0)} \in R_{\phi_i}$ and $\tilde{y}_{ij}^{(0)} = (\tilde{c}_{ij} - \phi_i\tilde{x}_{ij}^{(0)})/\phi_i$.

The general solution of corresponding equation of system (3.10) by the formula will have the following form:

$$\tilde{x}_{ij} = \tilde{x}_{ij}^{(0)} + \frac{\psi_i}{d_{ij}}r_i + \psi_i k_{ij}, \quad \tilde{y}_{ij} = \tilde{y}_{ij}^{(0)} + \frac{\psi_i}{d_{ij}}r_i + \phi_i k_{ij}, \quad i, j = 1, \dots, n, \tag{3.11}$$

where $d_{ij} = (\phi_i, \psi_i), r_i$ are arbitrary elements of $R_{d_{ij}}$, and k_{ij} are any elements of $R, i, j = 1, \dots, n$. The particular solution of matrix equation (3.9) is

$$\tilde{x}_0 = \|\tilde{x}_{ij}^{(0)}\|_1^n, \quad \tilde{y}_0 = \|\tilde{y}_{ij}^{(0)}\|_1^n, \tag{3.12}$$

where $\tilde{x}_{ij}^{(0)}, \tilde{y}_{ij}^{(0)}, i, j = 1, \dots, n$ is a particular solution of corresponding equation of system (3.10). Then

$$X_0 = V_A\tilde{X}_0, \quad Y_0 = V_B\tilde{Y}_0, \tag{3.13}$$

is a particular solution of matrix equation (1.3).

Theorem 3.1. *Let the pair of matrices (A, B) from matrix equation (1.3) be diagonalizable and its standard pair be the pair of matrices (Φ, Ψ) in the form (3.8). Let $\tilde{X}_0, \tilde{Y}_0$, be a particular solution of matrix equation (3.9). Then the general solution* *of matrix equation (3.9) is*

$$\tilde{X} = \tilde{X}_0 + diag(\frac{\psi_1}{d_{11}}r_1, \ldots, \frac{\psi_n}{d_{nn}}r_n)L + \Psi k,$$

$$\tilde{Y} = \tilde{Y}_0 + diag(\frac{\psi_1}{d_{11}}r_1, \ldots, \frac{\psi_n}{d_{nn}}r_n)L - \Phi k, \tag{3.14}$$

where $d_{ii} = (\phi_i, \psi_i), r_i$ *are arbitrary elements of* $R_{d_{ii}}, \ i = 1, \ldots, n$*;* $L = \|l_{ij}\|_1^n$*,* $l_{ij} = 1, \ i, j = 1, \ldots, n; k = \|k_{ij}\|_1^n$*,* k_{ij} *are arbitrary elements in R. The general solution of matrix equation (1.3) has the form* $X = V_A\tilde{X}, y = V_B\tilde{Y}$*.*

Example 2. *Consider the equation*

$$AX + BY = C. \tag{3.15}$$

For the matrices

$$A = \begin{Vmatrix} -2 & 4 \\ -6 & 8 \end{Vmatrix}, \quad B = \begin{Vmatrix} 1 & 3 \\ 5 & 7 \end{Vmatrix}, \quad C = \begin{Vmatrix} 9 & 12 \\ 11 & 10 \end{Vmatrix}, \tag{3.16}$$

are matrices over Z *and*

$$X = \begin{Vmatrix} x_{11} & x_{13} \\ x_{21} & x_{22} \end{Vmatrix}, \quad Y = \begin{Vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{Vmatrix} \tag{3.17}$$

are unknown matrices. The matrix equation (3.16) is solvable.

The pair of matrices (A, B) *from matrix equation (3.16) by a theorem is diagonalizable[6]. Let* $A, B \in M(n, R)$ *and* A *be a nonsingular matrix. Then the pair of matrices* (A, B) *is generalized equivalent to the pair of diagonal matrices* (D^A, D^B) *if, and only if, the matrices* $(adjA)B$ *and* $(adjD^A)D^B$ *are equivalent, where* $adjA$ *is an adjoint matrix[7].*

