ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **13** (2024), no.1, 1–26
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.1.1

ESTABLISH OF THE JENSEN TYPE (Γ_1, Γ_2)-FUNCTIONAL INEQUALITIES BASED ON JENSEN TYPE FUNCTIONAL EQUATION WITH $3k$-VARIABLES IN COMPLEX BANACH SPACE

Ly Van An

ABSTRACT. In this paper, I work on expanding the Jensen (Γ_1, Γ_2)-function inequalities by relying on the general Jensen functional equation with 3k-variables on the complex Banach space. That's the main result in this.

1. INTRODUCTION

Let $\mathbf{X}$ and $\mathbf{Y}$ be a normed spaces on the same field $\mathbb{K}$, and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping. We use the notations $\|\cdot\|_{\mathbf{X}}, \|\cdot\|_{\mathbf{Y}}$ are the norms on $\mathbf{X}$ and on $\mathbf{Y}$ respectively. In this paper, In this paper, I study the relationship between Jensen-type functional equations and Jensen-type (Γ_1, Γ_2)-function inequalities when $\left(\mathbf{X}, \|\cdot\|_{\mathbf{X}}\right)$ is a complex normed vector spaces and $\left(\mathbf{Y}, \|\cdot\|_{\mathbf{Y}}\right)$ is a complex normed vector Banach spaces.

In fact, when $\mathbf{X}$ is a complex normed vector spaces and and $\mathbf{Y}$ is a complex Banach space we solve and prove the Hyers-Ulam stability of following relationship between Jensen-type (Γ_1, Γ_2)-function inequalities and Jensen-type functional

2020 *Mathematics Subject Classification.* 39B22, 39B82, 46S10.
Key words and phrases. Generalized Jensen type (Γ_1, Γ_2)-functional inequality, Generalized Jensen type functional equations, Hyers-Ulam-Rassias stability, complex Banach space, complex normed vector spaces.
Submitted: 14.12.2024; *Accepted:* 30.12.2024; *Published:* 23.01.2024.

equations:

$$\begin{aligned}&\left\|f\left(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i+\sum_{i=1}^{k}z_i\right)+f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}z_i\right)\right.\\&\left.-2\sum_{i=1}^{k}f\Big(x_i\Big)-2\sum_{i=1}^{k}f\Big(y_i\Big)\right\|_{\mathbf{Y}}\end{aligned}$$

(1.1)

$$\begin{aligned}&\leq\left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f\Big(x_i\Big)-\sum_{i=1}^{k}f\Big(y_i\Big)-\sum_{i=1}^{k}f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}\\&+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f\Big(x_i\Big)-\sum_{i=1}^{k}f\Big(y_i\Big)+\sum_{i=1}^{k}f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}\end{aligned}$$

and

$$\begin{aligned}&\left\|f\left(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i+\sum_{i=1}^{k}z_i\right)-f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}z_i\right)\right.\\&\left.-2\sum_{i=1}^{k}f\Big(y_i\Big)-2\sum_{i=1}^{k}f\Big(z_i\Big)\right\|_{\mathbf{Y}}\end{aligned}$$

(1.2)

$$\begin{aligned}&\leq\left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f\Big(x_i+y_i\Big)-\sum_{i=1}^{k}f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}\\&+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}f\Big(x_i\Big)-\sum_{i=1}^{k}f\Big(y_i\Big)+\sum_{i=1}^{k}f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}\end{aligned}$$

based on following Jensen type functional equations with $3k$-variable

$$\begin{aligned}&f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i+\sum_{i=1}^{k}z_i\right)+f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i-\sum_{i=1}^{k}z_i\right)\\&-2\sum_{i=1}^{k}f\Big(x_i\Big)-2\sum_{k=1}^{k}f\Big(y_i\Big)=0\end{aligned}\tag{1.3}$$

and

$$f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \sum_{i=1}^{k} z_i\right) - f\left(\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i - \sum_{i=1}^{k} z_i\right)$$

$$-2\sum_{i=1}^{k} f\left(y_i\right) - 2\sum_{k=1}^{k} f\left(z_i\right) = 0 \tag{1.4}$$

Note: With k is a positive integer and Γ_1, Γ_2 are the fixed complex numbers for $\left|\Gamma_1\right| \leq \frac{1}{2}, \left|\Gamma_2\right| \leq \frac{1}{2}$.

The study of the functional equation stability originated from a question of S.M. Ulam [1], concerning the stability of group homomorphisms. Let $\left(\mathbb{G}, *\right)$ be a group and let $\left(\mathbb{G}', \circ, d\right)$ be a metric group with metric $d\left(\cdot, \cdot\right)$. Geven $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : \mathbb{G} \to \mathbb{G}'$ satisfies

$$d\left(f\left(x * y\right), f\left(x\right) \circ f\left(y\right)\right) < \delta$$

for all $x, y \in \mathbb{G}$ then there is a homomorphism $h : \mathbb{G} \to \mathbb{G}'$ with

$$d\left(f\left(x\right), h\left(x\right)\right) < \epsilon$$

for all $x \in \mathbb{G}$?, if the answer, is affirmative, we would say that equation of homomorphism $h\left(x*y\right) = h\left(y\right)\circ h\left(y\right)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation? Hyers [2] gave a first affirmative answers the question of Ulam as follows.

Let E_1 be a normed space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies inequality

$$\left\|f\left(x + y\right) - f\left(x\right) - f\left(y\right)\right\| \leq \epsilon,$$

for all $x, y \in \mathbb{E}_1$ where $\epsilon \geq 0$ is a constan. Then the limit $T\left(x\right) = \lim_{n\to\infty} 2^{-n} f\left(2^n x\right)$ exists for each $x \in E_1$ and T is the unique additive mapping

$$\left\|f\left(x\right) - T\left(x\right)\right\| \leq \epsilon, \forall x \in \mathbb{E}_1.$$

Also if for each x the functional $t \to f\Big(xt\Big)$ from $\mathbb{R}$ to $\mathbb{E}_2$ is continuous on $\mathbb{R}$. If f continuous at a single point of $\mathbb{E}_1$, then T is continuous everywhere in $\mathbb{E}_1$ Next Th. M. Rassias [3] provided a generalization of Hyers' Theorem as a special case. Suppose $\mathbb{E}$ and $\mathbb{E}'$ is normed space with $\mathbb{E}'$ a complete normed space, $f : \mathbb{E} \to \mathbb{E}'$ is a mapping such that for each fixed $x \in E$ the mapping $t \to f\Big(xt\Big)$ is continuous on $\mathbb{R}$. Assume that there exist $\epsilon > 0$ and $p \in [0, 1]$ such that

$$\left\|f\Big(x+y\Big)-f\Big(x\Big)-f\Big(y\Big)\right\| \leq \epsilon\Big(\Big\|x\Big\|^p+\Big\|y\Big\|^p\Big), \forall x, y \in \mathbb{E}.$$

Then there exists a unique linear $L : \mathbb{E} \to \mathbb{E}'$ satisfies

$$\left\|f\Big(x\Big)-L\Big(x\Big)\right\| \leq \frac{\epsilon}{1-2^{1-p}}\|x\|^p, x \in \mathbb{E}.$$

The case of the existence of a unique additive mapping had been obtained by Aoki [4], as it is recently noticed by Lech Maligranda. However, Aoki [4] had claimed the existence of a unique linear mapping, that is not true because he did not allow the mapping f to satisfy some continuity assumption. Th. M. Rassis [5], who independently introduced the unbounded difference was the first to prove that there exists a unique linear mapping T satisfying

$$\left\|f\Big(x\Big)-T\Big(x\Big)\right\| \leq \frac{\epsilon}{1-2^{1-p}}\|x\|^p, x \in \mathbb{E}.$$

In 1990, Th. M. Rassias [6] during the 27th International Symposium on Functional Equation asked the question whether such a theorem can also be proved for $p \geq 1$.

In 1991, Z. Gajda [7] following the same approach as in Th. M. Rassias [8], gave an affirmative solution to this question for $p > 1$.

It was proved by Gajda [8], as well as by Th. M. Rassias and P. Semrl [8] that one can not prove a Th. M. Rassias type theorem when $p = 1$.

