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ENERGETIC SOLUTIONS FOR REGULARIZED DAMAGE OF A VISCOELASTIC MATERIAL

Assempasa Komi Emmanuel[1], Gnatchao Eyana Roger, and Pindra Nadjime

ABSTRACT. A model for the evolution of damage for viscoelastic material in Kelvin voigt rhoelogie by using Generalized Standard Materials definition is addressed. Small strains and rate-depende evolution includind viscous and inertial effects are assumed. By using the methode of energetic solution develop by Alexander Mielke and Tomáš Roubíček.

1. INTRODUCTION

The goal of this work is to give a definition of energetic solutions for the damaging of viscoelastic material.

Damage is defined as all phenomena associated with the formation and growth of micro-cavities or micro-cracks in a material under machanical load. In other to describe the evolution of damage in material, Kachanov proposes in 1958 a contunuous variable α ranging the internal $[0;1]$ (as in [9]) and having an intuitive microscopical interpretation as density of microcracks or microvoids. In mathematical litterature, α decreasing means damaging and $\alpha = 0$ means maximal damaging.

[1]*corresponding author*

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In this paper, we consider isotropic with one scalar parameter under small strain as in [4], [5]. More over, the gradient of damage theory [6], [7], [8] expressing a nonlocallity in the sense that damage of a particular spot is influenced by its surrounding.

In the next section, energetic solutions are define in the frame of the linear viscosity of Kelvin-Voigt rheology and the energetic solutions [2].

2. REGULARIZED DAMAGE MODEL OF A VISCOELASTIC MATERIAL

We place ourselves in case of Generalized Standard Materials(GSM) [10], [11], [12] where to describe the evolution of the damage materials, we use two potentials:

- The free energy

$$\Psi\left(z, \underline{\underline{e}}(\underline{u}), \underline{\nabla} z\right) = \frac{1}{2} z \mathbb{C}_0 \underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u}) + \frac{k}{p} \|\underline{\nabla} z\|^p + I_+(z),$$

where $\mathbb{C}_0 \in \mathbb{R}^{d\times d\times d\times d}$ ($d \in \mathbb{N}$) is the elasticity tensor satisfying $\mathbb{C}_{ijkl} = \mathbb{C}_{ijkl} = \mathbb{C}_{ijkl}$, small-strain tensor $\underline{\underline{e}} = \frac{1}{2}\left(\underline{\underline{\nabla}}\, \underline{u} +^t \underline{\underline{\nabla}}\, \underline{u}\right)$, z scalar damage parameter and $\underline{\nabla} z$ the gradient of the damage. $k > 0$ is called factor of influence, $I_+(z)$ is the indicator function of the interval $[0; +\infty[$.So the free energy can be rewritte as:

$$\Psi\left(z, \underline{\underline{e}}(\underline{u}), \underline{\nabla} z\right) = \begin{cases} \frac{1}{2} z \mathbb{C}_0 \underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u}) + \frac{k}{p} \|\underline{\nabla} z\|^p & \text{if } z \geq 0, \\ +\infty & \text{otherwise}. \end{cases} \tag{2.1}$$

- The dissipation potential

$$\varphi\left(\dot{\underline{\underline{e}}}(\underline{u}), \dot{z}\right) = -a\dot{z} + \frac{1}{2} z \mathbb{D}_0 \dot{\underline{\underline{e}}}(\underline{u}) : \dot{\underline{\underline{e}}}(\underline{u}) + I_-(\dot{z}),$$

$$\varphi\left(\dot{\underline{\underline{e}}}(\underline{u}), \dot{z}\right) = \begin{cases} -a\dot{z} + \frac{1}{2} z \mathbb{D}_0 \dot{\underline{\underline{e}}}(\underline{u}) : \dot{\underline{\underline{e}}}(\underline{u}) & \text{if } \dot{z} \leq 0, \\ +\infty & \text{otherwise}. \end{cases} \tag{2.2}$$

Here, $+\infty$ means damage can only develop, but the material can never heal, $\mathbb{D}_0$ is positive definite viscosity tensor satisfying $D_{ijkl} = D_{jikl} = D_{klij}$, and $a > 0$ is activation threshold determines the phenomenology how much energy is dissipated by accomplishing the damage process.