Since the matrices

$$(adjA)B = \begin{Vmatrix} 8 & -4 \\ 6 & 6 \end{Vmatrix} \begin{Vmatrix} 1 & 3 \\ 5 & 7 \end{Vmatrix} = \begin{Vmatrix} -12 & -4 \\ -4 & 4 \end{Vmatrix} \tag{3.19a}$$

$$(adjD^A)D^B = \begin{Vmatrix} 8 & 0 \\ 0 & 6 \end{Vmatrix} \begin{Vmatrix} 6 & 0 \\ 0 & -8 \end{Vmatrix} = \begin{Vmatrix} 8 & 0 \\ 0 & -8 \end{Vmatrix}. \tag{3.19b}$$

From (3.19a) and (3.19 b) are equivalent. Therefore,

$$UAV_A = D^A = \Phi = diag(6,8), \quad \phi_1 = 6, \phi_2 = 8,$$

$$UBV_B = D^B = \Psi = diag(6,-8), \quad \phi_1 = 6, \phi_2 = -8, \tag{3.18}$$

where

$$U = \begin{Vmatrix} 0 & 1 \\ -1 & 5 \end{Vmatrix}, \quad V_A = \begin{Vmatrix} 5 & 2 \\ 2 & 1 \end{Vmatrix}, \quad V_B = \begin{Vmatrix} 3 & 5 \\ 1 & 2 \end{Vmatrix} \tag{3.19}$$

Then (3.16) is equivalent to the equation $\Phi\tilde{x} + \Psi\tilde{y} = \tilde{c}$, *where*

$$\begin{aligned} \tilde{X} &= V_A^{-1}X = \begin{Vmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{Vmatrix}, \\ \tilde{Y} &= V_B^{-1}Y = \begin{Vmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{21} & \tilde{Y}_{22} \end{Vmatrix}, \quad \tilde{C} = UC = \begin{Vmatrix} 11 & 10 \\ 46 & 38 \end{Vmatrix}. \end{aligned} \tag{3.20}$$

From matrix equation (3.22), we get the system of linear Diophantine equations:

$$6\tilde{x}_{11} + 6\tilde{y}_{11} = 11, \quad 6\tilde{x}_{12} + 6\tilde{y}_{12} = 10,$$

$$8\tilde{x}_{12} - 8\tilde{y}_{21} = 46, \quad 8\tilde{x}_{22} - 8\tilde{y}_{22} = 38. \tag{3.21}$$

The particular solution of each linear equation of system (3.24) has the following form

$$\tilde{x}_{11}^{(0)} = 30, \tilde{y}_{11}^{(0)} = 1, \quad \tilde{x}_{12}^{(0)} = 0, \tilde{y}_{12}^{(0)} = 2$$

$$\tilde{x}_{21}^{(0)} = -4, \tilde{y}_{21}^{(0)} = -10, \quad \tilde{x}_{22}^{(0)} = -4, \tilde{y}_{22}^{(0)} = -9. \tag{3.22}$$

The particular solution of matrix equation (3.22) is

$$\tilde{X}_0 = \begin{Vmatrix} 30 & 0 \\ -4 & -4 \end{Vmatrix}, \quad \tilde{Y}_0 = \begin{Vmatrix} 1 & 2 \\ -10 & -9 \end{Vmatrix}. \tag{3.23}$$

Then by (3.14) the general solution of matrix equation (3.22) is

$$\tilde{X} = \begin{Vmatrix} 30 & 0 \\ -4 & -4 \end{Vmatrix} + \begin{Vmatrix} r_1 & r_1 \\ -r_2 & -r_2 \end{Vmatrix} + \begin{Vmatrix} 6k_{11} & 6k_{12} \\ -8k_{21} & -8k_{22} \end{Vmatrix}$$

$$\tilde{Y} = \left\|\begin{matrix} 1 & 2 \\ -10 & -9 \end{matrix}\right\| - \left\|\begin{matrix} r_1 & r_1 \\ r_2 & r_2 \end{matrix}\right\| - \left\|\begin{matrix} 6k_{11} & 6k_{12} \\ 8k_{21} & 8k_{22} \end{matrix}\right\| \tag{3.24}$$

or

$$\tilde{X} = \left\|\begin{matrix} 30 + 6k_{11} & 6k_{12} \\ -4 - r_2 - 8k_{21} & -4 - r_2 - 8k_{22} \end{matrix}\right\|$$

$$\tilde{X} = \left\|\begin{matrix} 1 - 6k_{11} & 2 - 6k_{12} \\ -10 - r_2 - 8k_{21} & -9 - r_2 - 8k_{22} \end{matrix}\right\| \tag{3.25}$$

where r_i is from $Z_1 = \{0\}, r_2$ is arbtrary element of $Z_3 = \{0, 1, 2\}$, and $k_{ij},\ i, j = 1, 2$ is arbitrary element of Z.

