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SUPOUT TOPOLOGY ON DIRECTED FUZZY GRAPHS

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ABSTRACT. In this work, we introduce a topology, called supout topology and denoted $\mathcal{F}_G^o$, for a fuzzy directed graph. We study some properties of this topology and give some examples of open and closed sets. We demonstrate that two isomorphic fuzzy directed graphs have homeomorphic supout topologies. In addition, we prove that this topology is an Alexandroff one and then use minimal basis to characterize homeomorphic fuzzy directed graphs. Finally, we investigate the connectedness of this topology vs. the connectivity of the graph.

1. INTRODUCTION

Directed fuzzy graphs were introduced by Chen and Chang in [10]. After a few years, directed fuzzy graphs have been applied in many domains: Sensor networks, Communication networks, Social networks, Transportation networks, Air traffic control, [1, 19, 20, 24, 25, 28, 31]. They are used in medical field [22], economic [4, 21, 23]. In [1, 4, 6, 13, 14, 21, 26, 32], we find some topological characterizations of fuzzy graphs.

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Recall that topology can be used in many domains as examples: machine learning [8, 9, 11], image processing and computer vision [15, 17, 34] and networking [2,7,33]. In this paper, we are going to construct a topology for directed fuzzy graph and study their properties. We will prove that this topology is an Alexandroff topology, that is any intersection of open sets is an open set. Therefore, we have a minimal basis. This minimal basis helps us to characterize continuous functions, homeomorphisms, and to prove that two isomorphic directed fuzzy graphs are homeomorphic. This topology is called Supout Topology for the fuzzy graph since it is built by using out-neighbors for a given vertex x whose form with the vertex x an edge with positive membership.

The outline of the paper is as follows. In the second section, we recall some definitions and results in the fuzzy graph theory and the topology domain that we will use later. In section 3, we define our topology and prove that it is an Alexandroff one. We prove many results using the minimal basis. Section 4 is devoted to study homeomorphic directed fuzzy graphs. Finally section 5 studied the relation between the connectivity of the graph and the connectedness of the constructed topology.

2. Preliminaries

In the beginning of this paper, we take this section to recall the basic definitions and results that we will use along this paper, we can refer to [5,7,12,16,18,27,30] for more details.

Definition 2.1. *A fuzzy graph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is non empty set $\mathcal{V}$ with two maps $\rho : \mathcal{V} \to [0, 1]$ and $\nu : \mathcal{V} \times \mathcal{V} \to [0, 1]$ such that for all $x, y \in \mathcal{V}$, we have.*

(i) $\nu(x, y) = \nu(y, x)$.
(ii) $\nu(x, y) \leq \min(\rho(x), \rho(y))$.
(iii) $\nu(x, x) = 0$.

An element x of $\mathcal{V}$ is called a vertex of the graph $\mathcal{G}$ and we have the two supports

$$Supp(\rho) = \{x \in \mathcal{V};\ \rho(x) > 0\} \tag{2.1}$$

and

$$Supp(\nu) = \{(x, y) \in \mathcal{V} \times \mathcal{V};\ \nu(x, y) > 0\}. \tag{2.2}$$

Example 1. *In Figure 1, we have* $Supp(\rho) = \{a, b, c, d\} = \mathcal{V}$ *and* $Supp(\nu) = \{(a, b), (a, c), (b, c), (c, d)\}$ *but* $\nu(a, d) = \nu(b, d) = 0$.

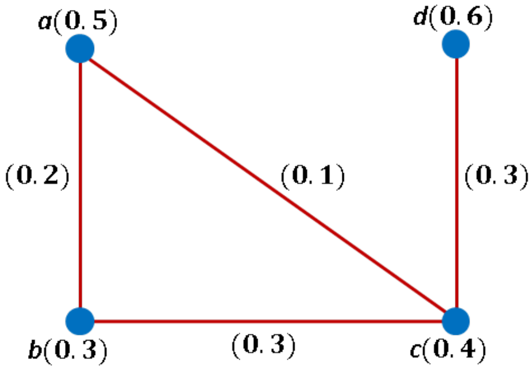

FIGURE 1. A fuzzy graph

Definition 2.2. *A fuzzy graph* $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ *is called directed if*

(i) $\nu(x, y) \leq \min(\rho(x), \rho(y))$.
(ii) $\nu(x, x) = 0$.

Example 2. *In a directed graph, the function* ν *is not needed to be symmetric and we precise the direction of the edge (the edges are directed) and we represent a directed fuzzy graph as in Figure 2.*

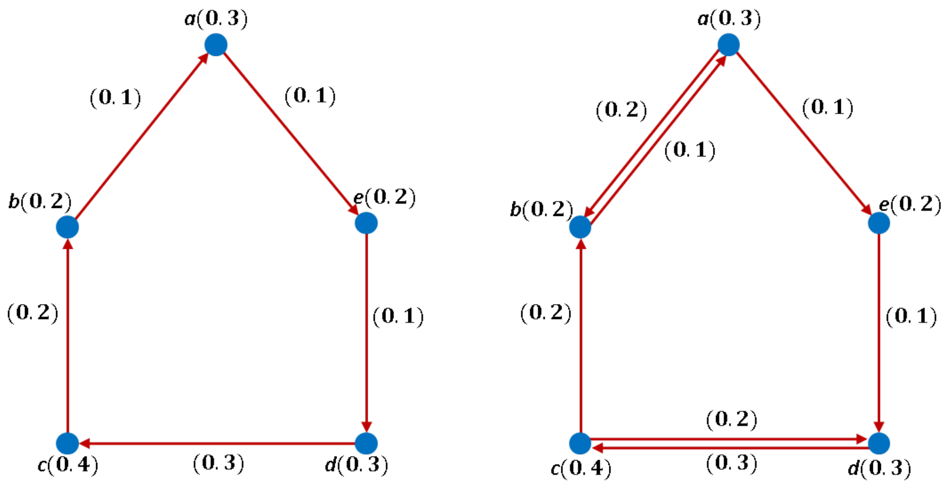

FIGURE 2. Examples of directed fuzzy graphs

Definition 2.3. *A fuzzy path in a directed fuzzy graph is any sequence of distinct vertices $a_0, a_1, \cdots, a_{n-1}, a_n$ with $\nu(a_{i-1}, a_i) > 0$ for $i = 1, \cdots, n$. We say that this path is length n.*

Definition 2.4. *We say that a directed fuzzy graph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is simple (or without loop) if for all $x \in \mathcal{V}$, $\nu(x, x) = 0$.*

For two distinct vertices a and b of a directed fuzzy graph, we denote $d(a, b)$ the length of the shortest path joining a and b. If there is no path between them, we say that $d(a, b) = \infty$.