In 1994, P. Găvruta [9] provided a further generalization of Th. M. Rassias theorem in which he replaced the bounded $\epsilon\Big(\Big\|x\Big\|^p+\Big\|y\Big\|^p\Big)$ by a general control function $\psi\Big(x, y\Big)$ for the existence of a unique linear mapping.

In the article, based on the idea of World Mathematics [1]- [46], I have built in general the Jensen-type functional inequality relationship with The multivariable

Jensen equation on a complex Banach space is intended to improve the classical Jensen equation form with a limited number of variables, and its only solution is also a general additive function.

Recently, the author has formulated general inequalities on spaces such as Banach spaces and non-Archimedean Banach spaces see [10] [11].

$$\begin{aligned} &\Big\|\sum_{j=1}^{k} f\big(x_j\big)+\sum_{j=1}^{k} f\big(y_j\big)+\sum_{j=1}^{k} f\big(z_j\big)\Big\|_{\mathbf{Y}} \\ &\leq\Big\|2kf\Big(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{2k}\Big)\Big\|_{\mathbf{Y}}, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\Big\|\sum_{j=1}^{k} f\big(x_j\big)+\sum_{j=1}^{k} f\big(y_j\big)+\sum_{j=1}^{k} f\big(z_j\big)\Big\|_{\mathbf{Y}} \\ &\leq\Big\|f\Big(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\Big)\Big\|_{\mathbf{Y}}, \end{aligned} \tag{1.6}$$

finally

$$\begin{aligned} &\Big\|\sum_{j=1}^{k} f\big(x_j\big)+\sum_{j=1}^{k} f\big(y_j\big)+2k\sum_{j=1}^{k} f\big(z_j\big)\Big\|_{\mathbf{Y}} \\ &\leq\Big\|2kf\Big(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j}{2k}+\sum_{j=1}^{k}z_j\Big)\Big\|_{\mathbf{Y}}, \end{aligned} \tag{1.7}$$

in Banach space, And

$$\begin{aligned} &\Big\|\sum_{j=1}^{n} f\Big(x_j\Big)+\sum_{j=1}^{n} f\Big(\frac{x_{n+j}}{n}\Big)\Big\|_{\mathbf{Y}} \\ &\leq\Big\|kf\Big(\frac{\sum_{j=1}^{n}x_j}{k}+\frac{\sum_{j=1}^{n}x_{n+j}}{n\cdot k}\Big)\Big\|_{\mathbf{Y}}, \Big|n\Big|>\Big|k\Big|. \end{aligned} \tag{1.8}$$

in non-Archimedean Banach spaces.

So that we solve and proved the Hyers-Ulam type stability for functional equation (1.1) and (1.2) ie the functional equations with $3k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the functional equations (1.1) or (1.2). Thus, the results in this paper are generalization of those in [10] [11]. for functional equations with $3k$-variables.

The paper is organized as follows. In section preliminaries we remind some basic notations in [12] such as Solutions of the inequalities

Section 3: Stability of the Jensen type (Γ_1, Γ_2)-functional inequalities (1.1) associated for functional equation of (1.3).

Section 4: Stability of the Jensen type (Γ_1, Γ_2)-functional inequalities (1.2) associated for functional equation of (1.4).

2. PRELIMINARIES

2.1. **Solutions of the inequalities.** The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the Cauchuy equation. In particular, every solution of the Cauchuy equation is said to be an additive mapping. The functional equations

$$f(\frac{x+y}{2}) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

is called the Jensen equations. In particular, every solution of the Jensen equation is said to be a Jensen additive mapping.

3. STABILITY (Γ_1, Γ_2)-FUNCTIONAL INEQUALITIES (1.1) RELATIVE TO FUNCTIONAL EQUATION (1.3)

3.1. **Condition for existence of solution of (1.1).** In this section, assume that $\mathbf{X}$ is a complex normed vector spaces, $\mathbf{Y}$ is a complex Banach space and Γ_1, Γ_2 are the fixed complex numbers for $\left|\Gamma_1\right| \le \frac{1}{2}, \left|\Gamma_2\right| \le \frac{1}{2}$. Under this setting, we can show that the mappings satisfying (1.1) is additive.

Lemma 3.1. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping and it satisfies the functional inequality*

$$\Bigg\| f\left(\sum_{i=1}^{k}(x_i) + \sum_{i=1}^{k} y_i + \sum_{i=1}^{k} z_i\right) + f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \sum_{i=1}^{k} z_i\right)$$

$$
\begin{aligned}
&-2\sum_{i=1}^{k} f\Big(x_i\Big) - 2\sum_{i=1}^{k} f\Big(y_i\Big)\Bigg\|_{\mathbf{Y}}\\
&\leq \Bigg\|\Gamma_1\Bigg(f\Bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \sum_{i=1}^{k} z_i\Bigg) - \sum_{i=1}^{k} f\Big(x_i\Big)\\
&-\sum_{i=1}^{k} f\Big(y_i\Big) - \sum_{i=1}^{k} f\Big(z_i\Big)\Bigg)\Bigg\|_{\mathbf{Y}}\\
&+\Bigg\|\Gamma_2\Bigg(f\Bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \sum_{i=1}^{k} z_i\Bigg) - \sum_{i=1}^{k} f\Big(x_i\Big)\\
&-\sum_{i=1}^{k} f\Big(y_i\Big) + \sum_{i=1}^{k} f\Big(z_i\Big)\Bigg)\Bigg\|_{\mathbf{Y}}
\end{aligned} \tag{3.1}
$$

For all $x_i, y_i, z_j \in \mathbf{X}, i = 1 \to k$ *then* f *is additive.*

Proof. We replacing $\big(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\big)$ by$\big(0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0\big)$ in (3.1), we have

$$\Big(2\Big|2k-1\Big| - \Big|\Gamma_1\Big|\Big|3k-1\Big| - \Big|\Gamma_2\Big|\Big|k-1\Big|\Big)\Big\|f\Big(0\Big)\Big\|_{\mathbf{Y}} \leq 0,$$

Thus $f\Big(0\Big) = 0$.

Next, by replacing $\big(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\big)$ by$\big(0, \ldots, 0, 0, \ldots, 0, z, \ldots, 0\big)$ in (3.1), we have

$$\Big\|f\Big(z\Big) + f\Big(-z\Big)\Big\| \leq \Big\|\Gamma_2\Big(f\Big(z\Big) + f\Big(-z\Big)\Big)\Big\|_{\mathbf{Y}}$$

So

$$f\Big(-z\Big) = -f\Big(z\Big)$$

It follows that f is an odd mapping.

Next, by replacing $\big(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\big)$ by$\big(x_1, \ldots, x_k, y_1, \ldots, y_k, 0, \ldots, 0\big)$ in (3.1), we have

$$\Bigg\|2f\Bigg(\sum_{i=1}^{n} x_i + \sum_{i=1}^{k} y_i\Bigg) - 2\sum_{i=1}^{k} f\Big(x_i\Big) - 2\sum_{i=1}^{k} f\Big(y_i\Big)\Bigg\|_{\mathbf{Y}}$$

$$\begin{aligned}
&\leq \left\|\Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right) - \sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(y_i\right)\right)\right\|_{\mathbf{Y}} \\
&+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right) - \sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(y_i\right)\right)\right\|_{\mathbf{Y}} \\
&= \left(\left|\Gamma_1\right| + \left|\Gamma_2\right|\right)\left\|f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i\right) - \sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(y_i\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{3.2}$$

and so

$$f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i\right) = \sum_{i=1}^{k} f\left(x_i\right) + \sum_{i=1}^{k} f\left(y_i\right)$$

for all $x_i, y_i, z_i \in \mathbf{X}$ for all $i = 1 \to k$. Hence f is additive as we expected. □

Corollary 3.1. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfying*

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \sum_{i=1}^{k} z_i\right) + f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i\right)\right. \\
&\left. -2\sum_{i=1}^{k} f\left(x_i\right) - 2\sum_{i=1}^{k} f\left(y_i\right)\right\|_{\mathbf{Y}} = \left\|\Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i\right)\right.\right. \\
&\left.\left. - \sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(y_i\right) - \sum_{i=1}^{k} f\left(z_i\right)\right)\right\| + \left\|\Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i\right.\right.\right. \\
&\left.\left.\left. - \sum_{i=1}^{n} z_i\right) - \sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(y_i\right) + \sum_{i=1}^{k} f\left(z_i\right)\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{3.3}$$

for all $x_i, y_i, z_i \in \mathbf{X}$ for all $i = 1 \to k$. ,then f is additive.

