Advances in Mathematics: Scientific Journal **13** (2024), no.4, 463–477
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.4.1

ON FS-LIFTING AND CFS-LIFTING SEMIMODULES OVER SEMIRINGS

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ABSTRACT. In this paper, we introduce the notions of fs-lifting and cfs-lifting semimodules as respectively the generalizations of finitely lifting (f-lifting for short) semimodules and cofinitely lifting (cf-lifting) semimodules. Under some conditions, we prove some results on fs-lifting and cfs-lifting semimodules for proving the equivalence between fs-lifting and cfs-lifting semimodules.

An R-semimodule M is fs-lifting if every finitely generated subtractive subsemimodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$; so if every coessential finitely generated subtractive subsemimodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$, M is called cfs-lifting.

1. INTRODUCTION

Extending semimodules are generalization of injective semimodules and, dually, lifting semimodules generalize projective supplemented semimodules ([2]).

Moreover let M be an R-semimodule. An equivalence relation ρ on M is an R-congruence relation if and only if: $m\rho m'$ and $n\rho n' \Rightarrow (m+n)\rho(m'+n')$ and $(rm)\rho(rm')$ for all $m, m', n, n' \in M$ and $r \in R$.

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2020 *Mathematics Subject Classification.* 28A51, 16D70, 13E05, 16Y60.
Key words and phrases. Subtractive semimodule, s-lifting semimodule, fs-lifting semimodule, cfs-lifting semimodule and V-semiring.
Submitted: 5.September. 2024; *Accepted:* 20.September. 2024; *Published:* 06.10.2024.

An R-congruence relation ρ is trivial if $m\rho m' \Leftrightarrow m = m'$.

Let N be a subsemimodule of a left R-semimodule M. N induces on M an R-congruence relation $\equiv_N$, known as the Bourne relation defined by: $\forall m, m' \in M; m \equiv_N m' \Leftrightarrow \exists n, n' \in N$ such that $m + n = m' + n'$.

The set of all the equivalences classes modulo $'' \equiv''_N$ denoted by M/N is such that $(M/N, +, .)$ is an R-semimodule which is called quotient semimodule where the operations are defined by:

$$''+'' : \overline{m} + \overline{m'} = \overline{m + m'},$$

$$''.'' : r\overline{m} = \overline{rm}.$$

Let $M_1,\ M_2$ two subsemimodules of a left R-semimodule M.

- An R-semimodule M is a direct weak sum of M_1 and M_2 (denoted:$M = M_1 \overline{\oplus} M_2$) if $M = M_1 + M_2$ and $M_1 \cap M_2 = \{0\}$.
- An R-semimodule M is direct strong sum of M_1 and M_2 (denoted by $M = M_1 \oplus M_2$) if and only if $M = M_1 + M_2$ and the restriction $'' \equiv''_{M_1}$ to M_2 and the restriction $'' \equiv_{M_2}$ to M_1 are trivial.

Let M be a left R-semimodule.

- A subsemimodule N of an R-semimodule M is called a **subtractive subsemimodule** (=k-subsemimodule) if $\forall\ x,\ y \in M,\ (x + y \in N,\ y \in N) \Rightarrow x \in N$.
- The subsemimodule N is called strongly subtractive if $\forall\ x,\ y \in M;\ (x+y \in N) \Rightarrow x \in N$ and $y \in N$.
- The R-semimodule M is called subtractive (resp.strongly subtractive) or subtractive completely if every subsemimodule of M is subtractive (resp.strongly subtractive).

Example 1. *Let $R = \{0; 1\}$ be the Boole semiring and the set $M = \{0; 1; a; b\}$. Define on M the operations as the following: $0_R = 0_M,\ 1_R = 1_M = 1,\ 1 + 1 = 1 + a = 1 + b = a + b = 1;\ a + 0 = a + a = a;\ b + 0 = b + b = b;\ 0 \times a = 0 \times b = a \times b = 0; 1 \times 1 = 1;\ 1 \times a = a \times a = a;\ 1 \times b = b \times b = b$. Then $(M, +, \times)$ is a commutative left R-semimodule which is finitely generated. In add, we have:*

- *$\{0; a\}$ is a subtractive subsemimodule of M but $\{0; 1; a\}$ is not subtracive (because $1 + b = 1 \in \{0; 1; a\}, 1 \in \{0; 1; a\}$ and $b \notin \{0; 1; a\}$).*

- $M = \{0;a\} + \{0;1;b\}, \{0;a\} \cap \{0;1;b\} = \{0\}$ *and* $1 = 0 + 1 = a + b$. *Since* $a \neq 0$ *and* $b \neq 1$, *the decomposition of 1 is not unique and hence* $M = \{0;a\}\overline{\oplus}\{0;1;b\}$.
- $M = \{0;a\} + \{0;b\}$ *and there does not exist* $x, y \in \{0;a\} \mid 0 + x = b + y$, *therefore* $m \equiv_{\{0;a\}} m' \Leftrightarrow m = m',\ \forall\, m, m' \in \{0;b\}$ *and hence the restriction of* $\equiv_{\{0;a\}}$ *to* $\{0;b\}$ *is trivial. Similarly, the restriction of* $\equiv_{\{0;b\}}$ *to* $\{0;a\}$ *is trivial.*

Thus $M = \{0;a\}\oplus\{0;b\}$.