Definition 2.5. *A fuzzy digraph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is complete if for every pair of directed adjacent vertices, we have*

$$\nu(x, y) = \min(\rho(x), \rho(y)).$$

Example 3. *The following Figure 3 represents a complete directed fuzzy graph.*

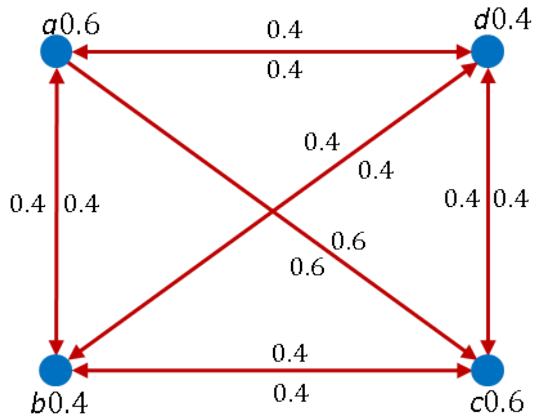

FIGURE 3. An example of a complete directed fuzzy graph

Definition 2.6. *The complement of a fuzzy digraph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is a the fuzzy digraph $\overline{\mathcal{G}} = (\mathcal{V}, \overline{\rho}, \overline{\nu})$, where $\overline{\rho} = \rho$ and for all x and y in $\mathcal{V}$, $\overline{\nu}(x, y) = \min(\rho(x), \rho(y)) - \nu(x, y)$.*

Example 4. *In the Figure 4, we have two complete directed fuzzy graphs and each one is the complement of the other.*

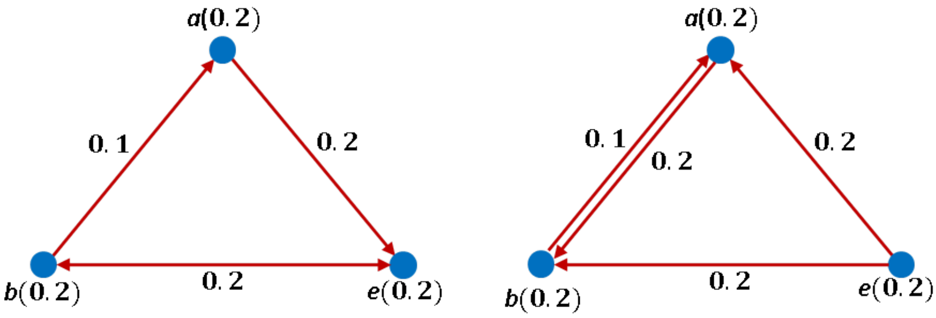

FIGURE 4. An example of a directed fuzzy graph and its complement

Next, we define

$$N_x^o = \{y \in \mathcal{V};\ \nu(x,y) > 0\}, \tag{2.3}$$

and

$$N_x^i = \{y \in \mathcal{V};\ \nu(y,x) > 0\}, \tag{2.4}$$

the set of out-neighbors and int-neighbors of x respectively. It is clear that

$$y \in N_x^o \text{ if and only if } x \in N_y^i.$$

In this paper, we define the isolated vertex as follows.

Definition 2.7. *If $\nu(y,x) = 0$ for all $y \in \mathcal{V}$, then x is called an isolated vertex in the fuzzy digraph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$. This is equivalent to $N_x^i = \emptyset$.*

Definition 2.8. *A fuzzy digraph $\mathcal{G}$ is said locally finite if N_x^i is a finite set, for all $x \in \mathcal{V}$.*

Also, in order to give some examples of open and closed subsets, we define the *supout-degree* of a vertex x as $d^{s+}(x) = card(N_x^o)$ and the *supint-degree* of x is $d^{s-}(x) = card(N_x^i)$.

Let

$$\delta^{s+}(G) = \min\{d^{s+}(x),\ x \in \mathcal{V}\},$$

and

$$\delta^{s-}(G) = \min\{d^{s-}(x),\ x \in \mathcal{V}\}$$

the minimum supout-degree and the minimum supint-degree of the fuzzy digraph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$. In the same way, we have the maximum supout-degree and maximum supint-degree of $\mathcal{G}$ given by

$$\Delta^{s+}(G) = \max\{d^{s+}(x),\ x \in \mathcal{V}\},$$

and

$$\Delta^{s-}(G) = \max\{d^{s-}(x),\ x \in \mathcal{V}\}.$$

We end this section by some definitions about topological spaces.

Definition 2.9. *A non empty set V is called a topological space if it has a family $\mathcal{F}$ of subsets satisfying*

(i) $\emptyset,\ V \in \mathcal{F}$.
(ii) *For all $A_1,\ A_2 \in \mathcal{F}$, $A_1 \cap A_2 \in \mathcal{F}$.*
(iii) *For all sequence $\{A_i\}_i$ of elements of $\mathcal{F}$, $\cup_{i \in I} A_i \in \mathcal{F}$.*

The elements of $\mathcal{F}$ are called open sets for the topology $\mathcal{F}$.
If all intersections of open sets are also open sets, then the topological space $(V, \mathcal{F})$ is said an Alexandroff space.