Finally, the general solution of matrix equation (3.16) is

$$\tilde{X} = V_A\tilde{X} = \left\|\begin{matrix} 142 - 2r_2 + 30k_{11} - 16k_{21} & -8 - 2r_2 + 30k_{12} - 16k_{12} \\ 56 - r_2 + 12k_{11} - 8k_{21} & -4 - r_2 + 12k_{11} - 8k_{22} \end{matrix}\right\|$$

$$\tilde{Y} = V_B\tilde{Y} = \left\|\begin{matrix} -47 - 5r_2 - 18k_{11} - 40k_{21} & -39 - 5r_2 - 18k_{12} - 40k_{12} \\ -19 - 2r_2 - 6k_{11} - 16k_{21} & -16 - 2r_2 - 6k_{12} - 16k_{22} \end{matrix}\right\| \tag{3.26}$$

3.3. **The Uniqueness of Particular Solutions of the Matrix Linear Unilateral Equation.**

The conditions of uniqueness of solutions of bounded degree (minimal solutions) of matrix linear polynomial equations We present the conditions of uniqueness of particular solutions of matrix linear equation over a commutative Bezout domain R [8].

Theorem 3.2. *The matrix equation (3.3) has a unique particular solution*

$$\tilde{X}_0 = \|\tilde{x}_{ij}^{(0)}\|_1^n, \qquad \tilde{Y}_0 = \|\tilde{y}_{ij}^{(0)}\|_1^n \tag{3.27}$$

such that $\tilde{x}_{ij}^{(0)} \in R_{\phi_i}, i, j = 1, \ldots, n$ if, and only if, $(detD^A, detT^B) = 1$.

Proof. From matrix equation (3.3), we get the system of linear equations (3.6). The solving of this system reduces to the successive solving of the linear Diophantine equations of the form (3.7). The matrix equation (2.3) has a unique particular solution $\tilde{X}_0 = \|\tilde{x}_{ij}^{(0)}\|_1^n$, $\tilde{Y}_0 = \|\tilde{y}_{ij}^{(0)}\|_1^n$ such that $\tilde{x}_{ij}^{(0)} \in R_{\phi_i}, i, j = 1, \ldots, n$ if, and only if, each linear Diophantine equations of the form (3.7) has a unique particular solution $\tilde{x}_{ij}^{(0)}, \tilde{y}_{ij}^{(0)}$, such that $\tilde{x}_{ij}^{(0)} \in R_{\phi_i}, i, j = 1, \ldots, n$. It follows that $(detD^A, detT^B) = 1$. □

Theorem 3.3. $\tilde{X}_0 = \|\tilde{x}_{ij}^{(0)}\|_1^n$, $\tilde{Y}_0 = \|\tilde{y}_{ij}^{(0)}\|_1^n$, *where* $\tilde{x}_{ij}^{(0)} \in R_{\phi_i}, i, j = 1, \ldots, n$ *be a unique particular solution of matrix equation (3.3). Then the general solution of matrix equation (3.3) is*

$$\tilde{X} = \tilde{X}_0 + \Psi K, \qquad \tilde{Y} = \tilde{Y}_0 + \Phi K, \tag{3.28}$$

where $\Phi = D^A$ *and* $\Psi = D^B$ *are canonical diagonal forms of* A *and* B *from matrix equation (1.3), respectively,* $K = \|\tilde{k}_{ij}^{(0)}\|_1^n$, k_{ij} *are arbitrary elements of* R, $\quad i, j = 1, \ldots, n$.

The general solution of matrix equation (1.3) is the pair of matrices

$$X = V_A \tilde{X}, \qquad Y = V_B \tilde{Y}. \tag{3.29}$$

Proof. The particular solution of the form (3.30) of (3.3) is unique if, and only if, $(\det D^A, \det T^B) = 1$, that is, $(detA, detB) = 1$. Then by corollary the pair of matrices (A, B) is diagonalizable and (1.3) gives us the equation of the form (3.9)

Thus by a Theorem we get the formula (3.31) of the general solution of (3.3) and the formula (3.32) for computation of general solution of (1.3) in the case where (3.3) has unique particular solution of the form (3.30). □