3.2. **Constructing a solution for the function inequality 1.1.** In this section, we will build a solution for 1.1. assume that $\mathbf{X}$ is a complex normed vector spaces, $\mathbf{Y}$ is a complex Banach space.

Notice that here: With k is a positive integer and Γ_1, Γ_2 are the fixed complex numbers for $\left|\Gamma_1\right| \leq \frac{1}{2}, \left|\Gamma_2\right| \leq \frac{1}{2}$.

Theorem 3.1. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping. Let a function $\varphi : \mathbf{X^{3k}} \to \Big[0, \infty\Big)$, $\varphi\Big(0, 0, \dots, 0\Big) = 0$ such that*

$$
\begin{aligned}
&\Bigg\|f\Bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i\Bigg) + f\Bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i\Bigg)\\
&-2\sum_{i=1}^{k} f\Big(x_i\Big) - 2\sum_{i=1}^{k} f\Big(y_i\Big)\Bigg\|_{\mathbf{Y}} \le \Bigg\|\Gamma_1\Bigg(f\Bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i\Bigg)\\
&-\sum_{i=1}^{k} f\Big(x_i\Big) - \sum_{i=1}^{k} f\Big(y_i\Big) - \sum_{i=1}^{k} f\Big(z_i\Big)\Bigg)\Bigg\|_{\mathbf{Y}} + \Bigg\|\Gamma_2\Bigg(f\Bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i\\
&-\sum_{i=1}^{n} z_i\Bigg) - \sum_{i=1}^{k} f\Big(x_i\Big) - sum_{i=1}^{k} f\Big(y_i\Big) + \sum_{i=1}^{k} f\Big(z_i\Big)\Bigg)\Bigg\|_{\mathbf{Y}}\\
&+\varphi\big(x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k\big)
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
&\widetilde{\varphi}\Big(x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k\Big) := \sum_{j=1}^{\infty} \frac{1}{\Big|2k\Big|^j}\\
&\varphi\big((2k)^j x_1, \dots, (2k)^j x_k, (2k)^j y_1, \dots, (2k)^j y_k, (2k)^j z_1, \dots, (2k)^j z_k\big) < \infty
\end{aligned} \tag{3.5}
$$

for all $x_i, y_i, z_i \in \mathbf{X}$ for all $i = 1 \to k$. Then there exisists a unique additive mapping

$$H : \mathbf{X} \to \mathbf{Y}$$

$$\Big\|f\Big(x\Big) - H\Big(x\Big)\Big\|_{\mathbf{Y}} \le \widetilde{\varphi}\Big(x, \dots, x, x, \dots, x, 0, 0, \dots, 0\Big) \tag{3.6}$$

for all $x \in \mathbf{X}$.

Proof. We replacing $\big(x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k\big)$ by$\big(0, \dots, 0, 0, \dots, 0, 0, \dots, 0\big)$ in (3.4), we have

$$\Big(2\Big|2k-1\Big| - \Big|\Gamma_1\Big|\Big|3k-1\Big| - \Big|\Gamma_2\Big|\Big|k-1\Big|\Big)\Big\|f\Big(0\Big)\Big\|_{\mathbf{Y}} \le 0,$$

Thus $f\Big(0\Big) = 0$. Next, by replacing $\big(x_1, \dots, x_k, y_1, \dots, y_k,\ z_1, \dots, z_k\big)$ by$\big(x, \dots, x, x, \dots, x, 0, \dots, 0\big)$ in (3.4), we have

$$\left\|2f\Big(2kx\Big) - 4kf\Big(x\Big)\right\|_{\mathbf{Y}} \leq \Big|\Gamma_1\Big|\left\|f\Big(2kx\Big) - 2kf\Big(x\Big)\right\|_{\mathbf{Y}} + \Big|\Gamma_2\Big|\left\|f\Big(2kx\Big) - 2kf\Big(x\Big)\right\|_{\mathbf{Y}}$$

$$+ \varphi\Big(x, \ldots, x, x, \ldots, x, 0, \ldots, 0\Big) \tag{3.7}$$

for all $x \in \mathbf{X}$. Thus

$$\left\|f\Big(x\Big) - \frac{f\Big(2kx\Big)}{2k}\right\|_{\mathbf{Y}} \leq \frac{1}{2k} \cdot \frac{1}{2 - \Big|\Gamma_1\Big| - \Big|\Gamma_2\Big|}\varphi\Big(x, \ldots, x, x \ldots, x, 0, \ldots, 0\Big)$$

$$\leq \varphi\Big(x, \ldots, x, x \ldots, x, 0, 0, \ldots, 0\Big) \tag{3.8}$$

for all $x \in \mathbf{X}$. Hence one may have the following formula for positive integer m, l with $m > l$,

$$\left\|\frac{1}{\Big(2k\Big)^l} f\Big(\Big(2k\Big)^l x\Big) - \frac{1}{\Big(2k\Big)^m} f\Big(\Big(2k\Big)^m x\Big)\right\|_{\mathbf{Y}} \tag{3.9}$$

$$\leq \sum_{j=l}^{m-1} \frac{1}{\Big|2k\Big|^j}\varphi\Big((2k)^j x_1, (2k)^j x_2, \ldots, (2k)^j x_{2k}, 0 \ldots, 0\Big) \tag{3.10}$$

for all $x \in \mathbf{X}$ It follows from (3.8) that the sequence $\left\{\frac{f((2k)^n x)}{(2k)^n}\right\}$ is Cauchy sequence. Since $\mathbf{Y}$ is complete, we conclude that $\left\{\frac{f((2k)^n x)}{(2k)^n}\right\}$ is convergent. So one may define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) = \lim_{n\to\infty} \frac{f\Big((2k)^n x\Big)}{(2k)^n}, \forall x \in \mathbf{X}. \tag{3.11}$$

By taking $m = 0$ and letting $l \to \infty$ in (3.9), we get (3.6)

$$\left\|H\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i + \sum_{i=1}^k z_i\right) + H\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i - \sum_{i=1}^k z_i\right)\right.$$

$$\left. - 2\sum_{i=1}^k H\Big(x_i\Big) - 2\sum_{i=1}^k H\Big(y_i\Big)\right\|_{\mathbf{Y}}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big(2k\Big)^n \bigg\| f\bigg(\frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} x_i + \frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} y_i + \frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} z_i\bigg) \\
&\quad + f\bigg(\frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} x_i + \frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} y_i - \frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} z_i\bigg) - 2\sum_{i=1}^{k} f\bigg(\frac{x_i}{\big(2k\big)^n}\bigg) \\
&\quad - 2\sum_{i=1}^{k} f\bigg(\frac{y_i}{\big(2k\big)^n}\bigg)\bigg\|_{\mathbf{Y}} \\
&\leq \bigg\|\Gamma_1\bigg(f\bigg(\frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} x_i + \frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} y_i + \frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} z_i\big) - \sum_{i=1}^{k} f\bigg(\frac{x_i}{\big(2k\big)^n}\bigg) \\
&\quad - \sum_{i=1}^{k} f\bigg(\frac{y_i}{\big(2k\big)^n}\bigg) - \sum_{i=1}^{k} f\bigg(\frac{z_i}{\big(2k\big)^n}\bigg)\bigg)\bigg\|_{\mathbf{Y}} \\
&\quad + \bigg\|\Gamma_2\bigg(f\bigg(\frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} x_i + \frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} y_i - \frac{1}{\big(2k\big)^n}\sum_{i=1}^{k} z_i\bigg) - \sum_{i=1}^{k} f\Big(\frac{x_i}{\big(2k\big)^n}\Big) \\
&\quad - \sum_{i=1}^{k} f\bigg(\frac{y_i}{\big(2k\big)^n}\bigg) + \sum_{i=1}^{k} f\bigg(\frac{z_i}{\big(2k\big)^n}\big)\bigg)\bigg\|_{\mathbf{Y}} \\
&\quad + \Big(2k\Big)^n \varphi\bigg(\frac{x_1}{\big(2k\big)^n}, \dots, \frac{x_k}{\big(2k\big)^n}, \frac{y_1}{\big(2k\big)^n}, \dots, \frac{y_k}{\big(2k\big)^n} \frac{z_1}{\big(2k\big)^n}, \dots, \frac{z_k}{\big(2k\big)^n}\bigg) \\
&= \bigg\|\Gamma_1\bigg(H\bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \sum_{i=1}^{k} z_i\bigg) + H\bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \sum_{i=1}^{k} z_i\bigg) \\
&\quad - \sum_{i=1}^{k} H\Big(x_i\Big) - \sum_{i=1}^{k} H\Big(y_i\Big) - \sum_{i=1}^{k} H\Big(z_i\Big)\bigg)\bigg\|_{\mathbf{Y}} \\
&\quad + \bigg\|\Gamma_2\bigg(H\bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \sum_{i=1}^{k} z_i\bigg) + H\bigg(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i - \sum_{i=1}^{k} z_i\bigg) \\
&\quad - \sum_{i=1}^{k} H\Big(x_i\Big) - \sum_{i=1}^{k} H\Big(y_i\Big) + \sum_{i=1}^{k} H\Big(z_i\Big)\bigg)\bigg\|_{\mathbf{Y}}
\end{aligned}
$$

