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SOME METRICAL AND TOPOLOGICAL PROPERTIES OF THE RIVER METRIC ON $\mathbb{R}^2$

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ABSTRACT. In this paper we consider some metrical and topological properties of the river metric d^* in the plane $\mathbb{R}^2$. We give the form of the metric segment and the set of all points that are equidistant from two points in $(\mathbb{R}^2, d^*)$. We also give the characterization of a compact sets in this space.

1. INTRODUCTION

When we consider the distance between two points in a plane, we actually think about the length of the shortest path connecting those two points. But, what are "length" and "shortest path"? For example, the shortest path connecting two points in the plane is given by the line segment. The length of this line segment gives us, what is called, the Euclidean distance and the usual way that we think about points, lines and angles in the plane is known as Euclidean geometry. What happens when we change the metric function? As expected, the metric specifies metric properties such as length and segment, but also shapes such as sphere, ellipse, hyperbola, etc ([1, 3]). It also dictates topological properties, openness of the sets, completeness, compactness of the sets, measure of non-compactness ([2, 5]) etc.

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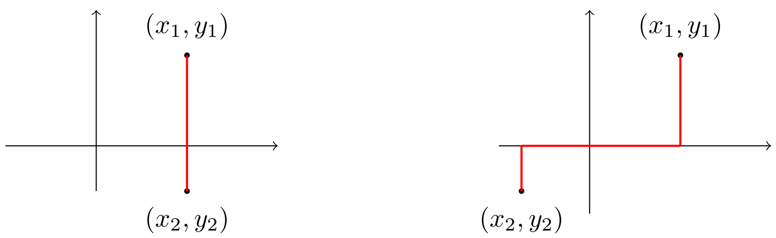
In this paper, we will replace the Euclidean metric in $\mathbb{R}^2$ with the so-called jungle river metric or barbed wire metric ([2]). In the first part, we will determine what a metric segment is in such a space, and in the second part, we will give a complete characterization of the compact sets in it.

2. The metric segment in the river metric

We will observe the set $\mathbb{R}^2$ with so-called river metric defined by

$$d^*(v_1, v_2) = \begin{cases} |y_1 - y_2| & , \quad x_1 = x_2, \\ |y_1| + |y_2| + |x_1 - x_2| & , \quad x_1 \neq x_2, \end{cases}$$

where $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$. Unfortunately, this space is not a normed space and not even a linear metric space. The definitions of some terms must be adapted to these structures, but for most terms they are standard. In the following, we will denote the river metric with d^* and with $(\mathbb{R}^2, d^*)$ the metric space with the river metric. We will denote the open ball in this space by $B((x_0, y_0), r) = \{(x, y) \in \mathbb{R}^2 \mid d^*((x, y), (x_0, y_0)) < r\}$, where $v = (x_0, y_0) \in \mathbb{R}^2$ is the center of the ball, and $r > 0$ is the radius. With $\overline{B}((x_0, y_0), r)$ we denote closed ball.



(a) Equal first coordinates

(b) Different first coordinates

FIGURE 1. The river metric in $\mathbb{R}^2$.

Let (X, d) be a metric space and let $x, y, z \in X$. We say that the point z is between the points x and y in a given metric space if and only if

$$d(x, z) + d(y, z) = d(x, y).$$

The set of all points located between points x and y is called the metric segment denoted by $[x, y]$.

Lemma 2.1. *The metric segment between points* $v_1 = (x_1, y_1),\ v_2 = (x_2, y_2) \in \mathbb{R}^2$ *with the river metric* d^* *is the set*

$$[v_1, v_2] = \{(x_1, a) \in \mathbb{R}^2 \mid a \in [0, y_1] \text{ (or } a \in [y_1, 0])\} \\ \cup \{(x_2, a) \in \mathbb{R}^2 \mid a \in [0, y_2] \text{ (or } a \in [y_2, 0])\} \cup \{(b, 0) \in \mathbb{R}^2 \mid x_1 \le b \le x_2\},$$

for $x_1 < x_2$ *(Figure 1 (b)),or the set*

$$[v_1, v_2] = \{(x_1, a) \in \mathbb{R}^2 \mid y_1 \le a \le y_2\},$$

for $x_1 = x_2$ *and* $y_1 \le y_2$ *(Figure 1 (a)).*

In a metric space, the set of all points that are equidistant from the one fixed point is a sphere, and thus we have a characterization of open and closed ball in a given space. The ball in the metric space $\mathbb{R}^2$ with the river metric can have several shapes depending on the center of the ball and its radius.

The ball centered at $(0,0)$ in $(\mathbb{R}^2, d^*)$, with radius $r > 0$ (Figure 2 (a)) is given with

$$B((0,0), r) = \left\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| < r\right\}.$$

If $v = (x^*, y^*)$ is arbitrary, then the ball centered at v and with the radius $r > 0$, such that $|y^*| < r$, has the form (Figure 2 (b) and (c))

$$B(v, r) = \left\{ (x, y) \in \mathbb{R}^2 \mid \left\{ \begin{array}{ccc} |y - y^*| < r & , & x = x^* \\ |x| + |y| < r - |y^*| & , & x \ne x^* \end{array} \right. \right\},$$

while for $|y^*| \ge r$ (Figure 2 (d)) has the form

$$B(v, r) = \{(x, y) \in \mathbb{R}^2 \mid x = x^* \ \wedge\ |y - y^*| < r\}. \tag{2.1}$$