4. The matrix linear bilateral equations $AX + YB = C$

Consider the matrix linear bilateral equation (1.2), where A, B, and C are matrices over a commutative Bezout domain R, and

$$U_A A V_A = D^A = \Phi = diag(\phi_1, \ldots, \phi_n), \phi_i | \phi_{i+1},$$

$$U_B B V_B = D^B = \Psi = diag(\psi_1, \ldots, \psi_n), \psi_i | \psi_{i+1}, i = 1, \ldots, n-1 \tag{4.1}$$

are the canonical diagonal forms of matrices A and B, respectively, and U_A, V_A, U_B, $V_B \in GL(n, R)$. Then (1.2) is equivalent to

$$\Phi \tilde{X} + \tilde{Y} = \tilde{C}, \tag{4.2}$$

where $\tilde{X} = V_A^{-1} X V_B, \tilde{Y} = U_A Y U_B^{-1}$ and $\tilde{C} = U_A C V_B$.

Such an approach to solving (1.2) where A, B and C are the matrices over a polynomial ring $F(\lambda)$, where F is a field, was applied in [3]. The equation (3.2) is equivalent to the system of linear Diophantine equations.

$$\phi_i \tilde{x}_{ij} + \psi_i \tilde{y}_{ij} = \tilde{c}_{ij}, \qquad i, j = 1, \ldots, n. \tag{4.3}$$

Theorem 4.1. *Let*

$$\tilde{X}_0 = \|\tilde{X}_{ij}^{(0)}\|_1^n, \qquad \tilde{Y}_0 = \|\tilde{Y}_{ij}^{(0)}\|_1^n, \tag{4.4}$$

be a particular solution of matrix equation (4.2) that is, $\tilde{X}_{ij}^{(0)}, \tilde{Y}_{ij}^{(0)}, i, j = 1, \ldots, n,$ *are particular solutions of linear Diophantine equation of system (3.3).*

The general solution of matrix equation (4.2) is

$$\tilde{X} = \tilde{X}_0 + W_\psi + k, \tilde{Y} = \tilde{Y}_0 + W_\Phi + k\Phi, \tag{4.5}$$

where $W_\psi = \|(\psi_j/d_{ij})w_{ij}\|_1^n, W_\Phi = \|(\phi_j/d_{ij})w_{ij}\|$, *where* w_{ij} *are arbitrary element of* $R_{d_{ij}}$ *and* $K = \|k_{ij}\|_1^n$, *where* k_{ij} *are arbitrary element of* $R, i, j = 1, \ldots, n.$

The general solution of matrix equation (1.2) is

$$X = V_A \tilde{X} V_B^{-1}, Y = U_A^{-1} \tilde{Y} U_B. \tag{4.6}$$

Similarly as for (3.3) we prove that particular solution of (4.2) is unique if, and only if, $(det\phi, det\psi) = 1.$ *Then by the same way as for (1.3) we write down the general solution of matrix equation (1.2).*

Theorem 4.2. *Suppose that*

$$\tilde{X}_0 = \|\tilde{X}_{ij}^{(0)}\|_1^n, \quad \tilde{Y}_0 = \|\tilde{Y}_{ij}^{(0)}\|_1^n,$$

where $\tilde{X}_{ij}^{(0)} \in R_{\psi_i}, i = 1, \ldots, n$ *is unique particular solution of matrix equation (4.2) and*

$$D^A = \Phi = diag(\phi_1, \ldots, \phi_n), D^B = \Psi = diag(\psi_1, \ldots, \psi_n), \tag{4.7}$$

are canonical diagonal forms of matrices A, B *from matrix equation (1.2), respectively. Then the general solution of matrix equation (4.2) is*

$$\tilde{X} = \tilde{X}_0 + K\Psi, \qquad \tilde{Y} = \tilde{Y}_0 + K\Phi, \tag{4.8}$$

where $K = \|k_{ij}\|_1^n, k_{ij}$ *are arbitrary elements of* $R, i, j = 1, \ldots, n.$

The general solution of matrix equation (1.2) is

$$X = V_A \tilde{X} V_B^{-1}, \quad Y = U_A^{-1} \tilde{Y} U_B. \tag{4.9}$$

Example 3. *Consider the equation*

$$AX + YB = C \tag{4.10}$$

for the matrices

$$A = \begin{Vmatrix} 8 & -6 \\ 4 & -2 \end{Vmatrix}, \quad B = \begin{Vmatrix} 7 & 5 \\ 3 & 1 \end{Vmatrix}, \quad C = \begin{Vmatrix} 10 & 11 \\ 12 & 9 \end{Vmatrix}$$

for the matrices over Z and

$$X = \begin{Vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{Vmatrix}, \quad Y = \begin{Vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{Vmatrix} \tag{4.11}$$

are unknown matrices. The matrix equation (4.10) is solvable. The Pair of matrices (A, B) from matrix equation (4.10) by a theorem is diagonalizable.