This paper is organized as follows:

- In Section 1: Basic notions, where more notions are defined.
- In Section 2, we study the notions of finitely subtractive lifting semimodule;
- In Section 3, we study the notion of cofinitely subtractive lifting semimodules.

In the following, R is always an associative, commutative semiring with unit and $1_R \neq 0_R$,the direct summands are the strong ones, the semimodules are left R-semimodules and we use Bourne relation for the semimodules quotients.

2. Basics Notions

Let M be a R-semimodule and N, H, L three subsemimodules of M such that $H \leq N$.

A proper subsemimodule S of M is called a small subsemimodule of M if for all subsemimodule T of M, $S + T = M$ implies that $T = M$. It is indicated by the notation $S \ll M$.

A semimodule M is called hollow if every proper subsemimodule of M is small in M.

A subsemimodule N of M is called a supplement of L in M if $N + L = M$ and $N \cap L \ll N$. In add, if N is subtractive it is trivial to see that N is a supplement of L in M if and only if it is minimal with the propriety of $N + L = M$.

A subsemimodule N of an R-semimodule M is called a weak supplement of L if $N + L = M$ and $N \cap L \ll M$ (see [1]).

If every subsemimodule of M has a supplement (resp a weak supplement), then M is called a supplemented semimodule (resp weakly supplemented semimodule). So M is amply supplemented if $M = L + N$ implies there exists a supplement K of L such that $K \leq N$.

If $N/H \ll M/H$, then H is called a coessential (or cosmall) subsemimodule of N in M and it is denoted by $H \leq^{ce} N$, and hence we say that N lies above H.

A subsemimodule N of M is coclosed in M (denoted by $N \leq^{cc} M$) if N has no proper coessential subsemimodule in M.

The subsemimodule H is called an s-closure of N in M if H is coessential subsemimodule of N and H is coclosed in M.

The R-semimodule M is called lifting if every subsemimodule of M lies above a direct summand of M i.e $\forall\ N \leq M$ there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$.

The R-semimodule M is k-simple (respectively k-noetherian) if it has no nontrivial k-subsemimodules (respectively if every k-subsemimodule of M is finitely generated). The R-semimodule M is k-semisimple if it is a direct sum of k-simple subsemimodules i.e, every k-subsemimodule of M is a direct summand of M.

A semiring R is a left V-semiring if $Rad(M) = 0$ for all R-semimodule M, where $Rad(M)$ is the Jacobson radical of M.

3. Fs-Lifting Semimodules

Definition 3.1. *A semimodule M is called finitely lifting or f-lifting for short , if every finitely generated subsemimodule of M lies above a direct summand of M.*

Example 2. *Let $(R = \{0, 1, \dots, n\} \cup \{+\infty\}; +; .)$ where the operation "+" and "." are define by: $x + y = \max(x, y)$, $x.y = xy = \min(x, y)$ and M be the set of all nonnegative integers. Define $a+b = \max(a, b)$ for each $a,\ b \in M$ and a mapping from $R \times M$ into M, sending (r, m) to $\min(r, m)$. Then M is an f-lifting R-semimodule. Indeed, we show that:*

I. *R is a semiring.*
   (1) *It is clear that $\max(0, x) = x$, $x + y = \max(x, y) = \max(y, x) = y + x \in R$, and $\max(\max(x, y), z) = \max(x, \max(y, z)) \Rightarrow (x + y) + z = x + (y + z), \forall\ x,\ y,\ z \in R$; then $(R, +)$ is a commutative monoid with identity element 0;*

(2) $xy = \min(x, y) \in R$, $\min(\min(x, y), z) = \min(x, \min(y, z)) \Rightarrow (xy)z = x(yz)$ *and* $x(+\infty) = \min(x, +\infty) = x$, $\forall\ x,\ y,\ z \in R$; *then* $(R, .)$ *is a monoid with identity element* $+\infty \neq 0$;

(3) $x(y+z) = \min(x, y+z) = \min(x, \max(y, z))$ *and* $xy+xz = \min(x, y) + \min(x, z) = \max(\min(x, y), \min(x, z))$

- *If* $\max(\min(x, y), \min(x, z)) = x$, *then* $\min(x, y) = x$ *or* $\min(x, z) = x$ *therefore* $\min(x, \max(y, z)) = x$;
- *If* $\max(\min(x, y), \min(x, z)) = y$, *then we have* $\min(x, y) = y$ *and* $\max(y, z) = y$ *therefore* $\min(x, \max(y, z)) = \min(x, y) = y$;
- *If* $\max(\min(x, y), \min(x, z)) = z$, *similarly of* y *case, we will have* $\min(x, \max(y, z)) = z$.