Definition 2.10. *If A is a subset of V and $(V, \mathcal{F})$ is a topological space, then.*

(i) *A is called a closed set if its complement $A^c := V \setminus A$ is an open set.*
(ii) *The closure of A, $\overline{A}$, is the smallest closed set containing A.*

Definition 2.11. *Suppose that $(V, \mathcal{F})$ is a topological space. If every open cover of $\mathcal{V}$ has a finite subcover, then we say that $\mathcal{V}$ is a compact space.*

3. First Results

In the sequel, we suppose that the directed fuzzy graph $\mathcal{G}$ is without isolated vertices, locally finite and $\rho > 0$. In order to introduce our topology, we set

$$\mathcal{N}_{\mathcal{G}}^{o} = \{N_x^o;\ x \in \mathcal{V}\}. \tag{3.1}$$

Theorem 3.1. *Suppose that $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is a fuzzy directed graph. Then, $\mathcal{N}_{\mathcal{G}}^{o}$ is a subbasis of a topology for $\mathcal{V}$.*

Proof. We have to prove that

$$\bigcup_{u\in\mathcal{V}} N_u^o = \mathcal{V}$$

Let $x \in \mathcal{V}$. We have that x is not isolated vertex and so $N_x^i \neq \emptyset$. Consider one element $y \in N_x^i$. Therefore, $x \in N_y^o$, and so $x \in \bigcup_{u\in\mathcal{V}} N_u^o$.

We get $\bigcup_{u\in\mathcal{V}} N_u^o \subset \mathcal{V}$ and so $\bigcup_{u\in\mathcal{V}} N_u^o = \mathcal{V}$. Hence $\mathcal{N}_\mathcal{G}^o$ is a subbasis for a topology of $\mathcal{V}$. □

The topology induced by the subbasis $\mathcal{N}_\mathcal{G}^o$ will be denoted by $\mathcal{F}_\mathcal{G}^o$ and will be called supout topology.

Theorem 3.2. *Suppose that $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is a fuzzy directed graph. Then, $(\mathcal{V}, \mathcal{F}_\mathcal{G}^o)$ is an Alexandroff topological space.*

Proof. Since the topology $\mathcal{F}_\mathcal{G}^o$ is defined by the subbasis $\mathcal{N}_\mathcal{G}^o$, we are going to prove that any intersection of elements in $\mathcal{N}_\mathcal{G}^o$ is an open set, see Definition 2.9.
Consider

$$\bigcap_{x\in M} N_x^o$$

where $M \subset \mathcal{V}$.

case 1. $\cap_{x\in M} N_x^o = \emptyset$, then $\cap_{x\in M} N_x^o$ is an open set.

case 2. $\cap_{x\in M} N_x^o \neq \emptyset$, then let y in $\cap_{x\in M} N_x^o$. So, $y \in N_x^o,\ \forall x \in M$.

We get for all $x \in M$, $x \in N_y^i$ and then $M \subset N_y^i$. Since the graph $\mathcal{G}$ is locally finite, N_y^i is a finite set and so M is also finite. Therefore, by the Definition 2.9, we get that $\cap_{x\in M} N_x^o$ is an open set. Hence the theorem is proved. □

We have an Alexandroff topological space $(\mathcal{V}, \mathcal{F}_\mathcal{G}^o)$. Let $x \in \mathcal{V}$, we denote D_x the smallest open set containing x, that is the intersection of all open sets containing x (Theorem 3.2). We set

$$\mathcal{D} = \{D_x;\ x \in \mathcal{V}\}. \tag{3.2}$$

$\mathcal{D}$ is called the minimal basis of $\mathcal{F}_\mathcal{G}^o$. For more details and results, we can refer to [3, 5, 12, 18, 29, 30]. We will call D_x the minimal open set containing x. The following result connects the minimal open sets with the subbasis.

Theorem 3.3. *Let $\mathcal{G}$ be a directed fuzzy graph. For all x a vertex of $\mathcal{G}$, we have*

$$D_x = \bigcap_{y \in N_x^i} N_y^o.$$

Proof. For x a vertex of the graph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$. First, we are going to prove that

$$D_x \subseteq \bigcap_{y \in N_x^i} N_y^o.$$

Since x is not isolated vertex, then there exists $y \in N_x^i$. By definition, N_y^o is an open set. In addition, $y \in N_x^i$ is equivalent $x \in N_y^o$, therefore N_y^o is an open set containing x. Then $D_x \subseteq N_y^o$ and hence $D_x \subseteq \bigcap_{y \in N_x^i} N_y^o$.

Now, since D_x is the small open set containing x, then there exists a subset H of $\mathcal{V}$ such that $D_x = \bigcap_{y \in H} N_y^o$. If $y \in H$, we have $x \in N_y^o$ and so $y \in N_x^i$ and so $H \subseteq N_x^i$. We get

$$\bigcap_{y \in N_x^i} N_y^o \subseteq \bigcap_{y \in H} N_y^o$$

and so

$$\bigcap_{y \in N_x^i} N_y^o \subseteq D_x.$$

The equality follows. □

As consequence, we can characterise and find the minimal open set D_x in some cases.

Proposition 3.1. *Suppose that x and y are two distinct vertices of a directed fuzzy graph $\mathcal{G}$.*

(a) *When $N_x^i = \{y\}$, we have $D_x = N_y^o$.*
(b) *Suppose that $y \in N_x^i$, we get $D_x \subset N_y^o$.*
(c) *If $D_y \subset N_x^i$, then $D_x \subset N_y^o$.*

Proof.

(a) $N_x^i = \{y\}$ and Theorem 3.3 give $D_x = N_y^o$.
(b) Since $D_x = \bigcap_{a \in N_x^i} N_a^o$, we get $D_x \subset N_y^o$.
(c) Follows from (b).

□

Proposition 3.2. *Suppose that x is a vertex of a directed fuzzy graph $\mathcal{G}$. Then,*

$$D_x = \{y \in \mathcal{V};\ N_x^i \subset N_y^i\}.$$

Proof. First, let $y \in D_x$, we have to prove that $N_x^i \subseteq N_y^i$. If $z \in N_x^i$ and since

$$D_x = \bigcap_{a \in N_x^i} N_a^o \subseteq N_z^o,$$

we get $y \in N_z^o$ and so $z \in N_y^i$. Therefore, $N_x^i \subset N_y^i$ and so $D_x \subseteq \{y \in \mathcal{V};\ N_x^i \subset N_y^i\}$.