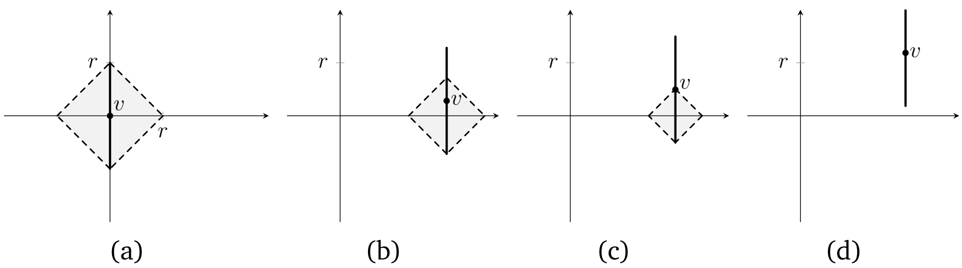


FIGURE 2. The shapes of $B(v, r)$ in the river metric, depending on the center v and the radius r.

We say that a metric space (X, d) is a geodesic metric space if any two points can be connected with a geodesic line, that is, for arbitrary $x, y \in X$ there is an isometric imbedding $\gamma : [\alpha, \beta] \to X$, $[\alpha, \beta] \subset \mathbb{R}$, so $\gamma(\alpha) = x$ and $\gamma(\beta) = y$. This fact is equivalent (see [1]) to the existence of a midpoint mapping for (X, d), that is, to the existence of a mapping $m : X \times X \to X$ such that for arbitrary $x, y \in X$ holds

$$d(m(x,y), x) = d(m(x,y), y) = \frac{1}{2} d(x,y).$$

Let $v_1 = (x_1, y_1), v_2 = (x_2, y_2) \in \mathbb{R}^2$, $x_1 < x_2$, be arbitrary. Then the mapping $\gamma : [0, d^*(v_1, v_2)] \to \mathbb{R}^2$, defined by

$$\gamma(t) = \tag{2.2}$$
$$\begin{cases} \begin{cases} (x_1, y_1 - t)\,; y_1 \geq 0 \\ (x_1, y_1 + t)\,; y_1 \leq 0 \end{cases} ; & 0 \leq t \leq |y_1| \\ (t - |y_1| + x_1, 0); & |y_1| < t \leq |y_1| + (x_2 - x_1) \\ \begin{cases} (x_2, t - (|y_1| + x_2 - x_1))\,; y_2 \geq 0 \\ (x_2, -(t - (|y_1| + x_2 - x_1)))\,; y_2 \leq 0 \end{cases} ; & \begin{array}{l} |y_1| + (x_2 - x_1) < t \\ \quad \leq d^*(v_1, v_2) \end{array} \end{cases} ,$$

is obviously isometry. Furthermore, $\gamma(0) = (x_1, y_1) = v_1$ and $\gamma(d^*(v_1, v_2)) = (x_2, y_2) = v_2$, so $(\mathbb{R}^2, d^*)$ is a geodesic metric space. With this we ensured the existence of midpoint mapping in this space. In addition, the existence of a mapping (2.2) gives us that $(\mathbb{R}^2, d^*)$ is so-called segment space ([4]).

Theorem 2.1. *Let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ be two points from $(\mathbb{R}^2, d^*)$ such that $x_1 \neq x_2$ and $|y_1| \pm x_1 \neq |y_2| \pm x_2$. If $\frac{d^*(v_1,v_2)}{2} \geq \max\{|y_1|, |y_2|\}$, then the set of all points that are equidistant from points v_1 and v_2 is the set*

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid x = \frac{|y_2| + x_2 - |y_1| + x_1}{2},\ y \in \mathbb{R} \right\},$$

and the middle of the metric segment $[v_1, v_2]$ is the point

$$(\overline{x}, \overline{y}) = \left(\frac{|y_2| + x_2 - |y_1| + x_1}{2}, 0 \right).$$

If $\frac{d^*(v_1,v_2)}{2} < \max\{|y_1|, |y_2|\}$, then the set of points that are equidistant from points v_1 and v_2 is a singleton and for the middle of metric segment $[v_1, v_2]$ we have, if $\max\{|y_1|, |y_2|\} = |y_i|$, $i \in \{1, 2\}$, the middle point is

$$(\overline{x}, \overline{y}) = \left(x_i, \frac{y_i - |y_j| - |x_1 - x_2|}{2} \right), \quad \textit{for } y_i > 0,$$

respectively

$$(\overline{x}, \overline{y}) = \left(x_i, \frac{y_i + |y_j| + |x_1 - x_2|}{2} \right), \quad \textit{for } y_i < 0,$$

where $j \in \{1, 2\}$, $j \neq i$.

Proof. Let $v_1 = (x_1, y_1),\ v_2 = (x_2, y_2) \in \mathbb{R}^2$, such that $x_1 \neq x_2$, $y_1 \neq y_2$. We can assume that $x_1 < x_2$. Let $(x, y) \in \mathbb{R}^2$ be arbitrary, such that $x \neq x_1$ and $x \neq x_2$. Then

$$d^*((x, y), (x_1, y_1)) = |y| + |y_1| + |x - x_1|,$$
$$d^*((x, y), (x_2, y_2)) = |y| + |y_2| + |x - x_2|.$$

If (x, y) is equidistant from points v_1 and v_2, $d^*((x, y), (x_1, y_1)) = d^*((x, y), (x_2, y_2))$, then following holds

$$|y_1| + |x - x_1| = |y_2| + |x - x_2|. \tag{2.3}$$

Let us consider three different cases.