Hence we conclude that $\min(x, \max(y, z)) = \max(\min(x, y), \min(x, z))$ *therefore* $x(y + z) = xy + xz$, $\forall\ x,\ y,\ z \in R$. *Similarly, we show that* $(x + y)z = xz + yz$, $\forall\ x,\ y,\ z \in R$;

(4) $\min(0, a) = \min(a, 0) = 0,\ \forall\ a \in R$, *then* $0a = a0 = 0,\ \forall\ a \in R$.

$(1), (2), (3), (4) \Rightarrow R$ *is a semiring.*

II. M *is an* R*-semimodule.*

By the previous demonstration, $(M, +)$ *is a commutative additive semigroup with identity element 0 (by(1)),* $(r + s)m = rm + sm,\ r(m + p) = rm + rp,\ \forall\ m,\ p \in M,\ r,\ s \in R$ *(by (3)). In add,* $r(sm) = \min(r, sm) = \min(r, \min(s, m)) = \min(\min(r, s), m) = (rs)m$, $0m = \min(0, m) = 0_M = \min(r, 0_M)\ \forall m \in M,\ r,\ s \in R$. *Hence* M *is a left* R*-semimodule.*

III. M *is f-lifting.*

It is clear that $M = \cup\{k \in \mathbb{N}\}$. *Let* L *be a finitely generated subsemimodule of* M. *Then there exists* $m \in \mathbb{N}$ *such that*

$$L = \cup\{k \in \mathbb{N} | 0 \leq k \leq m\}.$$

Let H *be a subsemimodule of* M *such that* $L+H = M$, $a+b = \max(a, b),\ \forall\ a,\ b \in M$, *then* $L + H = M \Rightarrow L = M$ *or* $H = M$. *Since* L *is finitely generated,* $L \neq M$ *therefore* $H = M$; *hence* $L \ll M$, *and*

$$L \ll M \Rightarrow L/\{0\} \ll M/\{0\},$$

and since $\{0\}$ *is a direct summand of* M, *we conclude that* L *lies above a direct summand of* M. *Hence* M *is f-lifting.*

Definition 3.2. *A semimodule M is called subtractive lifting semimodules or fs-lifting semimodule for short, if every finitely generated subtractive subsemimodule of M lies above a direct summand of M.*

Example 3. *Let $R = \{0; 1\}$ be the Boole semiring and the set $M = \{0; 1; a; b\}$.*

1) *Define on M the operations as the following: $0_R = 0_M,\ 1_R = 1_M = 1,\ 1+1 = 1 + a = 1 + b = a + b = 1;\ a + 0 = a + a = a;\ b + 0 = b + b = b;\ 0 \times a = 0 \times b = a \times b = 0; 1 \times 1 = 1;\ 1 \times a = a \times a = a;\ 1 \times b = b \times b = b$.*

   Then $(M, +, \times)$ is a fs-lifting R-semimodule.

   Indeed, clearly M is finitely generated and hence every subsemimodule of M is finitely generated. The only subtractive non trivial subsemimodules of M is $\{0, a\}$ and $\{0, b\}$. Clearly $\{0, a\}/\{0, a\} \ll M/\{0, a\}$, $\{0, b\}/\{0, b\} \ll M/\{0, b\}$ and since $\{0, a\}$ and $\{0, b\}$ are the direct summands of M (from Example 1), then $(M, +, \times)$ is fs-lifting.
2) *Define on M the operations as the following: $0_R = 0_M,\ 1_R = 1_M = 1,\ 1*1 = 1 * a = 1 * b = 1;\ a * a = a * 0 = 0 * a = a;\ b * b = b * 0 = 0 * b = b;\ a * b = 0;\ a.0 = 0.a = b.0 = 0.b = 0;\ 1.a = a;\ 1.b = b$.*

   Then $(M, *, .)$ is R-semimodule which is not fs-lifting.

   Indeed, consider the subsemimodule $N = \{0, a, b\}$ of M. Clearly N is subtractive and finitely generated, and the only direct summand of M contained in N is $\{0\}$. Since $N + \{0, 1\} = M$ and $\{0, 1\} \neq M$, $N \not\ll M$ therefore $N/\{0\} \not\ll M/\{0\}$ and hence $(M, *, .)$ is not fs-lifting.
3) *Define on M the operations as the following: $0_R = 0_M,\ 1_R = 1_M = 1,\ 1+1 = 1+a = 1+b = a+b = 0;\ a+a = a+0 = 0+a = a;\ b+b = b+0 = 0+b = b;\ a.0 = 0.a = b.0 = 0.b = 0;\ 1.a = a;\ 1.b = b$.*

   Then $(M, +, .)$ is an R-semimodule fs-lifting but it is not f-lifting.