The goal of this work is to give a definition of Energetic solutions. This concept base ont two inegalities: local or global (if the functionnal is convex) stability and power balance or Gibb's type energy balance see [2].

By applying the principle of virtual powers, we arrive at two systems of equations, one relating to damage and the other to the stability condition:

$$\begin{cases} \underline{div}(\underline{\underline{\sigma}}) + f &= \rho \dfrac{\partial^2 \underline{u}}{\partial t^2} \quad \text{in } \Omega, \\ \underline{\underline{\sigma}}.\underline{n} &= \underline{T} \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

$$\begin{cases} div\,(\underline{H}) - Y &= 0 \quad \text{in } \Omega, \\ \underline{H}.\underline{n} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Here, $\underline{\underline{\sigma}}$ is stress tensor; f is volumetric force; $\underline{T}$ is surface forces; Y, $\underline{H}$ thermodynamical force associate to the damage z, the gradient of damage. The thermodynamic forces are defined as:

$$\begin{aligned} \underline{\underline{\sigma}} &= \underline{\underline{\sigma}}^{\text{rev}} + \underline{\underline{\sigma}}^{\text{irr}} \\ &= \frac{\partial \Psi\left(z, \underline{\underline{e}}(\underline{u}), \underline{\nabla} z\right)}{\partial e} + \frac{\partial \varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right)}{\partial \dot{e}} \\ &= z\mathbb{C}_0 \underline{\underline{e}}(\underline{u}) + z\mathbb{D}_0 \underline{\underline{\dot{e}}}(\underline{u}) \end{aligned}$$

$$\boxed{\underline{\underline{\sigma}} = z\left(\mathbb{C}_0 \underline{\underline{e}}(\underline{u}) + \mathbb{D}_0 \underline{\underline{\dot{e}}}(\underline{u})\right)} \tag{2.5}$$

$$\begin{aligned} Y &= Y^{\text{rev}} + Y^{\text{irr}} \\ &= \frac{\partial \Psi\left(z, \underline{\underline{e}}(\underline{u}), \underline{\nabla} z\right)}{\partial z} + \frac{\partial \varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right)}{\partial \dot{z}} \end{aligned}$$

$$\boxed{Y = \frac{1}{2}\mathbb{C}_0 \underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u}) + \partial I_+(z) + \partial_{\dot{z}} \varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right)} \tag{2.6}$$

$$\begin{aligned} \underline{H} &= \underline{H}^{\text{rev}} + \underline{H}^{\text{irr}} \\ &= \frac{\partial \Psi\left(z, \underline{\underline{e}}(\underline{u}), \underline{\nabla} z\right)}{\partial \nabla z} + \frac{\partial \varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right)}{\partial \nabla \dot{z}} \\ &= \frac{k}{p}\left(p \underline{\nabla} z \|\underline{\nabla} z\|^{p-2}\right) \end{aligned}$$

$$\boxed{\underline{H} = k\underline{\nabla} z \|\underline{\nabla} z\|^{p-2}} \tag{2.7}$$

By replacing ((2.5)), ((2.6)), ((2.7)) in the systems ((2.3)) and ((2.4)), we obtain this new writing:

$$\begin{cases} \underline{div}\left(z\left(\mathbb{C}_0\underline{\underline{e}}(\underline{u}) + \mathbb{D}_0\underline{\underline{\dot{e}}}(\underline{u})\right)\right) + f & = \rho \dfrac{\partial^2 \underline{u}}{\partial t^2} \qquad \text{in } \Omega, \\ \underline{\underline{\sigma}}.\underline{n} & = \underline{T} \qquad \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

$$\begin{cases} \text{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right) - \dfrac{1}{2}\mathbb{C}_0\underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u}) & \\ \qquad -\partial I_+(z) - \partial_{\dot{z}}\varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right) & = 0 \text{ in } \Omega, \\ k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}.\underline{n} & = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.9}$$