(a) Let $x < x_1$. Using (2.3), we have

$$|y_1| - x + x_1 = |y_2| - x + x_2 \iff |y_1| + x_1 = |y_2| + x_2,$$

and by the assumption of the theorem there is no solution for x.

(b) Let $x > x_2$. Again, using (2.3), we have

$$|y_1| + x - x_1 = |y_2| + x - x_2 \iff |y_1| - x_1 = |y_2| - x_2,$$

so in this case, by the assumption of the theorem, there is no solution for x.

(c) Let $x_1 < x < x_2$. From (2.3) we get

$$|y_1| + x - x_1 = |y_2| - x + x_2 \iff x = \frac{|y_2| + x_2 - |y_1| + x_1}{2}.$$

Since the second coordinate (y) is arbitrary, every point

$$(x, y) = \left(\frac{|y_2| + x_2 - |y_1| + x_1}{2}, y \right), \ y \in \mathbb{R},$$

satisfies the condition (2.3). So, the set of all points that are equidistant from points v_1 and v_2 is actually the set A. Since the point $(\overline{x}, \overline{y})$ is the middle of the metric segment and satisfies the condition

$$d^*((\overline{x},\overline{y}),(x_1,y_1)) = d^*((\overline{x},\overline{y}),(x_2,y_2)) = \inf\{d^*((x,y),(x_1,y_1)) \mid (x,y) \in A\},$$

infimum of the expression

$$d^*((x,y),(x_1,y_1)) = |y| + |y_1| + \left|\frac{|y_2| + x_2 - |y_1| + x_1}{2} - x_1\right|,$$

where $(x,y) \in A$, is reached for $y = 0$. This means that the point

$$(\overline{x},\overline{y}) = \left(\frac{|y_2| + x_2 - |y_1| + x_1}{2}, 0\right),$$

is the middle of the metric segment $[v_1, v_2]$.

Let $v_1, v_2 \in \mathbb{R}^2$ be such that $x_1 < x_2$, $\frac{d^*(v_1,v_2)}{2} < \max\{|y_1|,|y_2|\}$ and without loss of generality let $y_1 > 0$ and $\max\{|y_1|,|y_2|\} = |y_1|$. The condition $\frac{d^*(v_1,v_2)}{2} < \max\{|y_1|,|y_2|\}$ is equivalent to the condition

$$|y_2| + x_2 < |y_1| + x_1. \tag{2.4}$$

Let us determine the set A of the points that are equidistant from points v_1 and v_2, i.e. the set $A = \{(x,y) \in \mathbb{R}^2 \mid d^*((x,y),v_1) = d^*((x,y),v_2)\}$. Based on the condition (2.4), we have that $x = x_1$ holds for the first coordinate of the point that belongs to the set A. Indeed, let $x \neq x_1$. We will consider following cases.

(a) Let $x < x_1 < x_2$. Then, the equality $d^*((x,y),v_1) = d^*((x,y),v_2)$ is equivalent to

$$|y| + |y_1| + |x - x_1| = |y| + |y_2| + |x - x_2|,$$

that is

$$|y_1| + x_1 = |y_2| + x_2,$$

and this is impossible given the assumption of the theorem.

(b) Let $x > x_2 > x_1$. The equality $d^*((x,y),v_1) = d^*((x,y),v_2)$ is equivalent to

$$|y_1| - x_1 = |y_2| - x_2,$$

which is again impossible, due to the assumption of the theorem.

(c) Let $x_1 < x \leq x_2$. If $x_1 < x < x_2$, then

$$d^*((x,y),v_1) = |y| + |y_1| + x - x_1,$$

$$d^*((x,y),v_2) = |y| + |y_2| + x_2 - x,$$

so, equalizing these distances we get

$$x = \frac{|y_2| + x_2 - |y_1| + x_1}{2}.$$

Using the condition (2.4) we conclude that $x < x_1$, and that is a contradiction to the assumption $x_1 < x$. On the other hand, if $x = x_2$, then

$$d^*((x,y),v_1) = |y| + |y_1| + |x_1 - x_2|,$$

$$d^*((x,y),v_2) = |y - y_2|,$$

and we get

$$|y| + |y_1| + x_2 - x_1 = |y - y_2|,$$

that is

$$y_1 + x_2 - x_1 = |y - y_2| - |y| \le |y_2| \le \max\{|y_1|, |y_2|\} = |y_1| = y_1.$$

Since the distances between the point (x,y) and points v_1 and v_2 are equal, we conclude that $x_2 \le x_1$, which is impossible due to assumption of this considered case $x_1 < x_2$.