   Indeed, clearly, the only subtractive subsemimodule of M is $\{0\}$, then M is fs-lifting. So the only direct summand of M contained in $\{0; 1\}$ is $\{0\}$ and it is clearly that $\{0; 1\}$ is finitely generated. Since $M = \{0; 1\} + \{0; a; b\}$ and $\{0; a; b\} \neq M$, then $\{0; 1\}/\{0\} \not\ll M/\{0\}$ therefore M is not f-lifting.

4. CFS-LIFTING SEMIMODULES

Definition 4.1. *A subsemimodule N of M is called a coessentialy finitely generated subsemimodule if there exist a finitely generated non zero subsemimodule H of M such that $N \leq^{ce} (H + N)$ in M.*

Definition 4.2. *An R-semimodule M is called co-finitely lifting or cf-lifting for short, if every coessentialy finitely generated subsemimodule of M lies above a direct summand of M.*

Example 4. *We consider the $\mathbb{N}$-semimodule ($\mathbb{N}/4\mathbb{N}$, $+, \times$) whose operations are the natural addition and multiplication. Then ($\mathbb{N}/4\mathbb{N}$, $+, \times$) is cf-lifting. Indeed, the only non trivial subsemimodule of $\mathbb{N}/4\mathbb{N}$ is $\{\bar{0}, \bar{2}\}$ which is a coessentialy finitely generated subsemimodule of $\mathbb{N}/4\mathbb{N}$. Since $\{\bar{0}, \bar{2}\} \ll \mathbb{N}/4\mathbb{N}$ and $\{\bar{0}\}$ is a direct of $\mathbb{N}/4\mathbb{N}$, then $\{\bar{0}, \bar{2}\}$ lies above a direct summand of $\mathbb{N}/4\mathbb{N}$ and hence $\mathbb{N}/4\mathbb{N}$ is cf-lifting.*

Remark 4.1. *Every lifting R-semimodule is f-lifting and every f-lifting R-semimodule is cf-lifting.*

Definition 4.3. *An R-semimodule M is called co-finitely subtractive lifting semimodule or cfs-lifting semimodule if every coessentialy finitely generated subtractive subsemimodule of M lies above a direct summand of M.*

Example 5. *Define on M the operations as the following: $0_R = 0_M,\ 1_R = 1_M = 1,\ 1 + 1 = 1 + a = 1 + b = a + b = 0;\ a + a = a + 0 = 0 + a = a;\ b + b = b + 0 = 0 + b = b;\ a.0 = 0.a = b.0 = 0.b = 0;\ 1.a = a;\ 1.b = b$.*

Then $(M, +, .)$ is an R-semimodule cfs-lifting but it is not cf-lifting.

Indeed, clearly, the only subtractive subsemimodule of M is $\{0\}$, then M is cfs-lifting. So the only direct summand of M contained in $\{0; 1\}$ is $\{0\}$ and it is clearly that $\{0; 1\}$ is a coessentialy finitely generated subsemimodule. Since $M = \{0; 1\} + \{0; a; b\}$ and $\{0; a; b\} \neq M$, then $\{0; 1\}/\{0\} \not\ll M/\{0\}$ therefore M is not cf-lifting.

Remark 4.2. *Every cf-lifting semimodule is cfs-lifting but it is clear that the converse is not true.*

Lemma 4.1. *(see [3] Lemma 1.4) Let M be a subtractive left R-semimodule and $H,\ K$ be subsemimodules of M such that $K \subset H$ and $H/K = M/K$. Then $M = H$.*

Theorem 4.1. *Any direct summand of a cfs-lifting semimodule is cfs-lifting.*

Proof. Let K be a direct summand of a cfs-lifting semimodule M. Then there is $K^{'} \leq M$ such that $M = K \oplus K^{'}$. Let N be a coessentialy finitely generated subtractive subsemimodule of K therefore it is a coessentialy finitely generated subtractive subsemimodule of M. Then there is a direct summand $N^{'}$ of M such that $N^{'} \leq N$ and $N/N^{'} \ll M/N^{'}$. We show that $N/N^{'} \ll K/N^{'}$.

Assume that there is $L \leq K$ such that $N^{'} \leq L$ and $K/N^{'} = N/N^{'} + L/N^{'}$. Then $N/N^{'} + L/N^{'} + (K^{'} + N^{'})/N^{'} = K/N^{'} + (K^{'} + N^{'})/N^{'} = M/N^{'}$. Since $N/N^{'} \ll M/N^{'}$, then $L/N^{'} + (K^{'} + N^{'})/N^{'} = M/N^{'}$. Clearly $L/N^{'} \leq K/N^{'}$ and $M/N^{'} = K/N^{'} \oplus (K^{'} + N^{'})/N^{'}$, then $L/N^{'} = K/N^{'}$. Hence $N/N^{'} \ll K/N^{'}$. Since $N^{'}$ is a direct summand of M and $N^{'} \leq K$, then $N^{'}$ is a direct summand of K. Thus K is a cfs-lifting semimodule. □