for all $x \in \mathbf{X}$. One can see that that H satisfies the inequality (3.1) and so it is additive by Lemma 3.1. Now, we show the uniqueness of $H : \mathbf{X} \to \mathbf{Y}$ be another additive mapping satisfying (3.2) then one has

$$\begin{aligned}
\Big\|H\Big(x\Big) - T\Big(x\Big)\Big\|_{\mathbf{Y}} &= \Bigg\|\frac{1}{\Big(2k\Big)^n}H\Big(\Big(2k\Big)^n x\Big) - \frac{1}{\Big(2k\Big)^n}T\Big(\Big(2k\Big)^n x\Big)\Bigg\|_{\mathbf{Y}} \\
&\leq \Bigg\|\frac{1}{\Big(2k\Big)^l}H\Big(\Big(2k\Big)^n x\Big) - \frac{1}{\Big(2k\Big)^n}f\Big(\Big(2k\Big)^n x\Big)\Bigg\|_{\mathbf{Y}} \\
&+ \Bigg\|\frac{1}{\Big(2k\Big)^n}f\Big(\Big(2k\Big)^l x\Big) - \frac{1}{\Big(2k\Big)^n}T\Big(\Big(2k\Big)^n x\Big)\Bigg\|_{\mathbf{Y}} \\
&\leq 2\frac{1}{\Big|2k\Big|^n}\widetilde{\varphi}\Big((2k)^n x, \ldots, (2k)^n x, (2k)^n x, \ldots, (2k)^n x, 0 \ldots, 0\Big) \qquad (3.12)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So we can couclude that $H\big(x\big) = T\big(x\big)$ for all $x \in \mathbf{X}$. □

Corollary 3.2. *Let $r < 1$ and θ be nonnegative real number, and suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that*

$$\begin{aligned}
&\Bigg\|f\Bigg(\sum_{i=1}^k x_i + \sum_{i=1}^n y_i + \sum_{i=1}^n z_i\Bigg) + f\Bigg(\sum_{i=1}^k x_i + \sum_{i=1}^n y_i - \sum_{i=1}^n z_i\Bigg) - 2\sum_{i=1}^k f\Big(x_i\Big) \\
&- 2\sum_{i=1}^k f\Big(y_i\Big)\Bigg\|_{\mathbf{Y}} \leq \Bigg\|\Gamma_1\Bigg(f\Bigg(\sum_{i=1}^k x_i + \sum_{i=1}^n y_i + \sum_{i=1}^n z_i\Bigg) - \sum_{i=1}^k f\Big(x_i\Big) \\
&- \sum_{i=1}^k f\Big(y_i\Big) - \sum_{i=1}^k f\Big(z_i\Big)\Bigg)\Bigg\|_{\mathbf{Y}} + \Bigg\|\Gamma_2\Bigg(f\Bigg(\sum_{i=1}^k x_i + \sum_{i=1}^n y_i - \sum_{i=1}^n z_i\Bigg) \\
&- \sum_{i=1}^k f\Big(x_i\Big) - \sum_{i=1}^k f\Big(y_i\Big) + \sum_{i=1}^k f\Big(z_i\Big)\Bigg)\Bigg\|_{\mathbf{Y}} + \theta\Bigg(\Big\|x_1\Big\|_{\mathbf{X}}^r \\
&+ \ldots + \Big\|x_k\Big\|_{\mathbf{X}}^r + \Big\|y_1\Big\|_{\mathbf{X}}^r + \ldots + \Big\|y_k\Big\|_{\mathbf{X}}^r + \Big\|z_1\Big\|_{\mathbf{X}}^r + \ldots + \Big\|z_k\Big\|_{\mathbf{X}}^r\Bigg) \qquad (3.13)
\end{aligned}$$

for all $x_i, y_i, z_i \in \mathbf{X}$ *for all* $i = 1 \to k$. *Then there exists a unique additive mapping* $H : \mathbf{X} \to \mathbf{Y}$

$$\left\| f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \leq \frac{2k\theta}{2k - \left(2k\right)^r} \left\|x\right\|_{\mathbf{X}}^r$$

for all $x \in \mathbf{X}$.

Theorem 3.2. *Suppose that* $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping. Let a function* $\varphi : \mathbf{X^{3k}} \to \left[0, \infty\right)$ *for* $\varphi\left(0, 0, \ldots, 0\right) = 0$ *such that*

$$\begin{aligned} &\left\| f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i\right) + f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i\right) \right. \\ &\left. - 2\sum_{i=1}^{k} f\left(x_i\right) - 2\sum_{i=1}^{k} f\left(y_i\right)\right\|_{\mathbf{Y}} \leq \left\|\Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i\right)\right.\right. \\ &\left.\left. - \sum_{i=1}^{k} f\left(x_i\right) - \sum_{i=1}^{k} f\left(y_i\right) - \sum_{i=1}^{k} f\left(z_i\right)\right)\right\|_{\mathbf{Y}} \\ &+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i\right) - \sum_{i=1}^{k} f\left(x_i\right)\right.\right. \\ &\left.\left. - \sum_{i=1}^{k} f\left(y_i\right) + \sum_{i=1}^{k} f\left(z_i\right)\right)\right\|_{\mathbf{Y}} + \varphi\left(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\right) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\widetilde{\varphi}\left(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\right) \\ &:= \sum_{j=1}^{\infty} \left|2k\right|^j \varphi\Big(\frac{x_1}{(2k)^j}, \ldots, \frac{x_k}{(2k)^j}, \frac{y_1}{(2k)^j}, \ldots, \frac{y_k}{(2k)^j}, \frac{z_1}{(2k)^j}, \ldots, \frac{z_k}{(2k)^j}\Big) < \infty \end{aligned} \tag{3.15}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k \in \mathbf{X}$. *Then there exisists a unique additive mapping*

$$H : \mathbf{X} \to \mathbf{Y}$$

$$\left\| f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \leq \widetilde{\varphi}\Big(\frac{x}{2k}, \ldots, \frac{x}{2k}, \frac{x}{2k}, \ldots, \frac{x}{2k}, 0, 0, \ldots, 0\Big)$$

for all $x \in \mathbf{X}$.

The proof is similar to Theorem 3.3.

Corollary 3.3. *Let $r < 1$ and θ be nonnegative real number, and suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that*

$$\begin{aligned}
&\Big\|f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}z_i\Big)+f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}z_i\Big)\\
&-2\sum_{i=1}^{k}f\big(x_i\big)-2\sum_{i=1}^{k}f\big(y_i\big)\Big\|_{\mathbf{Y}}\leq\Big\|\Gamma_1\Big(f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}z_i\Big)\\
&-\sum_{i=1}^{k}f\big(x_i\big)-\sum_{i=1}^{k}f\big(y_i\big)-\sum_{i=1}^{k}f\big(z_i\big)\Big)\Big\|_{\mathbf{Y}}\\
&+\Big\|\Gamma_2\Big(f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}z_i\Big)-\sum_{i=1}^{k}f\big(x_i\big)\\
&-\sum_{i=1}^{k}f\big(y_i\big)+\sum_{i=1}^{k}f\big(z_i\big)\Big)\Big\|_{\mathbf{Y}}+\theta\Big(\|x_1\|_{\mathbf{X}}^r+\ldots+\|x_k\|_{\mathbf{X}}^r\\
&+\|y_1\|_{\mathbf{X}}^r+\ldots+\|y_k\|_{\mathbf{X}}^r+\|z_1\|_{\mathbf{X}}^r+\ldots+\|z_k\|_{\mathbf{X}}^r\Big)
\end{aligned} \tag{3.16}$$

and for all $x_i, y_i, z_i \in \mathbf{X}$ for all $i = 1 \to k$. Then there exisists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$

$$\Big\|f\big(x\big)-H\big(x\big)\Big\|_{\mathbf{Y}}\leq\frac{\big(2k\big)^{r+1}\theta}{\big(2k\big)^{r}-1}\|x\|_{\mathbf{X}}^r$$

for all $x \in \mathbf{X}$.