Let $A, B \in M(n, R)$ and A be a nonsingular matrix. Then the pair of matrices (A, B) is generalized equivalent to the pair of diagonal matrices $(D^A D^B)$ if, and only if, the matrices $(adjA)B$ and $(adjD^A)D^B$ are equivalent, where $adjA$ is an adjoint matrix.

Since the matrices

$$(adjA)B = \begin{Vmatrix} -2 & 6 \\ -4 & 8 \end{Vmatrix} \begin{Vmatrix} 7 & 5 \\ 3 & 1 \end{Vmatrix} = \begin{Vmatrix} 4 & -4 \\ -4 & -12 \end{Vmatrix} \tag{4.12}$$

$$(adjD^A)D^B = \begin{Vmatrix} 8 & 0 \\ 0 & 6 \end{Vmatrix} \begin{Vmatrix} 6 & 0 \\ 0 & -8 \end{Vmatrix} = \begin{Vmatrix} 8 & 0 \\ 0 & -8 \end{Vmatrix}. \tag{4.13}$$

From (4.12) and (4.13) are equivalent. Therefore, $UAV_A = D^A = \Phi = diag(6, 8), \quad \phi_1 = 6, \quad \phi_2 = 8,$

$$UBV_B = D^B = \Psi = diag(6, -8), \phi_1 = 6.\phi_2 = -8, \tag{4.14}$$

where

$$U_A = \begin{Vmatrix} -1 & 0 \\ 1 & -1 \end{Vmatrix}, \quad U_B = \begin{Vmatrix} 0 & 1 \\ 1 & -2 \end{Vmatrix}$$

$$V_A = \left\|\begin{matrix} 1 & -2 \\ -1 & 3 \end{matrix}\right\|, \quad U_B = \left\|\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}\right\|. \tag{4.15}$$

Then (4.10) is equivalent to the equation

$$\Phi\tilde{x} + \Psi\tilde{y} = \tilde{c}, \tag{4.16}$$

where

$$\tilde{X} = V_A^{-1}XV_B = \left\|\begin{matrix} \tilde{x_{11}} & \tilde{x_{12}} \\ \tilde{x_{21}} & \tilde{x_{22}} \end{matrix}\right\| \left\|\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}\right\| = \left\|\begin{matrix} 2\tilde{x_{11}} + \tilde{x_{12}} & \tilde{x_{11}} + \tilde{x_{12}} \\ 2\tilde{x_{21}} + \tilde{x_{22}} & \tilde{x_{21}} + \tilde{x_{22}} \end{matrix}\right\|$$

$$\tilde{Y} = U_B^{-1}YU_A = \left\|\begin{matrix} \tilde{y_{11}} & \tilde{y_{12}} \\ \tilde{y_{21}} & \tilde{y_{22}} \end{matrix}\right\| \left\|\begin{matrix} -1 & 0 \\ 1 & -1 \end{matrix}\right\| = \left\|\begin{matrix} -\tilde{y_{11}} + \tilde{y_{12}} & 0 - \tilde{y_{12}} \\ -\tilde{y_{21}} + \tilde{y_{22}} & 0 - \tilde{y_{22}} \end{matrix}\right\|$$

$$\tilde{C} = U_A^{-1}CV_B = \left\|\begin{matrix} -1 & 0 \\ 1 & -1 \end{matrix}\right\| \left\|\begin{matrix} 10 & 11 \\ 12 & 9 \end{matrix}\right\| \left\|\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}\right\| = \left\|\begin{matrix} -31 & -21 \\ -2 & 0 \end{matrix}\right\|.$$

From matrix equation (4.16), we get the system of linear Diophantine equations:

$$\begin{aligned} 6\tilde{x_{11}} + 6\tilde{y_{11}} &= -31, \\ 6\tilde{x_{12}} + 6\tilde{y_{12}} &= -21, \\ 8\tilde{x_{21}} - 8\tilde{y_{21}} &= -2, \\ 8\tilde{x_{22}} - 8\tilde{y_{22}} &= 0. \end{aligned} \tag{4.17}$$