Conversely, suppose that $y \in \mathcal{V}$ and $N_x^i \subset N_y^i$. So, for all $z \in N_x^i$, $z \in N_y^i$. That is, $\forall z \in N_x^i$, $y \in N_z^o$ and then

$$y \in \bigcap_{z \in N_x^i} N_z^o.$$

From Theorem 3.3, $y \in D_x$. That is $\{y \in \mathcal{V};\ N_x^i \subset N_y^i\} \subseteq D_x$. So the proposition is proved. □

Theorem 3.4. *Suppose that $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is a fuzzy directed graph and $x, y \in \mathcal{V}$. Then,*

(1) $D_x \bigcap N_x^i = \emptyset$.
(2) *If $D_y \subset N_x^i$, then $D_x \bigcap D_y = \emptyset$.*

Proof.

(1) By contraction. Let $a \in D_x \bigcap N_x^i$. From the fact that $a \in D_x$ and Theorem 3.4, we get $N_x^i \subseteq N_a^i$. In addition, $a \in N_x^i$ and therefore $a \in N_a^i$. This is impossible since the directed fuzzy graph is simple. We deduce that $D_x \bigcap N_x^i = \emptyset$.
(2) Since $\emptyset = D_x \bigcap N_x^i \supseteq D_x \bigcap D_y$. So, $D_x \bigcap D_y = \emptyset$.

□

Proposition 3.3. *Suppose that $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is a fuzzy directed graph and $x \in \mathcal{V}$. Then,*

$$\overline{\{x\}} = \{y \in \mathcal{V};\ N_y^i \subset N_x^i\},$$

where $\overline{\{x\}}$ is the closure of the set $\{x\}$ (see the Definition 2.10).

Proof. Let $y \in \mathcal{V}$, we have $D_y = \{a \in \mathcal{V};\ N_y^i \subset N_a^i\}$, and $y \in \overline{\{x\}}$ this is equivalent to $A \cap \{x\} \neq \emptyset$, for all A open set containing y. That is, $D_y \cap \{x\} \neq \emptyset$. Therefore, this means $x \in D_y$. We get

$$y \in \overline{\{x\}} \Leftrightarrow x \in D_y \Leftrightarrow N_y^i \subset N_x^i.$$

The result follows. $\square$

4. FUNCTIONS AND $\mathcal{F}_{\mathcal{G}}^o$ TOPOLOGY

Definition 4.1. *Let $\mathcal{G}_1 = (\mathcal{V}_1, \rho_1, \nu_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \rho_2, \nu_2)$ be two directed fuzzy graphs. A morphism (homomorphism) from $\mathcal{G}_1$ to $\mathcal{G}_2$ is a map $f : \mathcal{V}_1 \to \mathcal{V}_2$ satisfying*

$$\rho_1(x) \leq \rho_2(f(x)) \text{ and } \nu_1(x,y) \leq \nu_2(f(x), f(y)), \forall x, y \in \mathcal{V}_1$$

We say the two directed fuzzy graphs $\mathcal{G}_1 = (\mathcal{V}_1, \rho_1, \nu_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \rho_2, \nu_2)$ are isomorphic if there is a bijective function $f : \mathcal{V}_1 \to \mathcal{V}_2$ verifying

$$\rho_1(x) = \rho_2(f(x)) \text{ and } \nu_1(x,y) = \nu_2(f(x), f(y)), \forall x, y \in \mathcal{V}_1$$

As topological spaces, we have the following equivalent relation.

Definition 4.2. *Consider the two topological spaces $(\mathcal{V}_1, \mathcal{F}_{\mathcal{G}_1}^o)$ and $(\mathcal{V}_2, \mathcal{F}_{\mathcal{G}_2}^o)$. A function $f : \mathcal{V}_1 \to \mathcal{V}_2$ is said continuous if*

$$f^{-1}(A) \in \mathcal{F}_{\mathcal{G}_1}^o, \forall A \in \mathcal{F}_{\mathcal{G}_2}^o.$$

The two topological spaces $(\mathcal{V}_1, \mathcal{F}_{\mathcal{G}_1}^o)$ and $(\mathcal{V}_2, \mathcal{F}_{\mathcal{G}_2}^o)$ are called homeomorphic if there is a bijective function $f : \mathcal{V}_1 \to \mathcal{V}_2$ such that f and f^{-1} are continuous.

Theorem 4.1. *If $\mathcal{G}_1 = (\mathcal{V}_1, \rho_1, \nu_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \rho_2, \nu_2)$ are two isomorphic fuzzy graphs, then $(\mathcal{V}_1, \mathcal{F}_{\mathcal{G}_1}^o)$ and $(\mathcal{V}_2, \mathcal{F}_{\mathcal{G}_2}^o)$ are homeomorphic.*

Proof. Denote $f : \mathcal{V}_1 \to \mathcal{V}_2$ an isomorphism. We have to prove that for all $A \in \mathcal{N}_{\mathcal{G}_2}^o$, $f^{-1}(A) \in \mathcal{F}_{\mathcal{G}_1}^o$ (see Theorem 3.1).

Let $A \in \mathcal{N}_{\mathcal{G}_2}^o$. Then, there exists $a \in \mathcal{V}_2$ such that $A = N_a^o$. We set $b = f^{-1}(a)$, then we have

$$\begin{aligned} f^{-1}(A) = f^{-1}(N_a^o) &= \{x \in \mathcal{V}_1,\ f(x) \in N_a^o\} = \{x \in \mathcal{V}_1,\ \nu_2(a, f(x)) > 0\} \\ &= \{x \in \mathcal{V}_1,\ \nu_2(f(b), f(x)) > 0\} = \{x \in \mathcal{V}_1,\ \nu_1(b, x) > 0\} \\ &= \{x \in \mathcal{V}_1,\ x \in N_b^o\} = N_b^o. \end{aligned}$$

We obtain

$$f^{-1}(A) = N_b^o \tag{4.1}$$

and

$$f^{-1}(N_{f(b)}^o) = N_b^o. \tag{4.2}$$

From (4.1), we deduce that $f^{-1}(A) \in \mathcal{N}_{\mathcal{G}_1}^o \subset \mathcal{F}_{\mathcal{G}_1}^o$. Therefore, f is continuous. From (4.2), we get

$$N_{f(b)}^o = f(N_b^o) = (f^{-1})^{-1}(N_b^o)$$

and hence, the function f^{-1} is continuous. □

The Figure 5 shows that the converse of the above theorem is not always true.