Theorem 4.2. *If an R-semimodule M is cfs-lifting then M is fs-lifting.*

Proof. Let N be a finitely generated subtractive subsemimodule of M then $N \leq^{ce} N$ in M hence there exist a direct summand K of M such that $K \leq^{ce} N$ in M then M is fs-lifting. □

Theorem 4.3. *Let M be an R-semimodule such that every k-subsemimodule of M is finitely generated. Then the following statements are equivalent:*

(1) *M is cfs-lifting*
(2) *M is fs-lifting.*

Proof.
$1. \Rightarrow 2.)$: From Theorem 4.2
$2. \Rightarrow 1.)$: Let N be a coessentialy finitely generated subtractive subsemimodule of M. Since N is subtractive, then it is finitely generated and hence by 2., N lies a bove a direct summand of M. Thus M is cfs-lifting. □

Definition 4.4. *A semimodule is k-noetherian if every k-subsemimodule is finitely generated.*

Theorem 4.4. *For a k-noetherian subtractive R-semimodule M, the following statements are equivalent:*

(1) *M is cfs-lifting*
(2) *M is fs-lifting*
(3) *M is cf-lifting*

(4) *M is f-lifting*
(5) *M is lifting*

Proof. $1) \Leftrightarrow 2) \Leftrightarrow 3) \Leftrightarrow 4) \Leftrightarrow 5)$

Indeed, since M is k-noetherian then every k-subsemimodule of M is finitely generated and hence, by the Theorem 4.3, we have $1) \Leftrightarrow 2)$. In addition, since M is subtractive, every subsemimodule of M is a k-subsemimodule; then every subsemimodule of M is finitely generated. Thus $2) \Leftrightarrow 3) \Leftrightarrow 4) \Leftrightarrow 5)$. □

For application we can consider the following example:

Example 6. *Let $(\mathbb{N};\ gcd;\ \mathrm{lcm})$ be a semiring. Then $\mathbb{N}$ is a k-noetherian subtractive $\mathbb{N}$-semimodule . Indeed, it is clear that $\mathbb{N}$ is an $\mathbb{N}$-semimodule and every k-subsemimodule of $\mathbb{N}$ is of the form $n\mathbb{N}$ where $n \in \mathbb{N}$.*

Let $m_1\mathbb{N} \subseteq m_2\mathbb{N} \subseteq \ldots \subseteq m_i\mathbb{N} \subseteq m_{i+1}\mathbb{N} \subseteq \ldots$ an increasing sequence of k-subsemimodules of $\mathbb{N}$.

We should show that this sequence is stationary: $m_1\mathbb{N} \subseteq m_2\mathbb{N} \Rightarrow m_2|m_1$ then $m_{i+1}|m_i|\ldots|m_1\ \forall\ i \in \mathbb{N}$. Since the divisors number of any inter is finite, there exists $t \in \mathbb{N}$ such that $m_t = m_n\ \forall\ t \leq n$ therefore there exists $t \in \mathbb{N}$ such that $m_t\mathbb{N} = m_n\mathbb{N},\ \forall\ k \leq n$. Hence the sequence is stationary so $\mathbb{N}$ is k-noetherian.

We proof that M is subtractive in showing every subsemimodule of M is a k-subsemimodule.

It is clear that every k-subsemimodule of M is of the form $n\mathbb{N},\ n \in \mathbb{N}$ and $\{0\}, M$ are trivial k-subsemimodule of M (because $\{0\} = 0\mathbb{N}$ and $M = 1\mathbb{N}$).

Let $N \neq \{0\}$ be a subsemimodule of $M = \mathbb{N}$ and $x \in N$. Then, $N \neq \{0\}$ is a subsemimodule of $M = \mathbb{N}$ then it has a non zero minimal element say m. Then, $x \in N$ and $m \in N \Rightarrow x + m = \gcd(x, m) \in N$, $\gcd(x, m)|m \Rightarrow 0 \neq \gcd(x, m) \leq m$, $\Rightarrow \gcd(x, m) = m$ (because m is a non zero minimal element of N). Hence $m|x$ therefore $x \in m\mathbb{N}$ so $N \subseteq m\mathbb{N}$ (1).

Let $y \in m\mathbb{N}$. Then there exists $\alpha \in \mathbb{N}$ such that $y = m\alpha = \mathrm{lcm}(m, \alpha)$. Since $m \in N,\ \alpha \in \mathbb{N}$ and N is a $\mathbb{N}$-subsemimodule of M, $\mathrm{lcm}(m, \alpha) \in N$ therefore $y \in N$. Hence $m\mathbb{N} \subseteq N$ (2), (1) and $(2) \Rightarrow N = m\mathbb{N}$ therefore N is a k-subsemimodule of M and M is subtractive.

