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SOME PROPERTIES OF HORADAM VECTORS

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ABSTRACT. In this study, Horadam vectors are defined for the first time by using the Horadam Binet-like formula and reduction relation. Then, geometric properties of Horadam vectors such as inner product, norm, and vector products are obtained. Additionally, the angles between the various vectors examined will be analyzed within the constraints of a two-dimensional.

1. INTRODUCTION

Number sequences have an important place in number theory. Many studies have been carried out by researchers on number sequences. They are used in power series, generating functions, time series analysis, queueing theory, and many other fields of science [7, 10, 13]. The first sequence that comes to mind as an integer sequence is the Fibonacci sequence.

Fibonacci and Lucas numbers are defined by the following recurrence relations

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 2$$

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 2$$

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Besides, the nth Fibonacci and Lucas number are formulized as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 0,$$

$$L_n = \alpha^n + \beta^n, \quad n \geq 0,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. [12]

Fibonacci numbers can be found in many areas of mathematics, such as Pascal's triangle, Pythagorean triples, computer algorithms, and graph theory. They can also be found in physics, finance, architecture, music, and art [2–4].

The sequence of $H_{n+2} = pH_{n+1} + qH_n$ second-order linear recurrence relations with initial values $H_0 = e$ and $H_1 = f$ is called the Horadam sequence due to the work of Alwyn Francis Horadam. The Horadam sequence can also be called a generalization of the Fibonacci sequence. Besides, the nth Horadam number is formulized as

$$H_n = \frac{(f - e\beta)\alpha^n - (f - e\alpha)\beta^n}{\alpha - \beta}, \quad n \geq 0, \tag{1.1}$$

where $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2+4q}}{2}$ [9]. Recurrence sequences are a central part of the study of number theory and are of great importance. These sequences appear in different fields of mathematics and other sciences [1, 5, 15, 16]. The abundance of references in the literature on sequences of the Horadam type makes this sequence worth analyzing.

In [17] the inner product of any two vectors in n-dimensional Euclidean space is

$$\langle k, l \rangle = k_1 l_1 + k_2 l_2 + \cdots + k_n l_n,$$

the length of a vector in n-dimensional Euclidean space is

$$\|\vec{k}\| = \sqrt{\langle k, k \rangle},$$

and the angle between k and l in n-dimensional Euclidean space is

$$\cos\delta = \frac{\langle k, l \rangle}{\|\vec{k}\| \cdot \|\vec{l}\|}. \tag{1.2}$$

Salter [14] expressed the inner product of any two Fibonacci vectors, any two Lucas vectors, and any Fibonacci vector with any Lucas vector in terms of the Fibonacci and Lucas numbers for arbitrary r. Güven and Nurkan [8] introduced new vectors known as dual Fibonacci vectors and presented their properties for application in the geometry of dual space. Yüce and Torunbalcı-Aydın [18] defined generalized dual Fibonacci vectors, including the inner product and cross product of two such vectors, as well as the triple scalar product of three generalized dual Fibonacci vectors. Kaya and Önder [11] looked at the vector products of taking into account two Fibonacci 3-vectors, two Lucas 3-vectors, and one of each vector using the vector version of the Binet's formula. Çetinberk and Yüce [6] studied vector products of Fibonacci 3-vectors, Fibonacci 4-vectors, and Fibonacci 7-vectors. They first described the corresponding antisymmetric matrix for the Fibonacci 3-vector and reconsidered the vector product with the aid of this matrix. Moreover, they examined certain properties of this vector product.

In this work, Horadam vectors will be defined, and a variety of their algebraic properties will be presented. After defining the Horadam vectors, the inner product and vector products of the Horadam vectors are calculated.

2. Horadam Vectors

Definition 2.1. *For all integers n, the Horadam vectors is defined by*

$$\vec{H}_n = [H_n \quad H_{n+1}]^T \tag{2.1}$$

where H_n is nth Horadam number.