Example 5. *For the first graph in the Figure 5, we have*

$$N_a^i = \{d, c\} \text{ and } N_a^o = \{b, c\},$$
$$N_b^i = \{a, d\} \text{ and } N_b^o = \{c, d\},$$
$$N_c^i = \{a, b\} \text{ and } N_c^o = \{a, d\},$$
$$N_d^i = \{b, c\} \text{ and } N_d^o = \{a, b\}.$$

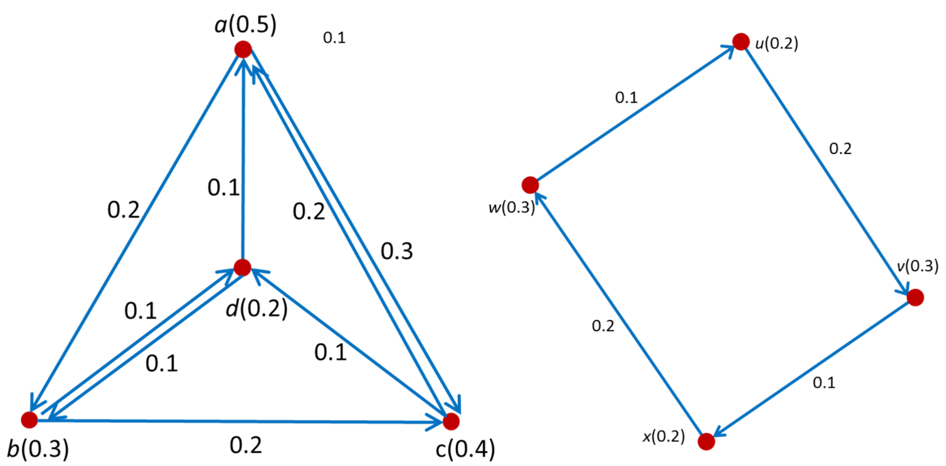

FIGURE 5. An example of two homeomorphic graphs but not isomorphic

Then,

$$D_a = \bigcap_{y \in N_a^i} N_y^o = \bigcap_{y \in \{d,c\}} N_y^o = N_d^o \cap N_c^o = \{a\},$$
$$D_b = \bigcap_{y \in N_b^i} N_y^o = \bigcap_{y \in \{a,d\}} N_y^o = N_a^o \cap N_d^o = \{b\},$$

$$D_c = \bigcap_{y\in N_c^i} N_y^o = \bigcap_{y\in\{a,b\}} N_y^o = N_a^o \cap N_b^o = \{c\},$$
$$D_d = \bigcap_{y\in N_d^i} N_y^o = \bigcap_{y\in\{b,c\}} N_y^o = N_b^o \cap N_c^o = \{d\}.$$

So, the minimal basis is

$$\mathcal{D} = \Big\{\{a\},\{b\},\{c\},\{d\}\Big\}.$$

For the second graph in the figure 5, we have

$N_u^i = \{w\}$ *and* $N_u^o = \{v\}$,
$N_v^i = \{u\}$ *and* $N_o^v = \{x\}$,
$N_x^i = \{v\}$ *and* $N_x^o = \{w\}$,
$N_w^i = \{x\}$ *and* $N_w^x = \{u\}$,
$D_x = \bigcap_{y\in N_x^i} N_y^o$,
$D_u = \bigcap_{y\in N_u^i} N_y^o = \bigcap_{y\in\{w\}} N_y^o = N_w^o = \{u\}$,
$D_v = \bigcap_{y\in N_b^i} N_y^o = \bigcap_{y\in\{u\}} N_y^o = N_u^o = \{v\}$,
$D_x = \bigcap_{y\in N_x^i} N_y^o = \bigcap_{y\in\{v\}} N_y^o = N_v^o = \{x\}$,
$D_w = \bigcap_{y\in N_w^i} N_y^o = \bigcap_{y\in\{x\}} N_y^o = N_x^o = \{w\}$.

We have

$$\mathcal{D} = \Big\{\{u\},\{v\},\{x\},\{w\}\Big\}.$$

Theorem 4.2. *If $\mathcal{G}_1 = (\mathcal{V}_1,\rho_1,\nu_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2,\rho_2,\nu_2)$ are two directed fuzzy graphs and $f : \mathcal{V}_1 \to \mathcal{V}_2$ a bijective function. Then, f is an homeomorphism between the two corresponding topological spaces $(\mathcal{V}_1,\mathcal{F}^o_{\mathcal{G}_1})$ and $(\mathcal{V}_2,\mathcal{F}^o_{\mathcal{G}_2})$ if and only if the two properties are equivalents*

(i) $N_a^i \subset N_b^i$.
(ii) $N_{f(a)}^i \subset N_{f(b)}^i$.

for all $a,b \in \mathcal{V}_1$.

Proof. First, we suppose that f is an homeomorphism. Let $a,b\in\mathcal{V}_1$.

Step 1. $(i) \Rightarrow (ii)$: Suppose that $N_a^i \subset N_b^i$. Recall that, for $x\in\mathcal{V}_1$, the minimal open set D_x is given by $D_x = \{y\in\mathcal{V}_1;\ N_x^i \subset N_y^i\}$ (see Proposition 3.2). Next, $N_a^i \subset N_b^i$ this is equivalent to

$$b \in D_a. \tag{4.3}$$

Or since the function f is continuous, $f^{-1}(D_{f(a)})$ is an open set containing a and so $D_a \subset f^{-1}(D_{f(a)})$. From (4.3), we obtain $b \in f^{-1}(D_{f(a)})$. But $b\in f^{-1}(D_{f(a)})$ gives $f(b)\in D_{f(a)}$. Hence, $N_{f(a)}^i \subset N_{f(b)}^i$.