Proposition 4.1. *Let I, J be R-semimodules, $f : I \longrightarrow J$ a surjective homomorphism and S a subsemimodule of J such that $S \ll J$. Then $f^{-1}(S) \ll I$. In addition, if f is an isomorphism, then $f(N) \ll J$ for all $N \ll I$.*

Proof. Indeed we show that $f^{-1}(S) \ll I$. Suppose that there exist $T \leq I$ such that $f^{-1}(S) + T = I$:

$$f^{-1}(S) + T = I \quad \Rightarrow \quad f(f^{-1}(S) + T) = f(I) = J$$

$$\Rightarrow \quad f(f^{-1}(S)) + f(T) = J \quad \Rightarrow S + f(T) = J.$$

Then $T \subset I \Rightarrow f(T) \subset f(I) \Rightarrow f(T) \subset J$.

Hence $S + f(T) = J$ and $f(T) \subset J$, contradiction then there exist not $T \leq I$ such that $f^{-1}(S) + T = I$ from where $f^{-1}(S) \ll S$.

Suppose f is an isomorphism and $N \ll I$. Let H be a subsemimodule of J such that $f(N) + H = J$. Then $f^{-1}(f(N) + H) = f^{-1}(J) = I \Rightarrow f^{-1}(f(N)) + f^{-1}(H) = N + f^{-1}(H) = I$ which is a contradiction hence $f(N) \ll J$ ($f^{-1}(f(N))$ come from of the fact that f is an isomorphism). □

Lemma 4.2. *Every supplement subsemimodule of subtractive semimodule M is coclosed in M.*

Proof. Let be N a supplement subsemimodule of M. Then there exists a subsemimodule L of such that N is minimal of the propriety $N + L = M$. Let $K \leq N$ such that $N/K \ll M/K$. Then

$$\begin{aligned} & N + L = M \Rightarrow N + (K + L) = M \\ & \Rightarrow (N + (K + L))/K = N/K + (K + L)/K = M/K \\ & \Rightarrow (K + L)/K = M/K \text{ (because } N/K \ll M/K) \\ & \Rightarrow K + L = M \text{ (from Lemma 4.1).} \end{aligned}$$

Since N is minimal with the propriety $N + L = M$, we conclude that $N = K$ therefore N is coclosed. □

Theorem 4.5. *An R-semimodule M is cfs-lifting if and only if for every coessentialy finitely generated subtractive subsemimodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$, $N \cap M_2 \ll M_2$.*

Proof. Assume that M is cfs-lifting. Let N be a coessentialy fifnitely generated subtractive subsemimodule of M. Since M is cfs-lifting, N lies above a direct summand M_1 of M. Then there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N, N/M_1 \ll M/M_1$ and $M_2 \leq M$.

Note that we want to verify $N \cap M_2 \ll M_2$. Consider the obvious isomorphism

$$f : M/M_1 \longrightarrow M_2$$

$$\bar{x} \longmapsto x_2$$

with $x = x_1 + x_2$ where $x_1 \in M_1$ and $x_2 \in M_2$. It is very easy to verify that $f(N/M_1) = N \cap M_2$.

Indeed, let $x_2 \in f(N/M_1)$. Then there is $\bar{x} \in N/M_1$ such that $x = x_1 + x_2$ and $f(\bar{x}) = x_2$ with $x_1 \in M_1,\ x_2 \in M_2$. Next, $\bar{x} \in N/M_1 \Rightarrow \exists\, x' \in N$ such that $\bar{x} = \overline{x'}$, and $\bar{x} = \overline{x'} \Rightarrow \exists\, m_1, m_2 \in M_1$ such that $x + m_1 = x' + m_2$. Since $m_2 \in M_1 \subseteq N$, $x' + m_2 \in N$ therefore $x + m_1 \in N$ and so $x \in N$ (because N is subtractive and $m_1 \in M_1 \subseteq N$).

Hence $x_1 + x_2 = x \in N$ therefore $x_2 \in N$ (because N is subtractive and $x \in M_1 \subseteq N$) whence $f(N/M_1) \subseteq N$. Since $f(N/M_1) \subseteq M_2$, we conclude that $f(N/M_1) \subseteq N \cap M_2$.

Let $x_2 \in N \cap M_2$. Then $x_2 \in M_2$ therefore there is an unique $\bar{x} \in M/M_1$ such that $x = x_1 + x_2$ where $x_1 \in M_1$, and $f(\bar{x}) = x_2$ (because f is an isomorphism). So, $x_2 \in N,\ x_1 \in M_1 \subseteq N \Rightarrow x = x_1 + x_2 \in N$ therefore $\bar{x} \in N/M_1$ whence $x_2 \in f(N/M_1)$.

The above implies that $f(N/M_1) = N \cap M_2$.

Since $N/M_1 \ll M/M_1$ and f is an isomorphism, $f(N/M_1) \ll M_2$ (from Proposition 4.1) therefore $N \cap M_2 \ll M_2$.

In sum, we have: $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll M_2$.