Lemma 2.1. *The Q matrix has characteristic polynomial $x^2 - px - q$. Thus, it has eigenvalues $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2+4q}}{2}$. The associated eigenvectors are respectively*

$$\vec{a} = \left[\alpha^0 \quad \alpha^1\right]^T \quad and \quad \vec{b} = \left[\beta^0 \quad \beta^1\right]^T,$$

where

$$P = \begin{bmatrix} 0 & 1 \\ q & p \end{bmatrix}.$$

Proposition 2.1. *For all integers n, the Horadam vectors is as*

$$\vec{H}_n = P\vec{H}_{n-1},$$

besides

$$\vec{H}_n = P^{n-k}\vec{H}_k.$$

Lemma 2.2. *Some properties of α and β roots of the Horadam numbers are given by*

$$\begin{aligned}
\alpha\beta &= -q,\\
\alpha+\beta &= p,\\
\alpha-\beta &= \sqrt{p^2+4q},\\
\alpha^2-q &= p\alpha,\\
\beta^2-q &= p\beta.
\end{aligned}$$

Theorem 2.1. *For all integers n, the Binet's formula of Horadam vector can be defined by*

$$\vec{H}_n = A\alpha^n \begin{bmatrix}\alpha^0 & \alpha^1\end{bmatrix}^T + B\beta^n \begin{bmatrix}\beta^0 & \beta^1\end{bmatrix}^T, \tag{2.2}$$

where $A = \frac{f-e\beta}{\alpha-\beta}$, $B = \frac{f-e\alpha}{\alpha-\beta}$, $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$, and $\beta = \frac{p-\sqrt{p^2+4q}}{2}$.

Proof. For the prove, we utilize induction principle on n. The equality holds for $n = 0$. Now assume that the equality is true for $n > 0$. Then, we can verify for $n+1$ as follows

$$\begin{aligned}
Q\vec{H}_n &= AQ\alpha^n \begin{bmatrix}\alpha^0 & \alpha^1\end{bmatrix}^T + BQ\beta^n \begin{bmatrix}\beta^0 & \beta^1\end{bmatrix}^T,\\
\vec{H}_{n+1} &= A\alpha^{n+1} \begin{bmatrix}\alpha^0 & \alpha^1\end{bmatrix}^T + B\beta^{n+1} \begin{bmatrix}\beta^0 & \beta^1\end{bmatrix}^T.
\end{aligned}$$

□

Some special equalities well-known for the Horadam sequence have also been calculated for the Horadam vectors. Then, the following equalities hold:

a) Tagiuri's Identity:

$$\begin{aligned}
&\langle \vec{H}_{m+k}, \vec{H}_{n-k}\rangle - \langle \vec{H}_m, \vec{H}_n\rangle\\
&= AB \begin{bmatrix}\alpha^0 & \alpha^1\end{bmatrix}^T \begin{bmatrix}\beta^0 & \beta^1\end{bmatrix}^T \left(\alpha^k - \beta^k\right)\left(\alpha^m\beta^{n-k} - \alpha^{n-k}\beta^m\right).
\end{aligned}$$

b) d'Ocagne's Identity:

$$\begin{aligned}
&\langle \vec{H}_{m+1}, \vec{H}_{n-1}\rangle - \langle \vec{H}_m, \vec{H}_n\rangle\\
&= AB \begin{bmatrix}\alpha^0 & \alpha^1\end{bmatrix}^T \begin{bmatrix}\beta^0 & \beta^1\end{bmatrix}^T (\alpha-\beta)\left(\alpha^m\beta^{n-1} - \alpha^{n-1}\beta^m\right).
\end{aligned}$$

c) Catalan's Identity:

$$\begin{aligned}&\langle \vec{H}_{n+k}, \vec{H}_{n-k}\rangle - \langle \vec{H}_n, \vec{H}_n\rangle \\ &= AB \begin{bmatrix}\alpha^0 & \alpha^1\end{bmatrix}^T \begin{bmatrix}\beta^0 & \beta^1\end{bmatrix}^T \left(\alpha^k - \beta^k\right)\left(\alpha^n \beta^{n-k} - \alpha^{n-k}\beta^n\right).\end{aligned}$$

d) Cassini's Identity:

$$\begin{aligned}&\langle \vec{H}_{n+1}, \vec{H}_{n-1}\rangle - \langle \vec{H}_n, \vec{H}_n\rangle \\ &= AB \begin{bmatrix}\alpha^0 & \alpha^1\end{bmatrix}^T \begin{bmatrix}\beta^0 & \beta^1\end{bmatrix}^T (\alpha - \beta)\left(\alpha^n \beta^{n-1} - \alpha^{n-1}\beta^n\right).\end{aligned}$$

Lemma 2.3. *The inner product of the vectors $\vec{a}$ and $\vec{b}$ with themselves, is obtained as*

(i) $\vec{a}\cdot\vec{a} = \frac{(p\alpha+q)^m-1}{p\alpha+q-1}$,
(ii) $\vec{b}\cdot\vec{b} = \frac{(p\beta+q)^m-1}{p\beta+q-1}$,
(iii) $\vec{a}\cdot\vec{b} = \frac{1-(-q)^m}{q+1}$,

where $\vec{a} = [\alpha^0 \quad \alpha^1 \quad \alpha^2 \quad \cdots \quad \alpha^{m-1}]^T$ *and* $\vec{b} = [\beta^0 \quad \beta^1 \quad \beta^2 \quad \cdots \quad \beta^{m-1}]^T$.