Step 2. $(ii) \Rightarrow (i)$: Suppose that $N^i_{f(a)} \subset N^i_{f(b)}$. We have $N^i_{f(a)} \subset N^i_{f(b)}$ and this means

$$f(b) \in D_{f(a)}. \tag{4.4}$$

D_a is an open set, since the function f^{-1} is continuous, $(f^{-1})^{-1}(D_a)$, which is $f(D_a)$, is an open set containing $f(a)$ and so $a \in D_{f(a)} \subset f(D_a)$.

From (4.4), we get $f(b) \in f(D_a)$. Then, $b \in f^{-1}(f(D_a))$, so $b \in D_a$ which equivalent to $N^i_a \subset N^i_b$.

For the other sense, suppose that (i) and (ii) are equivalent. We have to prove that f and f^{-1} are continuous.

Step 1. We have to prove that f is continuous. For this, let O an open set for $(\mathcal{V}_2, \mathcal{F}^o_{\mathcal{G}_2})$. Suppose that $a \in f^{-1}(O)$ and $b \in D_a$. we have $N^i_a \subset N^i_b$.
And so, by our hypothesis, we get $N^i_{f(a)} \subset N^i_{f(b)}$. Therefore, $f(b) \in D_{f(a)} \subset O$. That is, $b \in f^{-1}(O)$ and hence $D_a \subset f^{-1}(O)$ and then $f^{-1}(O)$ is an open for $(\mathcal{V}_2, \mathcal{F}^o_{\mathcal{G}_2})$. We deduce that f is a continuous function.

Step 2. In order to prove that $f^{-1} : \mathcal{V}_2 \to \mathcal{V}_1$ is continuous, let A be an open set for $(\mathcal{V}_1, \mathcal{F}^o_{\mathcal{G}_1})$. We are going to prove $(f^{-1})^{-1}(A) = f(A)$ is an open set for $(\mathcal{V}_2, \mathcal{F}^o_{\mathcal{G}_2})$. Let $f(x) \in f(A)$, for $x \in A$. If $y \in D_{f(x)}$, then $N^i_{f(x)} \subset N^i_y$. We set $y = f(z)$, since f is bijective and $y \in \mathcal{V}_2$. We get $N^i_{f(x)} \subset N^i_{f(z)}$ and so $N^i_x \subset N^i_z$. Therefore, $z \in D_x \subset A$ (A is an open set). We obtain $f(z) \in f(A)$, this means $y \in f(A)$, this is for all $y \in D_{f(x)}$. So, $D_{f(x)} \subset f(A)$. Hence, $f(A)$ is an open set for $\mathcal{V}_2$ and f^{-1} is continuous. □

Theorem 4.3. *In the particular case, if $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ is a directed fuzzy graph and $\nu(x, y) = \frac{1}{2}(\rho(x) \wedge \rho(y))$, then $\overline{\mathcal{G}} = (\mathcal{V}, \overline{\rho}, \overline{\nu})$ is self complementary (i.e $\overline{\mathcal{G}} \cong \mathcal{G}$) and $\mathcal{F}^o_{\overline{\mathcal{G}}} = \mathcal{F}^o_{\mathcal{G}}$.*

Proof. We have $\overline{\rho} = \rho$ by Definition 2.6 and also we have

$$\overline{\nu}(x, y) = \rho(x) \wedge \rho(y) - \nu(x, y) = \frac{1}{2}(\rho(x) \wedge \rho(y)) = \nu(x, y).$$

□

Theorem 4.4. *Let $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ be a directed fuzzy graph. Then, $\mathcal{V}$ is compact if and only if $\mathcal{V}$ is finite.*

Proof. Suppose that $\mathcal{V}$ is compact. Consider the open cover $\mathcal{D} = \{D_x\}_{x\in\mathcal{V}}$. This open cover has a finite subcover. Since it is a minimal basis, his subcover will be also equal $\mathcal{D}$. Therefore, $\mathcal{V}$ is finite.

Conversely, if $\mathcal{V}$ is finite then any open cover can be reduced to finite one and hence $\mathcal{V}$ is compact □

5. Connectivity and $\mathcal{F}_{\mathcal{G}}^{o}$ Topology

Definition 5.1. *Let $(V, \mathcal{F})$ be a topological space. The space V is called connected if whenever $V = X \cup Y$, with X, Y open sets and $X \cap Y = \emptyset$, we have $X = \emptyset$ or $Y = \emptyset$.*

For example, if $X = \{a, b, c\}$ and $\mathcal{F} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$, the space $(V, \mathcal{F})$ is a connected topological space.

Definition 5.2. *Let $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ be a directed fuzzy graph. The fuzzy graph $\mathcal{G}$ is called connected if for all $u, v \in \mathcal{V}$ there exists a fuzzy path from u to v and a fuzzy path from v to u (i.e. u and v are strongly connected). Other wise, the graph is called disconnected.*

Example 6.

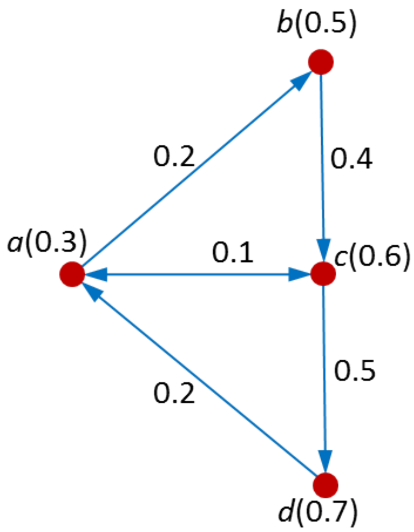

FIGURE 6. An example of connected directed fuzzy graph

The main problem of this section is: For a directed fuzzy graph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$, are there any relations between the connectedness of the the topological space $(\mathcal{V}, \mathcal{F}^o_{\mathcal{G}})$ and the connectivity of the directed fuzzy graph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$?