Therefore, assuming that $\frac{d^*(v_1,v_2)}{2} < \max\{|y_1|, |y_2|\} = |y_1|$, $y_1 > 0$, we have $x = x_1$, where (x) is the first coordinate of the point from the set of all points that are equidistant from points v_1 and v_2. In order to determine the points of the set A, we will actually search for the points $(x_1, y) \in \mathbb{R}^2$ such that $d^*((x_1,y),(x_1,y_1)) = d^*((x_1,y),(x_2,y_2))$. Since

$$d^*((x_1,y),(x_1,y_1)) = |y - y_1|,$$

$$d^*((x_1,y),(x_2,y_2)) = |y| + |y_2| + |x_1 - x_2|,$$

we are looking for $y \in \mathbb{R}$ such that

$$|y - y_1| = |y| + |y_2| + |x_1 - x_2|. \tag{2.5}$$

We will consider several cases.

(a) If $y = 0$, the condition (2.5) is equivalent to $|-y_1| = |y_2| + |x_1 - x_2|$, i.e. $|y_1| + x_1 = |y_2| + x_2$, which is impossible because of the assumption of the theorem. Hence $y \neq 0$.

(b) Let $y < 0$. The condition (2.5) is equivalent to $|y - y_1| = |y| + |y_2| + x_2 - x_1$, i.e. $|y_1| + x_1 = |y_2| + x_2$, which is impossible. So, $y < 0$ doesn't hold.

(c) Let $y > 0$ and $y > y_1$. Then, (2.5) is equivalent to $y - y_1 = y + |y_2| + x_2 - x_1$, i.e. $-|y_1| + x_1 = |y_2| + x_2$, we get $x_1 = y_1 + |y_2| + x_2 > x_2$, which is contrary to the initial assumption $x_1 < x_2$. Thus, let $0 < y < y_1$. The equality (2.5) becomes

$$y_1 - y = y + |y_2| + |x_1 - x_2|,$$

and we have

$$y = \frac{y_1 - |y_2| - |x_1 - x_2|}{2}.$$

Accordingly, the set $A = \left\{\left(x_1, \frac{y_1 - |y_2| - |x_1 - x_2|}{2}\right)\right\}$ is a singleton. So, if $\max\{|y_1|, |y_2|\} = |y_i|$, where $y_i > 0$, then the set of the points that are equidistant from points v_1 and v_2 is a singleton and $A = \left\{\left(x_i, \frac{y_i - |y_j| - |x_1 - x_2|}{2}\right)\right\}$, $j \in \{1, 2\}$, $j \neq i$.

If $\max\{|y_1|, |y_2|\} = |y_1|$ and $y_1 < 0$, in a similar way as in the previous case, using the condition $\frac{d^*(v_1, v_2)}{2} < \max\{|y_1|, |y_2|\}$, we conclude that the set of the points that are equidistant from points v_1 and v_2 is

$$A = \{(x_1, y) \mid d^*((x_1, y), v_1) = d^*((x_1, y), v_2)\}.$$

We will find the second coordinate $y \in \mathbb{R}$ from the condition (2.5). Let us consider following cases.

(a) Let $y = 0$. Then, the condition (2.5) is equivalent to $|y_1| + x_1 = |y_2| + x_2$, which is impossible due to assumption of the theorem. So, $y \neq 0$.

(b) If $y < 0$, that is $y < y_1$, (2.5) is equivalent to $y_1 = |y_2| + x_2 - x_1$, or $x_1 = |y_2| - y_1 + x_2 = |y_2| + |y_1| + x_2 > x_2$, which is also impossible.

(c) Let $y > 0$. Since $y_1 < 0$, we have $y > y_1$ and the condition (2.5) is equivalent to $-y_1 = |y_2| + x_2 - x_1$, that is $|y_1| + x_1 = |y_2| + x_2$, which is the opposite to the assumption of the theorem. If $y < 0$ and $y > y_1$, we have

$$y = \frac{y_1 + |y_2| + |x_1 - x_2|}{2},$$

thus the set

$$A = \left\{ \left(x_1, \frac{y_1 + |y_2| + |x_1 - x_2|}{2} \right) \right\}$$

is singleton.

Therefore, if $\dfrac{d^*(v_1, v_2)}{2} < \max\{|y_1|, |y_2|\}$, where $\max\{|y_1|, |y_2|\} = |y_i|$, $y_i > 0$, then

$$A = \left\{ \left(x_i, \frac{y_i + |y_j| + |x_1 - x_2|}{2} \right) \right\},$$

where $j \in \{1, 2\}$, $j \neq i$. □

The assumption $|y_1| \pm x_1 \neq |y_2| \pm x_2$ that we made in the Theorem 2.1 for points v_1 and v_2, eliminated several special cases. Let $v_1 = (a, b)$ and $v_2 = (b, a)$, where $a \neq b$. Without loss of generality, let $a < b$. For arbitrary $(x, y) \in \mathbb{R}^2$, $x \neq a, b$ we have

$$d^*((x, y), v_1) = |y| + |b| + |x - a|, \quad d^*((x, y), v_2) = |y| + |a| + |x - b|,$$

and from the equality of these two distances we get

$$|b| + |x - a| = |a| + |x - b|. \tag{2.6}$$

Let us now consider the possibilities for x.