Proof. The inner product of the vector $\vec{a}$ with itself in Euclidean n-space is

$$\begin{aligned}\vec{a}\cdot\vec{a} &= \langle \begin{bmatrix}\alpha^0 & \alpha^1 & \alpha^2 & \cdots & \alpha^{m-1}\end{bmatrix}, \begin{bmatrix}\alpha^0 & \alpha^1 & \alpha^2 & \cdots & \alpha^{m-1}\end{bmatrix}\rangle \\ &= (\alpha^0)^2 + (\alpha^1)^2 + (\alpha^2)^2 + \cdots + (\alpha^{m-1})^2 \\ &= \frac{(\alpha^{2m})-1}{(\alpha^2)-1} = \frac{(p\alpha+q)^m-1}{p\alpha+q-1}.\end{aligned}$$

Similarly, let's find the inner product of the vector $\vec{b}$ with itself:

$$\begin{aligned}\vec{b}\cdot\vec{b} &= \langle \begin{bmatrix}\beta^0 & \beta^1 & \beta^2 & \cdots & \beta^{m-1}\end{bmatrix}, \begin{bmatrix}\beta^0 & \beta^1 & \beta^2 & \cdots & \beta^{m-1}\end{bmatrix}\rangle \\ &= (\beta^0)^2 + (\beta^1)^2 + (\beta^2)^2 + \cdots + (\beta^{m-1})^2 \\ &= \frac{(\beta^{2m})-1}{(\beta^2)-1} = \frac{(p\beta+q)^m-1}{p\beta+q-1}.\end{aligned}$$

Finally, let's calculate the inner product of the vectors $\vec{a}$ and $\vec{b}$:

$$\begin{aligned}\vec{a}\cdot\vec{b} &= \langle \begin{bmatrix}\alpha^0 & \alpha^1 & \alpha^2 & \cdots & \alpha^{m-1}\end{bmatrix}, \begin{bmatrix}\beta^0 & \beta^1 & \beta^2 & \cdots & \beta^{m-1}\end{bmatrix}\rangle \\ &= (\alpha\beta)^0 + (\alpha\beta)^1 + (\alpha\beta)^2 + \cdots + (\alpha\beta)^{m-1} \\ &= \frac{(\alpha\beta)^m-1}{(\alpha\beta)-1} = \frac{(-q)^m-1}{-q-1} = \frac{1-(-q)^m}{q+1}.\end{aligned}$$

□

Corollary 2.1. *Since for any vector, the square of the length of the vector is the inner product of the vector with itself, the equations*

$$\|\vec{a}\| = \sqrt{\frac{(p\alpha+q)^m - 1}{p\alpha+q-1}},$$

$$\|\vec{b}\| = \sqrt{\frac{(p\beta+q)^m - 1}{p\beta+q-1}},$$

and,

$$\|\vec{a}\|\|\vec{b}\| = \sqrt{\frac{1-(-q)^m}{q+1}}$$

are obtained.

Definition 2.2. *The Horadam numbers, the matrix of m-dimensional Horadam vector is represented by*

$$\vec{H}_n^m = [H_n \quad H_{n+1} \quad \cdots \quad H_{n+m-1}]^T . \tag{2.3}$$

Theorem 2.2. *For all integers n, the Binet's formula of Horadam vector can be defined by*

$$\vec{H}_n^m = A\alpha^n\vec{a} + B\beta^n\vec{b} \tag{2.4}$$

where $A = \frac{f-e\beta}{\alpha-\beta}$, $B = \frac{f-e\alpha}{\alpha-\beta}$, $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$, $\vec{a} = [\alpha^0 \quad \alpha^1 \quad \cdots \quad \alpha^{m-1}]^T$, $\beta = \frac{p-\sqrt{p^2+4q}}{2}$, and $\vec{b} = [\beta^0 \quad \beta^1 \quad \cdots \quad \beta^{m-1}]^T$.