Taking the first equation of the system ((2.9)), we obtain:

$$\text{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right) - \frac{1}{2}\mathbb{C}_0\underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u}) - \partial I_+(z) - \partial_{\dot{z}}\varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right) = 0,$$

$$\underbrace{\text{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right) - \frac{1}{2}\mathbb{C}_0\underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u})}_{-\sigma_i} - \partial I_+(z) - \partial_{\dot{z}}\varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right) = 0,$$

$$-\sigma_i - \partial I_+(z) - \partial_{\dot{z}}\varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right) = 0,$$

$$\sigma_i + \partial I_+(z) + \partial_{\dot{z}}\varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right) = 0,$$

and $\sigma_i + \sigma_r + \partial_{\dot{z}}\varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right) \ni 0$, where $\sigma_r \in \mathcal{N}_{[0;+\infty[}(z)$ is the normal cone.

With the systeme (2.3) and (2.4), we consider formally the problem:

$$\underline{div}\left(z\left(\mathbb{C}_0\underline{\underline{e}}(\underline{u}) + \mathbb{D}_0\underline{\underline{\dot{e}}}(\underline{u})\right)\right) + f = \rho \frac{\partial^2 \underline{u}}{\partial t^2}, \tag{2.10}$$

$$\sigma_i + \sigma_r + \partial_{\dot{z}}\varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right) \ni 0; \sigma_r \in \mathcal{N}_{[0;+\infty[}(z), \tag{2.11}$$

where $\sigma_i = \text{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right) - \frac{1}{2}\mathbb{C}_0\underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u})$, and

$$-\left(\sigma_i + \sigma_r\right) \in \partial_{\dot{z}}\varphi\left(\underline{\underline{\dot{e}}}(\underline{u}), \dot{z}\right). \tag{2.12}$$

Definition 2.1. ***The subdifferential*** *of φ into $\dot{z}$ is defined by:*

$$\partial\varphi(\dot{z}) := \{\sigma \in \mathbb{R}, \forall \widetilde{z} \in \mathbb{R}, \varphi(\dot{z}) + (\widetilde{z} - \dot{z})\,\sigma \leq \varphi(\widetilde{z})\},$$

where $\sigma \in \partial\varphi(\dot{z}) \Longleftrightarrow \varphi(\dot{z}) + \sigma\,(\widetilde{z} - \dot{z}) \leq \varphi(\widetilde{z})$, for all $\widetilde{z} \in \mathbb{R}$.

Using the relation (2.12), we have:

$$-\left(\sigma_i+\sigma_r\right) \in \partial_{\dot z}\varphi\left(\underline{\dot{\underline{e}}}(\underline{u}), \dot z\right) \Longleftrightarrow \varphi\left(\underline{\dot{\underline{e}}}(\underline{u}), \dot z\right)+\left[-\left(\sigma_i+\sigma_r\right)(\widetilde z-\dot z)\right] \leq \varphi\left(\underline{\dot{\underline{e}}}(\underline{u}), \widetilde z\right)$$

$$\Longleftrightarrow \boxed{\varphi\left(\underline{\dot{\underline{e}}}(\underline{u}), \dot z\right) \leq \varphi\left(\underline{\dot{\underline{e}}}(\underline{u}), \widetilde z\right)+\left(\sigma_i+\sigma_r\right)(\widetilde z-\dot z)}. \tag{2.13}$$

By replacing $\left(\underline{\dot{\underline{e}}}(\underline{u}), \dot z\right)=(0,0)$, we get

$$\Longleftrightarrow \varphi(0,0) \leq \varphi(\widetilde z, 0)+\left(\sigma_i+\sigma_r\right)(\widetilde z-0).$$