Conversely, if N is a coessentialy fifnitely generated subtractive subsemimodule of M, then there a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$, $N \cap M_2 \ll M_2$ and in considering the reciprocal bijection f^{-1} of f, we have $f^{-1}(N \cap M_2) = N/M_1$. Since $N \cap M_2 \ll M_2$ and f^{-1} is a bijection, then $N/M_1 \ll M/M_1$ (by Proposition 4.1). Thus M is cfs-lifting. ☐

Theorem 4.6. *Let M be a cfs-lifting R-semimodule. Then:*

1) *Every coessentialy finitely generated subtractive subsemimodule N of M can be written as $N = N_1 \oplus N_2$ with N_1 is dierct summand of M and $N_2 \ll M_2$ with $M = N_1 \oplus M_2$.*

2) *Every coessentialy finitely generated subtractive coclosed subsemimodule of M is a direct summand of M.*

Proof.

1) By Theorem 4.5, we consider

$$N_1 = M_1,\ N_2 = N \cap M_2.$$

It is clear that N_1 is a direct summand of M and $N_2 \ll M_2$. In add $N = M_1 + N \cap M_2$ (because N is subtractive and $M_1 \subseteq N$) therefore $N = N_1 + N_2$. It is very trivial to see that $N = N_1 \oplus N_2$. Indeed let $x, y \in N_1$ such that $x \equiv_{N_2} y$. Then there exists $n_2, n_2^{'} \in N_2$ such that $x + n_2 = y + n_2^{'}$. Since $n_2, n_2^{'} \in M_2$ (because $N_2 \subseteq M_2$), $x \equiv_{M_2} y$ therefore $x = y$ (because $x, y \in N_1 = M_1$ and $M = M_1 \oplus M_2$) so $''\equiv_{N_2}''_{|N_1}$ is trivial.

Similarly, we prove $''\equiv_{N_1}''_{|N_2}$ is trivial and hence $N = N_1 \oplus N_2$ with N_1 a direct summand of M and $N_2 \ll M$.

2) Trivial □

Corollary 4.1. *Let M be a subtractive R-semimodule. Then the following statements are equivalent:*

1) *M is cfs-lifting.*
2) *For every coessentialy finitely generated subsemimodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$, $N \cap M_2 \ll M_2$.*
3) *Every coessentialy finitely generated subsemimodule N of M can be written as $N = N_1 \oplus N_2$ with N_1 is direct summand of M and $N_2 \ll M_2$ with $M = N_1 \oplus M_2$.*

Proof. $1) \Leftrightarrow 2) \Rightarrow 3)$ (From Theorem 4.5 and Theorem 4.6), while $3) \Rightarrow 1)$ is trivial. □

Proposition 4.2. *Let M_1 and M_2 be a cfs-lifting semimodules, and $M = M_1 \oplus M_2$. If every coessentialy finitely generated subtractive subsemimodule of M is fully invariant, then M is cfs-lifting.*

Proof. Let N be coessentialy finitely generated subtractive subsemimodule of M. Then N is fully invariant therefore it is very easy to verify that $N = (N \cap M_1) \oplus (N \cap M_2)$. Clearly $N' = N \cap M_1$ is a coessentialy finitely generated subtractive subsemimodule of M_1 . Since M_1 is cfs-lifting, then from the Theorem 4.6, there is a decomposition $N' = N'_1 \oplus N'_2$ such that $M_1 = N'_1 \oplus M'_1$ and $N'_2 \ll M'_1$.

Similarly, $N'' = N \cap M_2$ is a coessentialy finitely generated subtractive subsemimodule of M_2 and so $N'' = N''_1 \oplus N''_2$ such that $M_2 = N''_1 \oplus M'_2$ and $N''_2 \ll M'_2$.

Hence $M = (N'_1 \oplus M'_1) \oplus (N''_1 \oplus M'_2) = (N'_1 \oplus N''_1) \oplus (M'_1 \oplus M'_2)$ and $N = N' \oplus N'' = (N'_1 \oplus N''_1) \oplus (N'_2 \oplus N''_2)$.

Pose $N_1 = N'_1 \oplus N''_1$ and $N_2 = N'_2 \oplus N''_2$ therefore $M = N_1 \oplus (M'_1 \oplus M'_2)$ and $N = N_1 \oplus N_2$. Since $N'_2 \ll M'_1$ and $N''_2 \ll M'_2$ hence $N_2 = N'_2 \oplus N''_2 \ll M'_1 \oplus M'_2$. Clearly $M/N_1 \cong M'_1 \oplus M'_2$, $N/N_1 \cong N_2$ and $N_2 \ll M'_1 \oplus M'_2$, then $N/N_1 \ll M/N_1$ therefore M is cfs-lifting. □

Proposition 4.3. *(See [3]) A subsemimodule L of a subtractive R-semimodule M is coclosed if and only if for any proper subsemimodule $K \subsetneq L$, there is a subsemimodule N of M such that $L + N = M$ and $N + K \neq M$.*

Proof. (See [3]: Proposition 1.5) □

Definition 4.5. *A semiring R is a left V-semiring if $Rad(M) = 0$ for all R-semimodule M, where $Rad(M)$ is the Jacobson radical of M.*

Theorem 4.7. *A semiring R is a V-semiring if and only if every subsemimodule is coclcosed in M; for any subtractive R-semimodule M.*

Proof. Let M be a subtractive R-semimodule. We suppose that R is a V-semiring. Then

$$Rad(M) = \sum_{L \ll M} L = 0.$$

Let K be a subsemimodule of M and L be a proper subsemimodule of K. Since $Rad(M) = 0$ then $\{0\}$ is unique small subsemimodule of M therefore K is not small in M.