4. Stability of the Jensen type (Γ_1, Γ_2)-functional inequalities (1.2) associated for functional equation (1.3).

4.1. **Condition for existence of solution of (1.2).** In this section, assume that $\mathbf{X}$ is a complex normed vector spaces, $\mathbf{Y}$ is a complex Banach space and Γ_1, Γ_2 are the fixed complex numbers for $|\Gamma_1| \leq \frac{1}{2}, |\Gamma_2| \leq \frac{1}{2}$. Under this setting, we can show that the mappings satisfying (1.2) is additive.

Lemma 4.1. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping and satisfying the functional inequality*

$$
\begin{aligned}
&\Bigg\| f\Big(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\Big)-f\Big(\sum_{i=1}^{k} x_i-\sum_{i=1}^{k} y_i-\sum_{i=1}^{k} z_i\Big) \\
&-2\sum_{i=1}^{k} f\big(y_i\big)-2\sum_{i=1}^{k} f\big(z_i\big)\Bigg\|_{\mathbf{Y}} \\
\leq &\Bigg\|\Gamma_1\Bigg(f\Big(\sum_{i=1}^{k} x_i+\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\Big)-\sum_{i=1}^{k} f\big(x_i+y_i\big)-\sum_{i=1}^{k} f\big(z_i\big)\Bigg)\Bigg\|_{\mathbf{Y}} \\
&+\Bigg\|\Gamma_2\Bigg(f\Big(\sum_{i=1}^{k} x_i+\sum_{i=1}^{k} y_i-\sum_{i=1}^{k} z_i\Big)-\sum_{i=1}^{k} f\big(x_i\big) \\
&-\sum_{i=1}^{k} f\big(y_i\big)+\sum_{i=1}^{k} f\big(z_i\big)\Bigg)\Bigg\|_{\mathbf{Y}}
\end{aligned}
\tag{4.1}
$$

Then f is additive.

Proof. We replacing $(x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k)$ by$(0,\ldots,0,0,\ldots,0,0,\ldots,0)$ in (4.1), we have

$$\Big(\big|4k\big|-\big|\Gamma_1\big|\big|2k-1\big|-\big|\Gamma_2\big|\big|k-1\big|\Big)\Big\|f\big(0\big)\Big\|_{\mathbf{Y}}\leq 0,$$

Thus $f\big(0\big)=0$. Next, by replacing $(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k)$ by$(0, \ldots, 0, 0, \ldots, 0, z, \ldots, 0)$ in (4.1), we have

$$\Big\|f\big(z\big)+f\big(-z\big)\Big\|_{\mathbf{Y}}\leq\Big\|\Gamma_2\Big(f\big(z\big)+f\big(-z\big)\Big)\Big\|_{\mathbf{Y}}$$

So, $f(-z)=-f(z)$ and it follows that f is an odd mapping.

Next, by replacing $(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k)$ by$(0, \ldots, 0, y_1, \ldots, y_k, z_1, \ldots, z_k)$ in (4.1), we have

$$
\begin{aligned}
&\Bigg\|f\Big(\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\Big)-f\Big(-\sum_{i=1}^{k} y_i-\sum_{i=1}^{k} z_i\Big)-2\sum_{i=1}^{k} f\big(y_i\big)-2\sum_{i=1}^{k} f\big(z_i\big)\Bigg\|_{\mathbf{Y}} \\
\leq &\Bigg\|\Gamma_1\Bigg(f\Big(\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\Big)-\sum_{i=1}^{k} f\big(y_i\big)-\sum_{i=1}^{k} f\big(z_i\big)\Bigg)\Bigg\|_{\mathbf{Y}}
\end{aligned}
\tag{4.2}
$$

$$+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k} y_i-\sum_{i=1}^{k} z_i\right)+\sum_{i=1}^{k} f\left(y_i\right)+\sum_{i=1}^{k} f\left(z_i\right)\right)\right\|_{\mathbf{Y}}$$

for all $y_i, z_i \in \mathbf{X}$ for all $i=1 \rightarrow k$ Thus

$$\begin{aligned}
&\left(2-\left|\Gamma_1\right|\right)\left\|f\left(\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\right)-\sum_{i=1}^{k} f\left(y_i\right)-\sum_{i=1}^{k} f\left(z_i\right)\right\|_{\mathbf{Y}} \\
&\leq\left|\Gamma_2\right|\left\|f\left(\sum_{i=1}^{k} y_i-\sum_{i=1}^{k} z_i\right)-\sum_{i=1}^{k} f\left(y_i\right)+\sum_{i=1}^{k} f\left(z_i\right)\right\|_{\mathbf{Y}} \qquad (4.3)
\end{aligned}$$

for all $y_i, z_i \in \mathbf{X}$ for all $i=1 \rightarrow k$ Thus, Next, by replacing $\left(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\right)$ by$\left(0, \ldots, 0, y_1, \ldots, y_k,-z_1, \ldots,-z_k\right)$ in (4.1), we have

$$\begin{aligned}
&\left(2-\left|\Gamma_1\right|\right)\left\|f\left(\sum_{i=1}^{k} y_i-\sum_{i=1}^{k} z_i\right)-\sum_{i=1}^{k} f\left(y_i\right)+\sum_{i=1}^{k} f\left(z_i\right)\right\|_{\mathbf{Y}} \\
&\leq\left|\Gamma_2\right|\left\|f\left(\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\right)-\sum_{i=1}^{k} f\left(y_i\right)-\sum_{i=1}^{k} f\left(z_i\right)\right\|_{\mathbf{Y}} \qquad (4.4)
\end{aligned}$$

for all $y_i, z_i \in \mathbf{X}$ for all $i=1 \rightarrow k$ From and we get

$$\begin{aligned}
&\left(2-\left|\Gamma_1\right|\right)^2\left\|f\left(\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\right)-\sum_{i=1}^{k} f\left(y_i\right)-\sum_{i=1}^{k} f\left(z_i\right)\right\|_{\mathbf{Y}} \\
&\leq\left|\Gamma_2\right|^2\left\|f\left(\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\right)-\sum_{i=1}^{k} f\left(y_i\right)-\sum_{i=1}^{k} f\left(z_i\right)\right\|_{\mathbf{Y}} \qquad (4.5)
\end{aligned}$$

for all $y_i, z_i \in \mathbf{X}$ for all $i=1 \rightarrow k$. So, $f: \mathbf{X} \rightarrow \mathbf{Y}$ is additive. □

s

Corollary 4.1. *Suppose that $f: \mathbf{X} \rightarrow \mathbf{Y}$ be a odd mapping and satisfying the functional inequality*

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^{k}\left(x_i\right)+\sum_{i=1}^{k} y_i+\sum_{i=1}^{k} z_i\right)-f\left(\sum_{i=1}^{k} x_i-\sum_{i=1}^{k} y_i-\sum_{i=1}^{k} z_i\right)\right. \\
&\left.-2 \sum_{i=1}^{k} f\left(y_i\right)-2 \sum_{i=1}^{k} f\left(z_i\right)\right\|_{\mathbf{Y}}
\end{aligned}$$

$$= \left\|\Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i + \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\Big(x_i + y_i\Big) - \sum_{i=1}^{k} f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}$$

(4.6)

$$+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i - \sum_{i=1}^{k} z_i\right) - \sum_{i=1}^{k} f\Big(x_i\Big) + \sum_{i=1}^{k} f\Big(y_i\Big) + \sum_{i=1}^{k} f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}$$

for all $x_1, x_2, \ldots, x_{3k} \in \mathbf{X}$. *Hence* $f : X \to Y$ *is additive*

4.2. **Constructing a solution for the function inequality (1.2).** In this section, we will build a solution for (1.2). assume that $\mathbf{X}$ is a complex normed vector spaces, $\mathbf{Y}$ is a complex Banach space Notice that here: With k is a positive integer and Γ_1, Γ_2 are the fixed complex numbers for $\Big|\Gamma_1\Big| \leq \frac{1}{2}, \Big|\Gamma_2\Big| \leq \frac{1}{2}$.