The dissipation potential is a strictly convex homogeneous function of degree 1.

$$\Longleftrightarrow 0 \leq \varphi(\widetilde z, 0)+\left(\sigma_i+\sigma_r\right)\widetilde z \tag{2.14}$$

By writing $\widetilde z-z$ instead of $\widetilde z$, (2.14) becomes:

$$\Longleftrightarrow \boxed{0 \leq \varphi(\widetilde z-z, 0)+\left(\sigma_i+\sigma_r\right)(\widetilde z-z)}. \tag{2.15}$$

Using the definition of the subdifferential ($\sigma \in \varphi(x_0)$ for all $x \in \mathbb{R}$, $\varphi(x)-\varphi(x_0) \geq \sigma(x-x_0)$) to the function $\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right)$ and taking $x_0=(z, \underline{\nabla} z)$, we get:

$$\begin{aligned}
&\Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde z, \underline{\nabla}\widetilde z\right)-\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \\
&\geq \partial_z\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right)(\widetilde z-z)+\partial_{\underline{\nabla} z}\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right)\left(\underline{\nabla}\widetilde z-\underline{\nabla} z\right), \\
&\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \leq \Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde z, \underline{\nabla}\widetilde z\right) \\
&-\partial_z\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right)(\widetilde z-z)-\partial_{\underline{\nabla} z}\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right)\left(\underline{\nabla}\widetilde z-\underline{\nabla} z\right).
\end{aligned}$$

Using the data from the function $\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right)$ from equation ((2.1)), we have:

$$\begin{aligned}
\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \leq \Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde z, \underline{\nabla}\widetilde z\right)-\left(\frac{1}{2}\mathbb{C}_0\underline{\underline{e}}(\underline{u}):\underline{\underline{e}}(\underline{u})+\sigma_r\right)(\widetilde z-z) \\
-k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\left(\underline{\nabla}\widetilde z-\underline{\nabla} z\right)
\end{aligned} \tag{2.16}$$

The divergence formula allows us to write: $\operatorname{div}\left(a\overrightarrow{V}\right)=a\operatorname{div}\overrightarrow{V}+\overrightarrow{V}\,\overrightarrow{\operatorname{grad}}\,a$,

$$k\underline{\nabla} z\|\underline{\nabla}\|^{p-2}\left(\underline{\nabla}\widetilde z-\underline{\nabla} z\right)=\operatorname{div}\left((\widetilde z-z)\,k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right)-(\widetilde z-z)\operatorname{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right),$$

$$k\underline{\nabla} z\|\underline{\nabla}\|^{p-2}\left(\underline{\nabla}\widetilde z-\underline{\nabla} z\right) \geq -(\widetilde z-z)\operatorname{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right),$$

$$-k\underline{\nabla} z\|\underline{\nabla}\|^{p-2}\left(\underline{\nabla}\widetilde z-\underline{\nabla} z\right) \leq (\widetilde z-z)\operatorname{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right). \tag{2.17}$$

The relation ((2.17)) in ((2.16)) allows us to write:

$$\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \leq \Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde{z}, \underline{\nabla}\widetilde{z}\right) - \left(\frac{1}{2}\mathbb{C}_0\underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u}) + \sigma_r\right)(\widetilde{z} - z) + (\widetilde{z} - z)\,\mathrm{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right),$$

$$\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \leq \Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde{z}, \underline{\nabla}\widetilde{z}\right) - \left(\frac{1}{2}\mathbb{C}_0\underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u}) + \sigma_r - \mathrm{div}\left(k\underline{\nabla} z\|\underline{\nabla} z\|^{p-2}\right)\right)(\widetilde{z} - z),$$

$$\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \leq \Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde{z}, \underline{\nabla}\widetilde{z}\right) - (\sigma_i + \sigma_r)(\widetilde{z} - z), \tag{2.18}$$

with the relation (2.15), $0 \leq \varphi(\widetilde{z} - z, 0) + (\sigma_i + \sigma_r)(\widetilde{z} - z)$

$$-(\sigma_i + \sigma_r)(\widetilde{z} - z) \leq \varphi(\widetilde{z} - z, 0). \tag{2.19}$$