Proof. For the prove, we utilize induction princible on n. The equality holds for $n = 0$. Now assume that the equality is true for $n > 0$. Then, we can verify for $n+1$ as follows

$$Q\vec{H}_n = AQ\alpha^n\vec{a} + BQ\beta^n\vec{b},$$
$$\vec{H}_{n+1} = A\alpha^{n+1}\vec{a} + B\beta^{n+1}\vec{b}$$

where

$$Q=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & q & p \end{bmatrix}_{m\times m}.$$

□

Lemma 2.4. *The norm of the m-dimensional Horadam vector is calculated by*

$$\begin{aligned}\|\vec{H}_n^m\|^2 &= A^2\left(p\alpha+q\right)^n\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)+B^2\left(p\beta+q\right)^n\left(\frac{(p\beta+q)^m-1}{p\beta+q-1}\right)\\ &+2AB\left(-q\right)^n\left(\frac{1-(-q)^m}{q+1}\right).\end{aligned}$$

Proof.

$$\begin{aligned}\|\vec{H}_n^m\|^2 &= \vec{H}_n^m\cdot\vec{H}_n^m=\left(A\alpha^n\vec{a}+B\beta^n\vec{b}\right)\left(A\alpha^n\vec{a}+B\beta^n\vec{b}\right)\\ &= A^2\alpha^{2n}(\vec{a})^2+B^2\beta^{2n}(\vec{b})^2+2AB\alpha^n\beta^n\vec{a}\vec{b}\\ &= A^2\alpha^{2n}\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)+B^2\beta^{2n}\left(\frac{(p\beta+q)^m-1}{p\beta+q-1}\right)\\ &+2AB\alpha^n\beta^n\left(\frac{1-(-q)^m}{q+1}\right)\\ &= A^2\left(p\alpha+q\right)^n\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)+B^2\left(p\beta+q\right)^n\left(\frac{(p\beta+q)^m-1}{p\beta+q-1}\right)\\ &+2AB\left(-q\right)^n\left(\frac{1-(-q)^m}{q+1}\right).\end{aligned}$$

□

Lemma 2.5. *For all integers n,*

$$\begin{aligned}\vec{H}_n^m\vec{a} &= A\alpha^n\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)+B\beta^n\left(\frac{1-(-q)^m}{q+1}\right),\\ \vec{H}_n^m\vec{b} &= A\alpha^n\left(\frac{1-(-q)^m}{q+1}\right)+B\beta^n\left(\frac{(p\beta+q)^m-1}{p\beta+q-1}\right).\end{aligned}$$

Theorem 2.3. *Let $\phi_{n,m}$ denote the angle between the vectors $\vec{H}_n^m$ and $\vec{a}$, and $\varphi_{n,m}$ denote the angle between the vectors $\vec{H}_n^m$ and $\vec{b}$, and the following equations hold*

$$\cos\phi_{n,m} = \frac{A\alpha^n\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)+B\beta^n\left(\frac{1-(-q)^m}{q+1}\right)}{\sqrt{\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)}}$$
$$\times\frac{1}{\sqrt{A^2\left(p\alpha+q\right)^n\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)+B^2\left(p\beta+q\right)^n\left(\frac{(p\beta+q)^m-1}{p\beta+q-1}\right)+2AB\left(-q\right)^n\left(\frac{1-(-q)^m}{q+1}\right)}}$$

and

$$\cos\varphi_{n,m} = \frac{A\alpha^n\left(\frac{1-(-q)^m}{q+1}\right)+B\beta^n\left(\frac{(p\beta+q)^m-1}{p\beta+q-1}\right)}{\sqrt{\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)}}$$
$$\times\frac{1}{\sqrt{A^2\left(p\alpha+q\right)^n\left(\frac{(p\alpha+q)^m-1}{p\alpha+q-1}\right)+B^2\left(p\beta+q\right)^n\left(\frac{(p\beta+q)^m-1}{p\beta+q-1}\right)+2AB\left(-q\right)^n\left(\frac{1-(-q)^m}{q+1}\right)}}.$$

Proof. The desired is obtained using the inequality in (1.2). □

3. Conclusion

Our study introduced Horadam vectors for using a Horadam Binet-like formula and reduction relation. We computed the inner product, norm, and vector products of Horadam vectors and investigated the geometric properties of Horadam numbers.

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