Theorem 4.1. *Suppose that* $f : \mathbf{X} \to \mathbf{Y}$ *be a odd mapping. Let a function* $\varphi : \mathbf{X}^{\mathbf{3k}} \to \Big[0, \infty\Big)$ *with* $\varphi\Big(0, 0, \ldots, 0\Big) = 0$ *such that*

$$\left\|f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i\right) - f\left(\sum_{i=1}^{k} x_i - \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i\right)\right.$$

$$\left. - 2\sum_{i=1}^{k} f\Big(y_i\Big) - 2\sum_{i=1}^{k} f\Big(z_i\Big)\right\|_{\mathbf{Y}}$$

$$\leq \left\|\Gamma_1\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i\right) - \sum_{i=1}^{k} f\Big(x_i + y_i\Big) - \sum_{i=1}^{k} f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}$$

$$+ \left\|\Gamma_2\left(f\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i\right) - \sum_{i=1}^{k} f\Big(x_i\Big) - \sum_{i=1}^{k} f\Big(y_i\Big)\right.\right.$$

$$\left.\left. + \sum_{i=1}^{k} f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}} + \varphi\Big(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\Big) \tag{4.7}$$

and

$$\widetilde{\varphi}\left(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\right) := \sum_{j=1}^{\infty} \frac{1}{\Big|2k\Big|^j} \varphi\Big((2k)^j x_1, \ldots, (2k)^j x_k,$$

$$(2k)^j y_1, \ldots, (2k)^j y_k, (2k)^j z_1, \ldots, (2k)^j z_k\Big) < \infty \tag{4.8}$$

for all $x_1, x_2, \ldots, x_{3k} \in \mathbf{G}$*. Then there exisists a unique additive mapping* $H : \mathbf{X} \to \mathbf{Y}$

$$\left\| f\left(x\right) - H\left(x\right) \right\|_{\mathbf{Y}} \leq \widetilde{\varphi}\left(x, \ldots, x, 0, \ldots, 0, x, \ldots, x\right) \tag{4.9}$$

for all $x \in \mathbf{X}$.

Proof. We replacing $\left(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\right)$ by$\left(0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0\right)$ in (4.7), we have

$$\left(\left|4k\right| - \left|\Gamma_1\right|\left|2k - 1\right| - \left|\Gamma_2\right|\left|k - 1\right|\right)\left\| f\left(0\right) \right\|_{\mathbf{Y}} \leq 0,$$

Thus $f\left(0\right) = 0$. Next, by replacing $\left(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\right)$ by $\left(0, \ldots, 0, 0, \ldots, 0, z, \ldots, 0\right)$ in (4.7), we have

$$\left\| f\left(z\right) + f\left(- z\right) \right\|_{\mathbf{Y}} \leq \left\| \Gamma_2\left(f\left(z\right) + f\left(- z\right)\right) \right\|_{\mathbf{Y}}.$$

So,

$$f(-z) = -f(z).$$

It follows that f is an odd mapping. Next, by replacing $\left(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k\right)$ by$\left(x, \ldots, x, 0, \ldots, 0, x, \ldots, x\right)$ in (4.7), we have

$$\begin{aligned} \left\| f\left(2kx\right) - 2kf\left(x\right) \right\|_{\mathbf{Y}} &\leq \left|\Gamma_1\right| \left\| f\left(2kx\right) - 2kf\left(x\right) \right\|_{\mathbf{Y}} \\ &\quad + \varphi\left(x, \ldots, x, 0, \ldots, 0, x, \ldots, x\right) \end{aligned} \tag{4.10}$$

for all $x \in \mathbf{X}$. Thus

$$\begin{aligned} \left\| f\left(x\right) - \frac{f\left(2kx\right)}{2k} \right\|_{\mathbf{Y}} &\leq \frac{1}{2k} \cdot \frac{1}{1 - \left|\Gamma_1\right|}\varphi\left(x, \ldots, x, 0, \ldots, 0, x, \ldots, x\right) \\ &\leq \varphi\left(x, \ldots, x, 0, \ldots, 0, x, \ldots, x\right) \end{aligned} \tag{4.11}$$

for all $x \in \mathbf{X}$, $\left|\Gamma_1\right| < \frac{1}{2}$, $\frac{1}{1 - \left|\Gamma_1\right|} < 2$. Hence, one may have the following formula for positive integer m, l with $m > l$,

$$\left\| \frac{1}{\left(2k\right)^l} f\left(\left(2k\right)^l x\right) - \frac{1}{\left(2k\right)^m} f\left(\left(2k\right)^m x\right) \right\|_{\mathbf{Y}}$$

$$\leq \sum_{j=l}^{m-1} \frac{1}{\left|2k\right|^j}\varphi\Big((2k)^j x, \dots, (2k)^j x, 0, \dots, 0, (2k)^j x, \dots, (2k)^j x\Big) \tag{4.12}$$

for all $x \in \mathbf{X}$ It follows from (4.12) that the sequence $\left\{\frac{f((2k)^n x)}{(2k)^n}\right\}$ is Cauchy sequence. Since $\mathbf{Y}$ is complete, we conclude that $\left\{\frac{f((2k)^n x)}{(2k)^n}\right\}$ is convergent. So one may define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by $H(x) = \lim_{n\to\infty} \frac{f((2k)^n x)}{(2k)^n}, \forall x \in \mathbf{X}$. By taking $m = 0$ and letting $l \to \infty$ in (4.12), we get (4.9). It follows form (4.8) that

$$\begin{aligned}
&g\Bigg\|H\left(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i + \sum_{i=1}^k z_i\right) - H\left(\sum_{i=1}^k x_i - \sum_{i=1}^k x_{k+i} - \sum_{i=1}^k x_{2k+i}\right)\\
&- 2\sum_{i=1}^k H\Big(y_i\Big) - 2\sum_{i=1}^k H\Big(z_i\Big)\Bigg\|_{\mathbf{Y}}\\
&= \lim_{n\to\infty}\Big(2k\Big)^n\Bigg\|f\left(\frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k x_i + \frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k y_i + \frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k z_i\right)\\
&- f\left(\frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k x_i - \frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k y_i - \frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k z_i\right) - 2\sum_{i=1}^k f\left(\frac{y_i}{\Big(2k\Big)^n}\right)\\
&- 2\sum_{i=1}^k f\left(\frac{z_i}{\Big(2k\Big)^n}\right)\Bigg\|_{\mathbf{Y}}\\
&\leq \lim_{n\to\infty}\Big(2k\Big)^n\Bigg\|\Gamma_1\Bigg(f\left(\frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k x_i + \frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k y_i + \frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k z_i\right)-\\
&- \sum_{i=1}^k f\left(\frac{x_i + y_i}{\Big(2k\Big)^n}\right) - \sum_{i=1}^k f\left(\frac{z_i}{\Big(2k\Big)^n}\right)\Bigg)\Bigg\|_{\mathbf{Y}}\\
&+ \lim_{n\to\infty}\Big(2k\Big)^n\Bigg\|\Gamma_2\Bigg(f\left(\frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k x_i - \frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k y_i - \frac{1}{\Big(2k\Big)^n}\sum_{i=1}^k z_i\right)\\
&- \sum_{i=1}^k f\left(\frac{x_i}{\Big(2k\Big)^n}\right) + \sum_{i=1}^k f\left(\frac{y_i}{\Big(2k\Big)^n}\right) + \sum_{i=1}^k f\left(\frac{z_i}{\Big(2k\Big)^n}\right)\Bigg)\Bigg\|_{\mathbf{Y}}
\end{aligned}$$