We are going to give some examples in order to prove that all cases are possible.

Example 7. *This is an example for a disconnected directed fuzzy graph but the topological space $(\mathcal{V}, \mathcal{F}^o_{\mathcal{G}})$ is connected.*

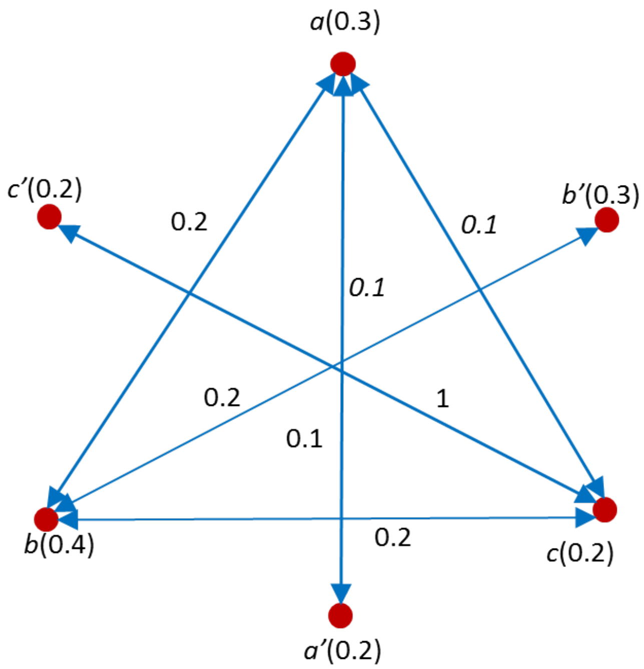

FIGURE 7. Disconnected directed fuzzy graph with connected topology

Indeed,

$$N^i_a = \{b, c\} \text{ and } N^o_a = \{b, c, a'\}$$
$$N^i_b = \{a, c\} \text{ and } N^o_b = \{a, c, b'\}$$
$$N^i_c = \{a, b\} \text{ and } N^o_c = \{a, b, c'\}$$
$$N^i_{a'} = \{a\} \text{ and } N^o_{a'} = \emptyset$$
$$N^i_{b'} = \{b\} \text{ and } N^o_{b'} = \emptyset$$
$$N^i_{c'} = \{c\} \text{ and } N^o_{c'} = \emptyset$$

$$D_x = \bigcap_{y \in N^i_x} N^o_y$$

$$D_a = \bigcap_{y \in N_a^i} N_y^o = \bigcap_{y \in \{b,c\}} N_y^o = N_b^o \cap N_c^o = \{a\}$$

In the same way

$$D_b = \{b\} \text{ and } D_c = \{c\}.$$

Also, we have

$$D_{a'} = \bigcap_{y \in N_{a'}^i} N_y^o = \bigcap_{y \in \{a\}} N_y^o = N_a^o = \{b, c, a'\}$$

For the same reason and by symmetry, we get

$$D_{b'} = \{a, c, b'\} \text{ and } D_{c'} = \{a, b, c'\}.$$

The minimal basis is $\mathcal{D} = \Big\{\{a\}, \{b\}, \{c\}, \{b, c, a'\}, \{a, c, b'\}, \{a, b, c'\}\Big\}$. *So,* $\mathcal{F}_{\mathcal{G}}^o$ *is connected*

Example 8. *For the following disconnected directed fuzzy graph, the topological space* $(\mathcal{V}, \mathcal{F}_{\mathcal{G}}^o)$ *is disconnected.*

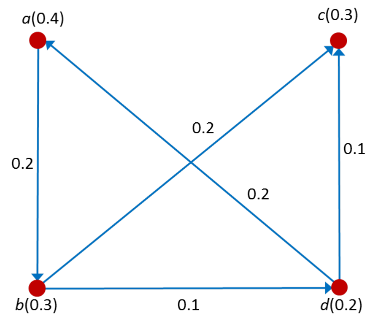

FIGURE 8. Disconnected directed fuzzy graph with disconnected topology

$$N_a^i = \{d\} \text{ and } N_a^o = \{b\}$$
$$N_b^i = \{a\} \text{ and } N_b^o = \{c, d\}$$
$$N_c^i = \{b, d\} \text{ and } N_c^o = \emptyset$$
$$N_d^i = \{b\} \text{ and } N_d^o = \{a, c\}$$

$$D_x = \bigcap_{y \in N_x^i} N_y^o$$

$$D_a = \bigcap_{y \in N_a^i} N_y^o = \bigcap_{y \in \{d\}} N_y^o = N_d^o = \{a, c\}$$

$$D_b = \bigcap_{y \in N_b^i} N_y^o = \bigcap_{y \in \{a\}} N_y^o = N_a^o = \{b\}$$
$$D_c = \bigcap_{y \in N_c^i} N_y^o = \bigcap_{y \in \{b,d\}} N_y^o = N_b^o \cap N_d^o = \{c\}$$
$$D_d = \bigcap_{y \in N_d^i} N_y^o = \bigcap_{y \in \{b\}} N_y^o = N_b^o = \{c, d\}.$$

We have $\mathcal{D} = \Big\{\{b\}, \{c\}, \{a, c\}, \{c, d\}\Big\}$ *and* $\mathcal{F}_{\mathcal{G}}^o = \Big\{\emptyset, \{b\}, \{c\}, \{a, c\}, \{c, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c, d\}\Big\}$. *The topology* $\mathcal{F}_{\mathcal{G}}^o$ *is disconnected.*

Example 9.