Since K is not small in M, then there is $H \leq M$ such that $K + H = M$ and $H \neq M$. Then $K + H = K + (H \backslash K) = M$. Let $N = H \backslash K^*$ and hence $K + N = M$; with $K^* = K \backslash \{0\}$. It is clear that $K \cap N = \{0\}$. Hence $M = K \overline{\oplus} N$; in this case $L + N \neq M$ (because L is a proper subsemimodule of K). Then we have: $K + N = M$ and $L + N \neq M$ therefore, by Proposition 4.3, K is coclosed in M.

Reciprocally we suppose that every subsemimodule of M is coclosed in M. Let L be a small subsemimodule of M. By the hypothesis, L is coclosed in M.

$$L \ll M \Rightarrow L/\{0\} \ll M/\{0\} \Rightarrow L = \{0\} \ (because\ L \leq^{cc} M).$$

Then $\{0\}$ is unique small subsemimodule of M therefore $Rad(M) = 0$ and hence R is a V-semiring. □

Definition 4.6. *An R-semimodule is k-simple (respectively k-semisimple) if it has no non-trivial k-subsemimodules (respectively if it is a direct sum of k-simple subsemimodules).*

Lemma 4.3. *Let R be a V-semiring. Then every subtractive lifting R-semimodule is k-semisimple.*

Proof. Let M be a subtractive lifting R-semimodule, where R is a V-semiring. By the Theorem 4.7, every subsemimodule of M is coclosed in M; since M is lifting, every coclosed subsemimodule of M is a direct summand of M therefore every subsemimodule of M is a direct summand of M.

First we show that a cyclic subsemimodule $Ra \neq 0$ of M contains a simple (i.e k-simple) subsemimodule. The mapping $\phi : r \mapsto ra$ is a semimodule homomorphism of ${}_RR$ onto Ra, whose kernel is a left ideal of R and is contained in a maximal ideal L of R (by Krull theorem). Then $La = \phi(L)$ is a maximal subsemimodule (i.e k-subsemimodule) of Ra, and Ra/La is k-simple. Since every subsemimodule (i.e k-subsemimodule) of M is a direct summand of M, $M = La \oplus H$ for some subsemimodule (i.e k-subsemimodule) H of M.

Since La is a direct summand of M and $La \subseteq Ra$, then La is direct summand of Ra therefore it is easy to verify that $Ra = La \oplus (Ra \cap H)$ (because M is subtractive).

Hence $Ra = La \oplus (Ra \cap H) \Rightarrow Ra \cap H \cong Ra/La$ is a simple (i.e k-simple) subsemimodule of Ra.

Now, let N be the sum of all the k-simple subsemimodules of M. Then $M = N \oplus N'$ for some subsemimodule N' of M. If $N' \neq \{0\}$ then N' contains a cyclic subsemimodule $Ra' \neq \{0\}$ containing a k-simple subsemimodule. Then N' has a k-simple subsemimodule S and hence $N \cap N' \supseteq S \neq \{0\}$ contradicting $M = N \oplus N'$ therefore $N' = \{0\}$ and $M = N$. Thus M is k-semisimple. □

Lemma 4.4. *Let M be a k-semisimple R-semimodule. Then every subtractive fully invariant subsemimodule is a direct summand of M.*

Proof. Let N be a subtractive subsemimodule of M which is fully invariant. Since M is semisimple, then $M = \bigoplus_{i\in I} M_i$ with M_i is a simple subsemimodule of M and for some index set I. Since N is fully invariant and subtractive, then it is very easu yo verify that $N = \bigoplus_{i\in I}(N \cap M_i)$ and hence $N = \{0\}$ or there is $i \in I$ such that $N = M_i$ because M_i is simple and $N \cap M_i \leq M_i,\ \forall i \in I$. Thus N is a direct summand of M. □

Theorem 4.8. *Let R be a V-semiring and M be a k-noetherian subtractive R-semimodule. If every coessentialy finitely generated subsemimodule of M is fully invariant, then M is cfs-lifting if and only if it is semisimple.*

Proof. Assume that M is cfs-lifting. Since M is k-noetherian subtractive, then by Theorem 4.4, M is lifting and by Lemma 4.3, M is semisimple.

Conversely, by Lemma 4.4, every coessentialy finitely generated subtractive subsemimodule of M is a direct summand of M and hence M is semisimple. □

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