(a) For $x < a$, we have $x < b$, so (2.6) implies $a + |b| = |a| + b$. According to the initial assumption ($a < b$), the last equality is satisfied only for the case when $a, b > 0$. In this case, the set of the points that are equidistant from points v_1 and v_2 is given by

$$A = \{(x, y) \in \mathbb{R}^2 \mid x < a, \ y \in \mathbb{R}\}.$$

(b) For $x > b$, we have $x > a$, so (2.6) implies $|b| - a = |a| - b$. Using the initial assumption $a < b$, the last condition will be satisfied if and only if $a, b < 0$. The set of the points that are equidistant from points v_1 and v_2 is given by

$$A = \{(x, y) \in \mathbb{R}^2 \mid x > b, \ y \in \mathbb{R}\}.$$

(c) For $a < x < b$, using (2.6) we get

$$|b| + x - a = |a| - x + b \iff x = \frac{|a| + a + b - |b|}{2}.$$

Such x will exist if and only if $a < 0$ i $b > 0$, which means that $x = 0$, therefore

$$A = \{(0, y) \mid y \in \mathbb{R}\}.$$

Theorem 2.2. *Let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ are two points from $(\mathbb{R}^2, d^*)$, such that $x_1 = x_2$ and $y_1 \neq -y_2$. The set of all points that are equidistant from points v_1 and v_2 is a singleton and the middle of the metric segment $[v_1, v_2]$ is the point*

$$(\overline{x}, \overline{y}) = \left(x_1, \frac{y_1 + y_2}{2}\right).$$

If $y_1 = -y_2$, then the set of all points that are equidistant from the points v_1 and v_2 is the set

$$A = \mathbb{R}^2 \setminus \{(x_1, y) \mid y \in \mathbb{R},\ y \neq 0\},$$

and the middle of the metric segment is the point $(x_1, 0)$.

Proof. Let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ be such that $x_1 = x_2$ and $y_1 = -y_2$. Let $(x, y) \in \mathbb{R}^2$ be arbitrary and $x \neq x_1$. Then

$$d^*((x, y), (x_1, y_1)) = |y| + |y_1| + |x - x_1|,$$

$$d^*((x, y), (x_2, y_2)) = |y| + |y_2| + |x - x_2|.$$

The equality of the distances gives $|y_1| = |y_2|$. This equality is true with respect to the initial assumption, so every point $(x, y) \in \mathbb{R}^2$, where $x \neq x_1$ is equidistant from points v_1 and v_2.

Specially, we have

$$d^*((x_1, 0), (x_1, y_1)) = |y_1| = |y_2| = d^*((x_1, 0), (x_2, y_2)).$$

Now, let us consider the point (x_1, y), $y \in \mathbb{R}$.

$$d^*((x_1, y), (x_1, y_1)) = |y - y_1|,\ d^*((x_1, y), (x_2, y_2)) = |y - y_2|.$$

The equality of these two distances gives us

$$|y - y_1| = |y - y_2| \iff |y + y_2| = |y - y_2|,$$

and we conclude that the solution is possible only for $y = 0$. Consequently, we have that the set of all points that are equidistant from points v_1 and v_2 is

$$A = \mathbb{R}^2 \setminus \{(x_1, y) \mid y \in \mathbb{R},\ y \neq 0\}.$$

For arbitrary $(x, y) \in A$ we have

$$\begin{aligned} d^*((x,y),v_1) = d^*((x,y),v_2) &= |y| + |y_1| + |x - x_1| \\ &\geq |y_1| = d^*((x_1,0),v_1) = d^*((x_1,0),v_2), \end{aligned}$$

so, the middle of the metric segment $[v_1, v_2]$ is the point $(x_1, 0)$.

Now, let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ be such that $x_1 = x_2$ and $y_1 \neq -y_2$. For arbitrary $(x, y) \in \mathbb{R}^2$, where $x \neq x_1$ we have

$$d^*((x,y),v_1) = |y| + |y_1| + |x - x_1|, \quad d^*((x,y),v_2) = |y| + |y_2| + |x - x_1|.$$

The equality of these two distances gives us $|y_1| = |y_2|$, that is $y_1 = y_2$. This means that $v_1 = v_2$ or $y_1 = -y_2$, which is impossible due to the initial assumption. Thus, neither one of points $(x, y) \in \mathbb{R}^2$, such that $x \neq x_1$, is not equidistant from v_1 and v_2. For that reason, let us consider points $(x_1, y) \in \mathbb{R}^2$. It holds

$$d^*((x_1,y),v_1) = |y - y_1|, \quad d^*((x_1,y),v_2) = |y - y_2|.$$

The equality of these two distances gives us $|y - y_1| = |y - y_2|$. So, the equality $y - y_1 = y - y_2$ can hold, but using this we conclude there is no solution for y. On the other side, the equality $y - y_1 = -(y - y_2)$ can hold. This gives us the solution $y = \dfrac{y_1 + y_2}{2}$. Accordingly, the point $\left(x_1, \dfrac{y_1 + y_2}{2}\right)$ is equidistant from points v_1 and v_2 and it is unique. □

3. The compactness in the river metric

In a metric space, open balls are open sets, and the base in that metric space consists of all possible open balls.

The topology induced by the river metric on $\mathbb{R}^2$ is not equal to Euclidean topologies on $\mathbb{R}^2$ that are induced by Euclidean metrics d_p ($1 \leq p \leq \infty$). The reason for this lies in the characterization of topologically equivalent metrics ([7]). Namely, for the arbitrary ball B_{d^*} in the river metric, in the general case, there is no ball B_{d_p} in the Euclidean metric such that $B_{d_p} \subset B_{d^*}$. For example, if we consider the ball $B_{d^*}((2, 1.2), 1)$ (Figure 2 (d)), it is obvious that there is no ball in $(\mathbb{R}^2, d_p)$ ($1 \leq p \leq \infty$) which is contained in B_{d^*}.