The particular solution of each linear equation of system (4.18) has the following form

$$\begin{aligned} \tilde{x}_{11}^{(0)} &= 30, & \tilde{y}_{11}^{(0)} &= -35, \\ \tilde{x}_{12}^{(0)} &= 18, & \tilde{y}_{12}^{(0)} &= -22, \\ \tilde{x}_{21}^{(0)} &= -4, & \tilde{y}_{21}^{(0)} &= -3, \\ \tilde{x}_{22}^{(0)} &= 0, & \tilde{y}_{22}^{(0)} &= 0. \end{aligned} \tag{4.18}$$

The partricular solution of matrix equation (4.19) is

$$\tilde{X}_0 = \left\|\begin{matrix} 30 & 18 \\ -4 & 0 \end{matrix}\right\|, \quad \tilde{Y}_0 = \left\|\begin{matrix} -35 & -22 \\ -3 & 0 \end{matrix}\right\|. \tag{4.19}$$

Then by (3.14) the general solution of matrix equation (3.22) is

$$\tilde{X} = \begin{Vmatrix} 30 & 18 \\ -4 & 0 \end{Vmatrix} + \begin{Vmatrix} W_{11} & W_{12} \\ -W_{21} & -W_{22} \end{Vmatrix} + \begin{Vmatrix} 6k_{11} & 6k_{12} \\ -8k_{21} & -8k_{22} \end{Vmatrix}$$

$$\tilde{Y} = \begin{Vmatrix} -35 & -22 \\ -3 & 0 \end{Vmatrix} - \begin{Vmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{Vmatrix} - \begin{Vmatrix} 6k_{11} & 6k_{12} \\ 8k_{21} & 8k_{22} \end{Vmatrix} \tag{4.20}$$

$$\tilde{X} = \begin{Vmatrix} 30 + W_{12} + 6k_{11} & 18 + W_{12} + 6k_{12} \\ -4 - W_{21} - 8k_{21} & 0 - W_{22} - 8k_{22} \end{Vmatrix}$$

$$\tilde{Y} = \begin{Vmatrix} -35 - W_{11} - 6k_{11} & -22 - W_{12} - 6k_{12} \\ -3 - W_{21} - 8k_{21} & 0 - W_{22} - 8k_{22} \end{Vmatrix}, \tag{4.21}$$

where r_1 is from $Z_1 = \{0\}, r_2$ is arbitrary element of $Z_3 = \{0, 1, 2\}$ and $k_{ij}, i, j = 1, 2$ is arbitrary element of Z. Finally, the general solution of matrix equation (3.16) is

$$\tilde{X} = V_A \tilde{X} V_B^{-1} \tag{4.22}$$

$$= \begin{Vmatrix} 110 + 3W_{21} + 18k_{11} - 5W_{12} + 40k_{21} & 54 + 3W_{12} + 18k_{12} + 5W_{22} + 40k_{22} \\ -148 - 4W_{12} - 24k_{11} - 7W_{21} - 56k_{21} & -72 - 4W_{12} - 24k_{12} - 7W_{22} - 56k_{22} \end{Vmatrix}$$

$$\tilde{Y} = U_A^{-1} \tilde{Y} U_B = \begin{Vmatrix} 3 + 3W_{21} + 8k_{21} & W_{22} + 8k_{22} \\ 32 + W_{11} + 6k_{11} - W_{21} - 8k_{21} & 2 + W_{12} + 6k_{12} - W_{22} - 8k_{22} \end{Vmatrix}.$$

5. Conclusion

Hence we conclude that the method of solving matrix linear equations are over a commutative bezout domain. This method is based on the use of standard form of a pair of matrices with respect to generalized equivalence introduced and on congruences. Now the notion of particular solution of such matrix equations. We establish the criterions of uniqueness of particular solutions and write down the formulas of general solutions of such equations.

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DEPARTMENT OF MATHEMATICS,
MARUDUPANDIYAR COLLEGE,
PILLAIYARPATTI, THANJAVUR, TAMILNADU,
INDIA.
Email address: mathseswari@gmail.com

DEPARTMENT OF MATHEMATICS,
MARUDUPANDIYAR COLLEGE,
PILLAIYARPATTI, THANJAVUR, TAMILNADU,
INDIA.
Email address: andavanmathsramesh@gmail.com