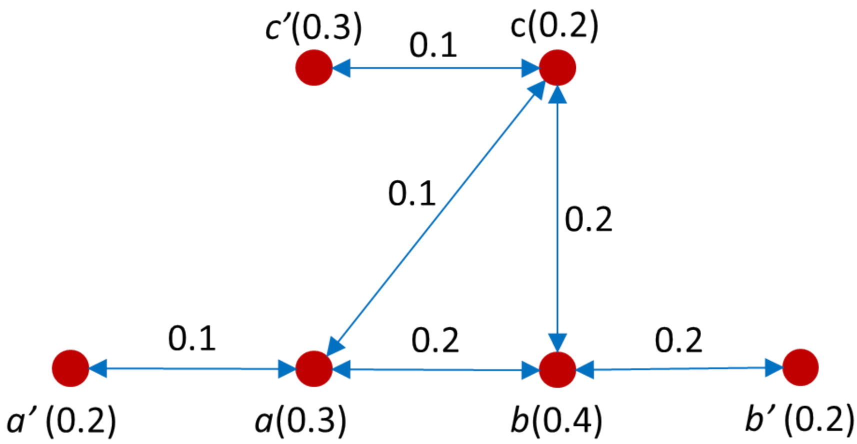

FIGURE 9. Connected directed fuzzy graph with connected topology

$$N_a^i = \{a', b, c\} \text{ and } N_a^o = \{a', b, c\}$$
$$N_b^i = \{a, b', c\} \text{ and } N_b^o = \{a, b', c\}$$
$$N_c^i = \{a, b, c'\} \text{ and } N_c^o = \{a, b, c'\}$$
$$N_{a'}^i = \{a\} \text{ and } N_{a'}^o = \{a\}$$
$$N_{b'}^i = \{b\} \text{ and } N_{b'}^o = \{b\}$$
$$N_{c'}^i = \{c\} \text{ and } N_{c'}^o = \{c\}$$
$$D_x = \bigcap_{y \in N_x^i} N_y^o$$
$$D_a = \bigcap_{y \in N_a^i} N_y^o = \bigcap_{y \in \{a', b, c\}} N_y^o = N_{a'}^o \cap N_b^o \cap N_c^o = \{a\}$$

Also,

$$D_b = \{b\} \text{ and } D_c = \{c\}$$

and we have

$$D_{a'} = \bigcap_{y \in N^i_{a'}} N^o_y = \bigcap_{y \in \{a\}} N^o_y = N^o_a = \{a', b, c\}.$$

For the same reason and by symmetry, we get

$$D_{b'} = \{a, b', c\} \text{ and } D_{c'} = \{a, b, c'\}.$$

We obtain $\mathcal{D} = \Big\{\{a\}, \{b\}, \{c\}, \{b, c, a'\}, \{a, c, b'\}, \{a, b, c'\}\Big\}$*. So, The topology* $\mathcal{F}^o_{\mathcal{G}}$ *is connected.*

Example 10.

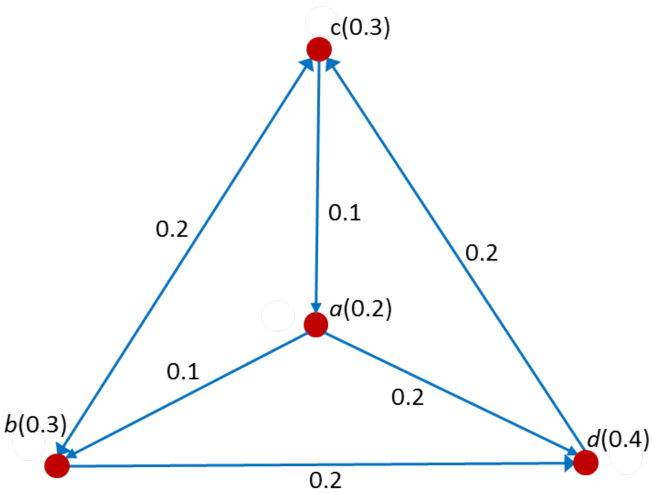

FIGURE 10. Connected directed fuzzy graph with disconnected topology

$$N^i_a = \{c\} \text{ and } N^o_a = \{b, d\}$$
$$N^i_b = \{a, c\} \text{ and } N^o_b = \{c, d\}$$
$$N^i_c = \{b, d\} \text{ and } N^o_c = \{a, b\}$$
$$N^i_d = \{a, b\} \text{ and } N^o_d = \{c\}$$
$$D_x = \bigcap_{y \in N^i_x} N^o_y$$
$$D_a = \bigcap_{y \in N^i_a} N^o_y = \bigcap_{y \in \{c\}} N^o_y = N^o_c = \{a, b\}$$

$$D_b = \bigcap_{y \in N_b^i} N_y^o = \bigcap_{y \in \{a,c\}} N_y^o = N_a^o \cap N_c^o = \{b\}$$

$$D_c = \bigcap_{y \in N_c^i} N_y^o = \bigcap_{y \in \{b,d\}} N_y^o = N_b^o \cap N_d^o = \{c\}$$

$$D_d = \bigcap_{y \in N_d^i} N_y^o = \bigcap_{y \in \{a,b\}} N_y^o = N_a^o \cap N_b^o = \{d\}.$$

We have $\mathcal{D} = \Big\{\{b\}, \{c\}, \{d\}, \{a,b\}\Big\}$ *and so the topology is not connected.*

6. Conclusions

Along this work, we investigated fuzzy directed graph and topology. We defined a topology $\mathcal{F}_{\mathcal{G}}^o$ for the vertices set of a fuzzy directed graph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ using a subbasis depended on the out-neighbors. We have proved that $\mathcal{F}_{\mathcal{G}}^o$ is an Alexandroff topology and characterize its minimal basis. In addition, we proved some results about continuous functions and isomorphic fuzzy directed graphs. We proved that the space $(\mathcal{V}, \mathcal{F}_{\mathcal{G}}^o)$ is compact if and only if $\mathcal{V}$ is finite. For the connectedness of the topology vs. the connectivity of the graph, we have given four examples and so proved that all situations can be realized. As open question, we can look for some necessary or sufficient conditions on the graph $\mathcal{G} = (\mathcal{V}, \rho, \nu)$ for the connectivity of the topology $\mathcal{F}_{\mathcal{G}}^o$.

Furthermore, we can use the membership of the edges (or the weight) in order to define the connectivity of the graph using strong edges. And then, see the relation between the connectedness of the topology and the connectivity of the graph.

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