Let $B_{d_p}((x_0, y_0), r)$ be arbitrary ball in $(\mathbb{R}^2, d_p)$ ($1 \leq p \leq \infty$). Let us consider the ball $B_{d^*}((x_0, y_0), r_0)$, where

$$r_0 = \begin{cases} r & ; \quad |y_0| \geq r \\ \dfrac{r}{n} & ; \quad |y_0| < r \end{cases},$$

and $n \in \mathbb{N}$ such that $|y_0| \geq \dfrac{r}{n}$. If $|y_0| \geq r$, then

$$B_{d^*}((x_0, y_0), r_0) = \{(x_0, y) \in \mathbb{R}^2 \mid |y - y_0| < r\},$$

and if $|y_0| < r$, then

$$B_{d^*}((x_0, y_0), r_0) = \{(x_0, y) \in \mathbb{R}^2 \mid |y - y_0| < \frac{r}{n}\}.$$

In both cases we have $B_{d^*}((x_0, y_0), r_0) \subset B_{d_p}((x_0, y_0), r)$. This means that the topology induced by the river metric is finer topology than the topology induced by the Euclidean metric on $\mathbb{R}^2$ ([7]). The balls given by (2.1) (see Figure 2 (d)) are open balls in the river topology, and are not open in Euclidean topologies on $\mathbb{R}^2$. This confirms previous assertion.

The final conclusion is that the river topology is not topologically equivalent to the Euclidean topology on $\mathbb{R}^2$. Moreover, the river metric is not even uniformly equivalent to any of the metrics d_p ($1 \leq p \leq +\infty$) on $\mathbb{R}^2$ ([6]).

It was proved in [6] that the space $(\mathbb{R}^2, d^*)$ is complete metric space and hence we have complete characterization of Cauchy sequences in this space.

Lemma 3.1 ([6], Lemma 3.3). *A sequence $((x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy sequence in $(\mathbb{R}^2, d^*)$ if and only if $(x_n)_{n\in\mathbb{N}}$ is convergent, and $(y_n)_{n\in\mathbb{N}}$ is zero-sequence or $(x_n)_{n\in\mathbb{N}}$ is constant sequence, starting from some index, and $(y_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$.*

As we noted, the topology induced by the river metric is finer than the Euclidean topology on $\mathbb{R}^2$. This will cause the decrement of the number of compact sets in $\mathbb{R}^2$ with the river metric to decrease.

Example 1. *Let $a, b, c, d \in \mathbb{R}$, $a < b$ and $c < d$. Let us consider the set $K = [a, b] \times [c, d]$. Let us define the sequence $((x_n, y_n))_{n\in\mathbb{N}} \subset \mathbb{R}^2$ with*

$$x_n = a + \frac{b-a}{n},\ y_n = c + \frac{d-c}{k} = \alpha,\ n \in \mathbb{N},\ \textit{and } k \in \mathbb{N} \textit{ is fixed}.$$

Now, let $n, m \in \mathbb{N}$ be arbitrary. Then

$$
\begin{aligned}
d^*((x_n, y_n), (x_{n+m}, y_{n+m})) &= |\alpha| + |\alpha| + \left| a + \frac{b-a}{n} - \left(a + \frac{b-a}{n+m} \right) \right| \\
&= 2|\alpha| + (b-a)\frac{m}{n(n+m)} \geq 2|\alpha|.
\end{aligned}
$$

Thus, considered sequence is not Cauchy sequence, so it has no accumulation points. Hence, there exists no convergent subsequence of the given sequence, so the set K is not compact.

Theorem 3.1. *Let $a, b \in \mathbb{R}$, $a < b$ and let $f : [a, b] \to \mathbb{R}$, $f \neq 0$ be continuous function. If closed set $K \subset \mathbb{R}^2$ contains the set $I = \{(x, f(x)) \mid x \in [a, b]\}$, then the set K is not compact in $\mathbb{R}^2$ with the river metric.*

Proof. Since $a < b$, let us consider the sequence of the points defined by $x_n = a + \dfrac{b-a}{n} \in [a, b]$, $n \in \mathbb{N}$. For arbitrary $n, m \in \mathbb{N}$, $n \neq m$, we have $x_n \neq x_m$. It is obvious that $x_n \to a \in [a, b]$ when $n \to \infty$. Since f is continuous we have $f(x_n) \to f(a)$, $n \to \infty$. Notice that both of these convergence are in $\mathbb{R}$ with the standard metric. Without loss of generality assume that $f(a) \neq 0$. Consider the sequence $(x_n, f(x_n))_{n\in\mathbb{N}} \subset \mathbb{R}^2$. For arbitrary $n, m \in \mathbb{N}$ we have

$$d^*((x_n, f(x_n)), (x_m, f(x_m))) = |f(x_n)| + |f(x_m)| + |x_n - x_m|.$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{R}$ and f is continuous on $[a, b]$, so we conclude that

$$d^*((x_n, f(x_n)), (x_m, f(x_m))) \to 2|f(a)|,\ n, m \to \infty.$$

Therefore, the sequence $(x_n, f(x_n))_{n\in\mathbb{N}}$ is not Cauchy sequence which means it has no accumulation points, i.e. there is no convergent subsequence of this sequence. So, the set K is not compact.