The inequality (2.19) in (2.18) allows us to write:

$$\Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \leq \Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde{z}, \underline{\nabla}\widetilde{z}\right) + \varphi(\widetilde{z} - z, 0).$$

Integrating the last relation over the entire domain, we have:

$$\int_\Omega \Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \mathrm{d}\Omega \leq \int_\Omega \Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde{z}, \underline{\nabla}\widetilde{z}\right) \mathrm{d}\Omega + \int_\Omega \varphi(\widetilde{z} - z, 0)\,\mathrm{d}\Omega, \tag{2.20}$$

for all $\widetilde{z} \in W^{1,p}(\Omega)$, $z \geq 0$.

If $z(t)$ satisfies (2.20), we say that $z(t)$ is partially stable at t. This relation will be used to give a definition of energetic solutions.

The system (2.8) on Ω and the initial condition can be written as:

$$\rho\frac{\partial^2 u}{\partial t^2} - div\,(\sigma_\nu + \sigma_e) = f, \tag{2.21}$$

$$\begin{aligned} &u(0,.) = u_0 \in W^{1,2}(\Omega; \mathbb{R}^d), \quad \frac{\partial u}{\partial t}(0,.) = \dot{u}_0 \in L^2(\Omega; \mathbb{R}^d), \\ &z(0,.) = z_0 \in W^{1,p}(\Omega). \end{aligned} \tag{2.22}$$

This is indeed to be understood only formally because in the completely damage part $z = 0$, the deplacement $\underline{u}$ as well as strain $\underline{\underline{e}}(\underline{u})$ lose any sense. let us take $\forall\varepsilon > 0$ and consider the regularized equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - div\left((z + \varepsilon)\,\mathbb{D}_0 e\left(\frac{\partial u}{\partial t}\right) + (z + \varepsilon)\,\mathbb{C}_0 e\,(u)\right) = f.$$

The variational formulation: multiplying the last equation by the test function ν and integrating the result on the domain Ω and from 0 to T, after using the divergence formula, we obtain:

$$\int_0^T \left(\langle \rho \frac{\partial^2 u_\varepsilon}{\partial t^2}, \nu \rangle + \int_\Omega (z_\varepsilon + \varepsilon) \left(\mathbb{D}_0 e \left(\frac{\partial u_\varepsilon}{\partial t} \right) + \mathbb{C}_0 e(u_\varepsilon) \right) : e(\nu) \right. \\ \left. - f.\nu d\Omega \right) dt = 0, \tag{2.23}$$

for all $\nu := \frac{\partial u_\varepsilon}{\partial t} \in L^2\left(I; W^{1,2}\left(\Omega; \mathbb{R}^d\right)\right)$ and $\langle ., . \rangle$ the duality product between $W^{1,2}\left(\Omega; \mathbb{R}^d\right)$ and $W^{1,2}\left(\Omega; \mathbb{R}^d\right)^*$.

Gibbs-type energy balance:

$$\mathcal{G}_\varepsilon(T, u(T), z(T)) + Var_\varrho(z; 0, T) \leq \mathcal{G}_\varepsilon(0, u(0), z(0)) \\ + \int_0^T \int_\Omega \sigma_{\underline{\underline{e}}} : \underline{\underline{e}} \left(\frac{\partial w}{\partial t} \right) \mathrm{d}x \mathrm{d}t \tag{2.24}$$