$$+\lim_{n\to\infty}\Big(2k\Big)^n\varphi\left(\frac{x_1}{\Big(2k\Big)^n},\frac{x_2}{\Big(2k\Big)^n},\ldots,\frac{x_{3k}}{\Big(2k\Big)^n}\right)$$
$$=\left\|\Gamma_1\left(H\left(\sum_{i=1}^k x_i+\sum_{i=1}^k y_i+\sum_{i=1}^k z_i\right)-\sum_{i=1}^k H\Big(x_i+y_i\Big)-\sum_{i=1}^k H\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}$$
$$+\left\|\Gamma_2\left(H\left(\sum_{i=1}^k x_i+\sum_{i=1}^k y_i-\sum_{i=1}^k z_i\right)-\sum_{i=1}^k H\Big(x_i\Big)\right.\right.$$
$$\left.\left.-\sum_{i=1}^k H\Big(y_i\Big)+\sum_{i=1}^k H\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}$$

for all $x\in\mathbf{X}$. One can see that that H satisfies the inequalitty (4.1) and so it is additive by Lemma 4.1. Now, we show the uniqueness of H. Let $H:\mathbf{X}\to\mathbf{Y}$ be another additive mapping satisfying (4.7) then one has

$$\Big\|H\Big(x\Big)-T\Big(x\Big)\Big\|_{\mathbf{Y}}$$
$$=\left\|\frac{1}{\Big(2k\Big)^n}H\Big(\Big(2k\Big)^n x\Big)-\frac{1}{\Big(2k\Big)^n}T\Big(\Big(2k\Big)^n x\Big)\right\|_{\mathbf{Y}}$$
$$\le\left\|\frac{1}{\Big(2k\Big)^l}H\Big(\Big(2k\Big)^n x\Big)-\frac{1}{\Big(2k\Big)^n}f\Big(\Big(2k\Big)^n x\Big)\right\|_{\mathbf{Y}}$$
$$+\left\|\frac{1}{\Big(2k\Big)^n}f\Big(\Big(2k\Big)^l x\Big)-\frac{1}{\Big(2k\Big)^n}T\Big(\Big(2k\Big)^n x\Big)\right\|_{\mathbf{Y}}$$
$$\le 2\frac{1}{\Big|2k\Big|^n}\widetilde{\varphi}\Big((2k)^n x,\ldots,(2k)^n x,0,0,\ldots,0,(2k)^n x,\ldots,(2k)^n x\Big) \tag{4.13}$$

which tends to zero as $n\to\infty$ for all $x\in\mathbf{X}$. So we can couclude that $H\Big(x\Big)=T\Big(x\Big)$ for all $x\in\mathbf{X}$. □

Corollary 4.2. *Let $r<1$ and θ be nonnegative real number, and suppose that $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that*

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}z_i\right)-f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}z_i\right)\right.\\
&\left.-2\sum_{i=1}^{k}f\left(y_i\right)-2\sum_{i=1}^{k}f\left(z_i\right)\right\|_{\mathbf{Y}}\\
&\leq\left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}z_i\right)-\sum_{i=1}^{k}f\left(x_i+y_i\right)-\sum_{i=1}^{k}f\left(z_i\right)\right)\right\|_{\mathbf{Y}}\\
&+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}z_i\right)-\sum_{i=1}^{k}f\left(x_i\right)-\sum_{i=1}^{k}f\left(y_i\right)\right.\right.\\
&\left.\left.+\sum_{i=1}^{k}f\left(z_i\right)\right)\right\|_{\mathbf{Y}}+\theta\left(\left\|x_1\right\|_{\mathbf{X}}^{r}+\ldots+\left\|x_k\right\|_{\mathbf{X}}^{r}\right.\\
&\left.+\left\|y_1\right\|_{\mathbf{X}}^{r}+\ldots+\left\|y_k\right\|_{\mathbf{X}}^{r},\left\|z_1\right\|_{\mathbf{X}}^{r}+\ldots+\left\|z_k\right\|_{\mathbf{X}}^{r}\right)
\end{aligned}\tag{4.14}$$

and for all $x_i, y_i, z_i \in \mathbf{X}$, for all $i = 1 \to k$. Then there exisists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$

$$\left\|f\left(x\right)-H\left(x\right)\right\|_{\mathbf{Y}}\leq\frac{2k\theta}{2k-\left(2k\right)^{r}}\left\|x\right\|_{\mathbf{X}}^{r}$$

for all $x \in \mathbf{X}$.

Theorem 4.2. *Suppose that $f : \mathbf{X} \to \mathbf{Y}$ be a mapping. If there a function $\varphi : \mathbf{X}^{\mathbf{3k}} \to \left[0,\infty\right)$ with $\varphi\left(0,0,\ldots,0\right) = 0$ such that*

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}z_i\right)-f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}z_i\right)\right.\\
&\left.-2\sum_{i=1}^{k}f\left(y_i\right)-2\sum_{i=1}^{k}f\left(z_i\right)\right\|_{\mathbf{Y}}\\
&\leq\left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}z_i\right)-\sum_{i=1}^{k}f\left(x_i+y_i\right)-\sum_{i=1}^{k}f\left(z_i\right)\right)\right\|_{\mathbf{Y}}
\end{aligned}\tag{4.15}$$

$$+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}z_i\right)-\sum_{i=1}^{k}f\Big(x_i\Big)-\sum_{i=1}^{k}f\Big(y_i\Big)\right.\right.$$
$$\left.\left.+\sum_{i=1}^{k}f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}+\varphi\Big(x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k\Big)$$

and

$$\widetilde{\varphi}\Big((x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k\Big):=\sum_{j=1}^{\infty}\Big|2k\Big|^{j}\varphi\Big(\frac{x_1}{(2k)^j},\ldots,\frac{x_k}{(2k)^j},$$
$$\frac{y_1}{(2k)^j},\ldots,\frac{y_k}{(2k)^j},\frac{z_1}{(2k)^j},\ldots,\frac{z_k}{(2k)^j}\Big)<\infty \tag{4.16}$$

for all $x_i, y_i, z_i \in \mathbf{X}$*. Then there exisists a unique additive mapping* $H:\mathbf{X}\to\mathbf{Y}$

$$\left\|f\Big(x\Big)-H\Big(x\Big)\right\|_{\mathbf{Y}}\le\widetilde{\varphi}\Big(\frac{x}{2k},\ldots,\frac{x}{2k},0,\ldots,0,\frac{x}{2k},\ldots,\frac{x}{2k}\Big)$$

for all $x\in\mathbf{X}$*.*

The proof is similar to Theorem 4.3.

Corollary 4.3. *Let* $r<1$ *and* θ *be nonnegative real number, and suppose that* $f:\mathbf{X}\to\mathbf{Y}$ *be a mapping such that*

$$\left\|f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}z_i\right)-f\left(\sum_{i=1}^{k}x_i-\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}z_i\right)\right.$$
$$\left.-2\sum_{i=1}^{k}f\Big(y_i\Big)-2\sum_{i=1}^{k}f\Big(z_i\Big)\right\|_{\mathbf{Y}}$$
$$\le\left\|\Gamma_1\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}z_i\right)-\sum_{i=1}^{k}f\Big(x_i+y_i\Big)-\sum_{i=1}^{k}f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}$$
$$+\left\|\Gamma_2\left(f\left(\sum_{i=1}^{k}x_i+\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}z_i\right)-\sum_{i=1}^{k}f\Big(x_i\Big)-\sum_{i=1}^{k}f\Big(y_i\Big)+\sum_{i=1}^{k}f\Big(z_i\Big)\right)\right\|_{\mathbf{Y}}$$
$$+\theta\left(\Big\|x_1\Big\|_{\mathbf{X}}^{r}+\ldots+\Big\|x_k\Big\|_{\mathbf{X}}^{r}+\Big\|y_1\Big\|_{\mathbf{X}}^{r}+\ldots+\Big\|y_k\Big\|_{\mathbf{X}}^{r},\Big\|z_1\Big\|_{\mathbf{X}}^{r}+\ldots+\Big\|z_k\Big\|_{\mathbf{X}}^{r}\right)$$

and for all $x_i, y_i, z_i \in \mathbf{X}$*, for all* $i = 1 \to k$*. Then there exists a unique additive mapping* $H : \mathbf{X} \to \mathbf{Y}$

$$\left\|f\Big(x\Big) - H\Big(x\Big)\right\|_{\mathbf{Y}} \leq \frac{(2k)^{r+1}\theta}{\Big(2k\Big)^r - 1}\Big\|x\Big\|_{\mathbf{X}}^r$$

for all $x \in \mathbf{X}$*.*

5. CONCLUSION

In this paper, I build the correlation of functional equations and (Γ_1, Γ_2)-functional inequalities as an inseparable link on spaces in general, then I show their solutions. It is an ideal of Modern Mathematics.