The consequence of the above statement is that balls $B((x^*, y^*), r)$ in $\mathbb{R}^2$ with the river metric, for $|y^*| < r$, are not compact sets. Based on the Bolzano-Weierstrass theorem, balls $B((x^*, y^*), r)$, for $|y^*| \geq r$, are compact sets. Furthermore, based on the same theorem, sets given by $[a, b] \times \{0\}$ are also compact sets. With the following theorem we give the description of compact sets in considered space.

Theorem 3.2. *Let $\mathbb{R}^2$ be equipped with the river metric d^*. The set $K \subset \mathbb{R}^2$ is compact if and only if it is closed and satisfies*

$$K \subseteq \{\alpha\} \times [c, d]\ ,\ \ \alpha, c, d \in \mathbb{R}$$

or

$$K \subseteq ([a, b] \times \{0\}) \cup \bigcup_{n=1}^{\infty} (\{\alpha_n\} \times [-\beta_n, \beta_n]),$$

where $a, b \in \mathbb{R}$, $a \leq b$, $\alpha_n \in [a, b]$ and $(\beta_n)_{n\in\mathbb{N}}$ is sequence in $\mathbb{R}$ such that $\beta_n \to 0$ ($n \to \infty$).

Proof. Let K be closed set and $K \subseteq \{\alpha\} \times [c, d]$. Let $(v_n)_{n\in\mathbb{N}} \subseteq K$ be arbitrary. Then $v_n = (\alpha, y_n)$ and $(y_n)_{n\in\mathbb{N}} \subset [c, d]$. Based on Weierstrass theorem, there exists convergent subsequence $(y_{n_k})_{k\in\mathbb{N}} \subseteq (y_n)_{n\in\mathbb{N}}$. By the Lemma 3.1, the sequence $(v_{n_k})_{k\in\mathbb{N}}$ is Cauchy sequence, so it is convergent in $(\mathbb{R}^2, d^*)$. Thus, from arbitrary sequence in K we have found convergent subsequence, so K is compact set.

Now, let

$$K \subseteq ([a, b] \times \{0\}) \cup \bigcup_{n=1}^{\infty} (\{\alpha_n\} \times [-\beta_n, \beta_n]),$$

where a, b α_n and β_n ($n \in \mathbb{N}$) satisfy assumptions of the theorem. Let $(v_k)_{k\in\mathbb{N}} \subset K$ be arbitrary. We will consider following cases.

(a) Let infinitely many points of the sequence $v_k = (x_k, y_k)$ be in $[a, b] \times \{0\}$. This means that $(x_k)_{k\in\mathbb{N}} \subset [a, b]$ and $y_k = 0$ for all $k \in \mathbb{N}$. Based on the Weierstrass theorem, we can find convergent subsequence $(x_{k_l})_{l\in\mathbb{N}}$ of the sequence $(x_k)_{k\in\mathbb{N}}$. Now, using Lemma 3.1, we have that the sequence $v_{k_l} = (x_{k_l}, 0)$ is Cauchy sequence, i.e. convergent sequence. Therefore, K is compact.

(b) For some $n_0 \in \mathbb{N}$ let infinitely many points of the sequence $v_k = (x_k, y_k)$ be in the set $\{\alpha_{n_0}\} \times [-\beta_{n_0}, \beta_{n_0}]$. This means that the sequence $(x_k)_{k\in\mathbb{N}}$ is constant sequence starting from the index n_0, and $(y_k)_{k\in\mathbb{N}} \subset [-\beta_{n_0}, \beta_{n_0}]$. Then, we can find convergent subsequence $(y_{k_l})_{l\in\mathbb{N}}$ of the sequence $(y_k)_{k\in\mathbb{N}}$. Based on the Lemma 3.1, the sequence $v_{k_l} = (\alpha_{n_0}, y_{k_l})$ is Cauchy sequence, so it is convergent. Hence, we have found convergent subsequence in the arbitrary sequence in K, that is K is compact.

(c) Let only finitely many points from the sequence $(v_k)_{k\in\mathbb{N}}$ belong to each of sets $[a, b] \times \{0\}$ and $\{\alpha_n\} \times [-\beta_n, \beta_n]$ ($n \in \mathbb{N}$). From each of the sets $\{\alpha_n\} \times [-\beta_n, \beta_n]$

that contain points of sequence $(v_k)_{k\in\mathbb{N}}$, let us choose one member of the given sequence. Notice, that there are infinitely many such sets. In this way, we constructed the subsequence $v_{k_n} = (x_{k_n}, y_{k_n})$ $(n \in \mathbb{N})$ of the sequence $(v_k)_{k\in\mathbb{N}}$. Actually, we formed the subsequence of the given sequence, such that the sequence of the first coordinates satisfies $(x_{k_n})_{n\in\mathbb{N}} \subset [a,b]$, so we can find the convergent subsequence $(x'_{k_n})_{n\in\mathbb{N}}$. If we consider appropriate subsequence of second coordinates $(y'_{k_n})_{n\in\mathbb{N}} \subset (y_{k_n})$, we have $(y'_{k_n})_{n\in\mathbb{N}} \subset [-\beta_n, \beta_n]$. Based on the "sandwich" theorem this sequence is convergent, furthermore $y'_{k_n} \to 0$ $(n \to \infty)$. Thus, using Lemma 3.1 we have that the subsequence $(v'_{k_n})_{n\in\mathbb{N}}$ is Cauchy sequence, so it converges. In this way, we proved that K is compact.