The Gibb's energy is given by:

$$\mathcal{G}_\varepsilon(t, \underline{u}, z) := \begin{cases} \frac{1}{2}(z+\varepsilon)\, \mathbb{C}_0\, \underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u}) + \frac{k}{p} ||\nabla z||^p & \text{if } \underline{u}|_\Gamma = w(t,.), \\ & \text{and if} 0 \leq z \text{in} \Omega \\ +\infty & \text{otherwise}, \end{cases}$$

where Var_ϱ is the variation of z define by:

$$Var_\varrho(\zeta; t_1, t_2) = \begin{cases} a \int_\Omega z(t_1, x) - z(t_2, x)\, \mathrm{d}x + \varphi_0 & \text{if } z(.,x) \text{is non-increasing on} \\ & [t_1, t_2]\, \forall x \in \Omega, \\ +\infty & \text{otherwise }, \end{cases}$$

with $\varphi_0 = \frac{1}{2} z \mathbb{D}_0 \dot{\underline{\underline{e}}}(\underline{u}) : \dot{\underline{\underline{e}}}(\underline{u})$

3. Energetics solution for viscodynamique material

Let us denote by "B" and "BV" the Banach space of everywhere defined bounded measurable and bounded-variation functions, respectively. Moreover, let us abbreviate $I : (0, T)$; $\overline{I} : [0, T]$.

Definition 3.1. *(Weak/energetic solution) We call* $(u_\varepsilon, \zeta_\varepsilon)$ *with* $u_\varepsilon \in B\left(\overline{I}; W^{1,2}(\Omega;\mathbb{R}^d)\right)$ *and* $\zeta_\varepsilon \in B\left(\overline{I}; W^{1,p}(\Omega;\mathbb{R}^d)\right) \cap BV\left(\overline{I}; L^1(\Omega)\right)$ *a weak/energetic solution to the original problem* (2.10)-(2.11) *with the initial and boundary conditions if:*

$i)$ *The partial stability* (2.20) *holds for all* $t \in [0;T]$

$$\int_\Omega \Psi\left(\underline{\underline{e}}(\underline{u}), z, \underline{\nabla} z\right) \mathrm{d}\Omega \le \int_\Omega \Psi\left(\underline{\underline{e}}(\underline{u}), \widetilde{z}, \underline{\nabla}\widetilde{z}\right) \mathrm{d}\Omega + \int_\Omega \varphi\left(\widetilde{z} - z, 0\right) \mathrm{d}\Omega,$$

for all $\widetilde{z} \in W^{1,p}(\Omega)$, $z \ge 0$.

$ii)$ *The energy inequality* (2.24) *holds with* $(u_\varepsilon, z_\varepsilon)$ *in the place of* $(u; z)$.

$iii)$ *the variational formulation* (2.23) *is satisfied*

$$\int_0^T \left(\langle \rho \frac{\partial^2 u_\varepsilon}{\partial t^2}, \nu \rangle + \int_\Omega (z_\varepsilon + \varepsilon) \left(\mathbb{D}_0 e\left(\frac{\partial u_\varepsilon}{\partial t}\right) \right.\right.$$
$$\left.\left. + \mathbb{C}_0 e\left(u_\varepsilon\right)\right) : e\left(\nu\right) - f.\nu d\Omega \right) dt = 0$$

for all $\nu := \dfrac{\partial u_\varepsilon}{\partial t} \in L^2\left(I; W^{1,2}\left(\Omega;\mathbb{R}^d\right)\right)$ *and* $\langle .,. \rangle$ *the duality product between* $W^{1,2}\left(\Omega;\mathbb{R}^d\right)$ *and* $W^{1,2}\left(\Omega;\mathbb{R}^d\right)^*$.

The existence of a weak/energetic solution $(u_\varepsilon, z\varepsilon)$ is guaranteed for any $\varepsilon > 0$ by assuming $p > d$ and the stability of z_0 (see [2,3]).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOMÉ
BP 1515, LOMÉ,
TOGO.
Email address: keassempasa@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOMÉ
BP 1515, LOMÉ,
TOGO.
Email address: gnatchaoe@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOMÉ
BP 1515, LOMÉ,
TOGO.
Email address: nadjimepindra@gmail.com