REFERENCES

[1] S.M. ULAM: *A collection of Mathematical problems*, volume 8, Interscience Publishers, New York, 1960.

[2] DONALD H. HYERS: *On the stability of the functional equation*, Proceedings of the National Academy of the United States of America, **27** (4) (1941), 222.

[3] TH. M. RASSIAS: *On the stability of the linear mapping in Banach spaces*, Proceedings of the American Mathematical Society, **72** (2) (1978), 297-300.

[4] T. AOKI: *On the Stability of the Linear Transformation in Banach Space*, Journal of the Mathematical Society of Japan, **2**(1950), 64-66.

[5] T. M. RASSIAS: *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., **251** (2000), 264-284.

[6] TH. M. RASSIAS: *Problem 16; 2, Report of the 27th International Symp. on Functional Equations*, Aequationes Mathematicae, **39**(309) (1990), 292-293.

[7] Z. GAJDA: *On stability of additive mappings*, International Journal of Mathematics and Mathematical Sciences, **14**(3) (1991), 431-434.

[8] TH. M. RASSIAS, P. SEMRL: *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proceedings of the American Mathematical Society, **114**(4) (1992), 989-993.

[9] P. GăVRUT: *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, Journal of Mathematical Analysis and Applications, **184**(3) (1994), 431-436.

[10] LY VAN AN: *Generalized Hyers-Ulam stability of the additive functional inequalities with 2n-variables in non-Archimedean Banach space*, Bulletin of mathemtics and statistics research, **9**(3) (2021), 1-8.

[11] LY VAN AN: *Generalized Stability of Functional Inequalities with 3k-Variables Associated for Jordan-von Neumann-Type Additive Functional Equation*, Open Access Library Journal, **10**(1) (2023), 1-17.

[12] Jung Rye Lee: *Choonkil Park and Dong Yun Shin Additive and Quadratic Functional in Equalities in Non-Archimedean Normed Spaces*, Int. Journal of Math. Analysis, **8**(25) (2014), 1233 - 1247.

[13] J. M. Rassias: *On approximation of approximately linear mappings by linear mappings*, Journal of Functional Analysis, **46**(1) (1982), 126-130.

[14] J. Acze'l, J. Dhombres: *Functional equations in several variables, with applications to mathematics, information theory and to the natural and social sciences*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1989.

[15] L. Cadariu, V. Radu: *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math., **4** (2003), 7 pages.

[16] I.S. Chang, M. Eshaghi Gordji, H. Khodaei, H.M. Kim: *Nearly quartic mappings in ß-homogeneous F-spaces*, Results Math., **63** (2013), 529541.

[17] Y.J. Cho, C.K. Park, R. Saadati: *Functional inequalities in non-Archimedean Banach spaces*, Appl. Math. Lett., **23** (2010), 1238-1242.

[18] P.W. Cholewa: *Remarks on the stability of functional equations*, Aequationes Math., **27** (1984), 76-86.

[19] J.B. Diaz, B. Margolis: *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305-309.

[20] A. Ebadian, N. Ghobadipour, T.M. Rassias, M. Eshaghi Gordji: *Functional inequalities associated with Cauchy additive functional equation in non-Archimedean spaces*, Discrete Dyn. Nat. Soc., **2011**(14) (2011), 14 pages.

[21] G. Isac, T. M. Rassias: *On the Hyers-Ulam stability of additive mappings*, J. Approx. Theory, **72** (1993), 131-137.

[22] V. Lakshmikantham, S. Leela, J. Vasundhara Devi: *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.

[23] S.B. Lee, J.H. Bae, W.G. Park: *On the stability of an additive functional inequality for the fixed point alternative*, J. Comput. Anal. Appl., **17** (2014), 361-371.

[24] G. Lu, C. K. Park: *Hyers-Ulam Stability of Additive Set-valued Functional Equations*, Appl. Math. Lett., **24**(1) (2011), 1312-1316.

[25] C.K. Park, Y.S. Cho, M.-H. Han: *Functional inequalities associated with Jordan-von Neumann-type additive functional equations*, J. Inequal. Appl., **2007**(1) (2007), 13 pages.

[26] T.M. Rassias: *Functional equations and inequalities*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 2000.

[27] T. M. Rassias: *On the stability of functional equations and a problem of Ulam*, Acta Math. Appl., **62** (2000), 23-130.

[28] J.M. Rassias: *Refined Hyers-Ulam approximation of approximately Jensen type mappings*, Bull. Sci. Math., **131** (2007), 89-98.

[29] J.M. Rassias, M.J. Rassias: *On the Ulam stability of Jensen and Jensen type mappings on restricted domains*, J. Math. Anal. Appl., **281**(1) (2003), 516-524.

[30] K.-W. JUN, Y.-H. LEE: *A generalization of the Hyers-Ulam-Rassias stability of the pexiderized quadratic equations*, Journal of Mathematical Analysis and Applications, **297**(1) (2004), 70-86.

[31] A. GILANYI: *Eine zur Parallelogrammgleichung aquivalente Ungleichung*, Aequationes Mathematicae, **62**(3) (2001), 303-309.

[32] A. GILA'NYI: *On a problem by K. Nikodem*, Mathematical Inequalities Applications, **5**(4) (2002), 707-710.

[33] A. BAHYRYCZ, M. PISZCZEK: *Hyers stability of the Jensen function equation*, Acta Math. Hungar., **142** (2014), 353-365.

[31] M. BALCEROWSKI: *On the functional equations related to a problem of z Boros and Z. Dróczy*, Acta Math. Hungar., **138** (2013), 329-340.

[34] W. FECHNER: *Stability of a functional inequlities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149-161.

[35] J. SCHWAIGER: *A system of two inhomogeneous linear functional equations*, Acta Math. Hungar **140** (2013), 377-406.

[36] L. MALIGRANDA: *Tosio Aoki (1910-1989)*, in International symposium on Banach and function spaces, Yokohama Publishers, 2008.

[37] A. NAJATI, G.Z. ESKANDANI: *Stability of a mixed additive and cubic functional equation in quasi-Banach spaces*, J. Math. Anal. Appl. **342**(2) (2008), 1318-1331.

[38] ATTILA GILáNYI: *On a problem by K. Nikodem*, Math. Inequal. Appl., **5** (2002), 707-710.

[39] Jürg RäTZ: *On inequalities assosciated with the jordan-von neumann functional equation*, Aequationes matheaticae, **66** (1) (2003), 191-200.

[40] W. FECHNER: *On some functional inequalities related to the logarithmic mean*, Acta Math. Hungar., **128**(31-45) (2010), 303-309.

[41] C. PARK: *Additive β-functional inequalities*, Journal of Nonlinear Science and Appl. **7** (2014), 296-310.

[42] LY VAN AN: *Hyers-Ulam stability of functional inequalities with three variable in Banach spaces and Non-Archemdean Banach spaces*, International Journal of Mathematical Analysis, **13**(11) (2019), 296-310.

[43] Y.J. CHO, C. PARK, R. SAADATI: *Functional in equalities in Non-Archimedean normed spaces*, Applied Mathematics Letters **23** (2010), 1238-1242.

[44] Y. ARIBOU, S. KABBAJ: *Generalized functional in inequalities in Non-Archimedean normed spaces*, Applied Mathematics Letters **2** (2018), 61-66.

[45] LY VAN AN: *Hyers-Ulam stability additive β-functional inequalities with three variable in Banach spaces and Non-Archemdean Banach spaces*, International Journal of Mathematical Analysis, **14**(5-8) (2020), 296-310. .

[46] LY VAN AN: *Establish an additive (s; t)-function inequlities fixed point method and direct method with n-variables Banach space*, Jounal of mathematics, **9**(1) (2023).

Faculty of Mathematics Teacher Education,
Tay Ninh University,
Tay Ninh, Vietnam.
Email address: lyvanan145@gmail.com, lyvananvietnam@gmail.com