This completes the first part of the proof. □

Now, let us assume that K is compact set. Then K is closed. Let us assume that the condition of the theorem is not satisfied, i.e.

$$K \not\subseteq \{\alpha\} \times [c,d], \tag{3.1}$$

and

$$K \not\subseteq ([a,b] \times \{0\}) \cup \bigcup_{n=1}^{\infty} (\{\alpha_n\} \times [-\beta_n, \beta_n]), \tag{3.2}$$

where a, b, c, d, α_n and β_n $(n \in \mathbb{N})$ satisfy the assumptions of the theorem. Since the conditions (3.1) and (3.2) hold, we have three different possibilities.

(a) There exists infinite set $A \subseteq K$ such that

$$A = \{(x_n, 0) \in \mathbb{R}^2 \mid (x_n)_{n\in\mathbb{N}} \text{ is unbounded sequence}\}.$$

Then, there exists $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$, such that $x_{n_k} \to +\infty$ (or $x_{n_k} \to -\infty$). However, there is no convergent subsequence of the sequence $(v_k)_{k\in\mathbb{N}} \subset A$, where $v_k = (x_{n_k}, 0)$. Thus, K is not compact, which contradicts the initial assumption.

(b) There exists infinite set $A \subseteq K$ such that

$$A = \{(\alpha_n, y) \in \mathbb{R}^2 \mid n \in \mathbb{N},\ |y| \geq \delta > 0\},$$

and the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is non-constant sequence. We can eventually find convergent subsequence $(\alpha_{n_k})_{k\in\mathbb{N}}$ of the sequence $(\alpha_n)_{n\in\mathbb{N}}$, but because of the condition for the set A, there is no subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of the second coordinates such that $y_{n_k} \to 0$ $(k \to \infty)$. This means that neither one of the subsequences $(v_k)_{k\in\mathbb{N}} \subset A$, where $v_k = (\alpha_{n_k}, y_{n_k})$, is not Cauchy sequence in $\mathbb{R}^2$ with the river metric (Lemma 3.1). Therefore, there are no accumulation points of the set A, i.e. the set K is not compact. This is contradiction with the initial assumption.

(c) There exists infinite set $A \subseteq K$ such that

$$A = \{(\alpha, y_n) \in \mathbb{R}^2 \mid y_n \to +\infty \ (y_n \to -\infty) \ n \to \infty\}.$$

It is obvious that this set A has no accumulation points. Hence, the set K is not compact, which contradicts assumption about compactness of the set K.

In this way, we considered all possible cases when the conditions (3.1) and (3.2) are satisfied. Based on the reductio ad absurdum principle, we conclude that if K is compact, then following holds:

$$K \subseteq \{\alpha\} \times [c, d] \ , \quad \alpha, c, d \in \mathbb{R}$$

or

$$K \subseteq ([a, b] \times \{0\}) \cup \bigcup_{n=1}^{\infty} (\{\alpha_n\} \times [-\beta_n, \beta_n]).$$

□

Corollary 3.1. *The set $K \subset \mathbb{R}^2$ is compact if and only if*

(a) $K = \overline{B}((x^*, y^*), r)$*, for* $|y^*| \geq r$.

(b) $K = [a, b] \times \{0\}$*, for* $a, b \in \mathbb{R}$*,* $a \leq b$.

(c) *A set K is the most countable union of the sets from (a), that is*

$$K = \bigcup_{n=1}^{N} \overline{B}((x_n, y_n), r_n) \ (N \in \mathbb{N}) \ \ \textit{or} \quad K = \bigcup_{n=1}^{\infty} \overline{B}((x_n, y_n), r_n),$$

such that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, $y_n \to 0$ $(n \to \infty)$ and $|y_n| \geq r_n$ $(n \in \mathbb{N})$.

The set $A \subseteq \mathbb{R}^2$, such that $A = [a, b] \times \{c\}$, where $a < b$, $c \neq 0$ is not totally bounded set. Indeed, let us assume that set A is totally bounded. Then for $\varepsilon = |c|$ there exists covering $\{A_1, A_2, \ldots, A_n\}$ of the set A, i.e.

$$[a, b] \times \{c\} \subseteq \bigcup_{i=1}^{n} A_i.$$

Since there are finitely many sets A_i, then there exists at least one set A_{i_0}, $i_0 \in \{1, 2, \ldots, n\}$ such that $(a', c), (b', c) \in A_{i_0}$, where $a', b' \in [a, b]$, $a' < b'$. This means that $diamA_{i_0} \geq d^*((a', c), (b', c)) = 2|c| + b' - a'$. On the other hand, $diamA_{i_0} = |c|$, so we conclude that $|c| \geq 2|c| + b' - a'$, i.e. $|c| \leq a' - b' < 0$ which is contradiction.

Obviously, sets that contain sets of the form $A = [a, b] \times \{c\} \subseteq \mathbb{R}^2$, $c \neq 0$ and $a < b$, are not totally bounded. Based on that, balls of the form $B((x_0, y_0), r)$, where $|y_0| < r$ are not totally bounded sets in $\mathbb{R}^2$ with the river metric.

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