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A CRITERIA FOR UNIVALENCE ,STARLIKENESS AND CONVEX OF A SPECIAL TYPE ANALYTIC FUNCTION

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ABSTRACT. In this study, univalence, starlike and convex conditions of some special types of analytic functions are determined.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{D} = \{z : z \in \mathbb{C}, |z| < 1\}$ and satisfying the conditions $f(0) = 0$ and $f'(0) = 1$. Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the form (1.1) which are univalent in $\mathcal{D}$.

The set P is the set of all functions of the form

$$f(z) = 1 + p_1 z + p_z z^2 + p_3 z^3 + \ldots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

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that are analytic in D, and such that for $z \in D$,

$$Ref(z) > 0.$$

Any function in P is called a function with positive real part in D.

Let E be a set in $\mathbb{C}$. We say that E is starlike (with respect to orijin) if the closed line segment joining the origin to each point $w \in E$ lies entirely in E. We say that E is convex if for all $w_1, w_2 \in E$, the closed line segment between w_1 and w_2 lies entirely in E.

Let ST denote the subclass of S which consists all the starlike functions with respect to origin and let CV denote the subclass of S which consists all the convex functions. Closely related to the classes ST and CV is the class P containing all the function g holomorphic and having positive real part in $D = \{z \in \mathbb{C} : |z| < 1\}$, with $g(0) = 1$.

Theorem 1.1. *([1]) Let f be holomorphic in D, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in ST$ if and only if $\frac{zf'}{f} \in P$.*

Theorem 1.2. *([1]) Let f be holomorphic in D, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in CV$ if and only if $\left\{1 + \frac{zf''}{f'}\right\} \in P$.*

Example 1. *$f(z) = z + az^2$ is in ST if and only if $|a| \leq \frac{1}{2}$.*

Example 2. *$f(z) = z + az^2$ is in CV if and only if $|a| \leq \frac{1}{4}$.*

For the proof of Examples 1 and 2, please refer to ([2]).

There are several examples of starlike and convex functions. The first one is the famous Koebe function:

$$k(z) = \frac{z}{(1 - z)^2}$$

is in ST but not in CV.

The following results were given in the thesis titled "A study on univalent functions and their Geometrical properties" prepared by Wei Dik KAI [3] in 2017.

Corollary 1.1. *$f(z) = z + az^3$ is in S if and only if $|a| \leq \frac{1}{3}$.*

Corollary 1.2. *$f(z) = z + az^3$ is in ST if and only if $|a| \leq \frac{1}{3}$.*

Corollary 1.3. *$f(z) = z + az^3$ is in CV if and only if $|a| \leq \frac{1}{9}$.*

Corollary 1.4. *$f(z) = z + az^{m+1}$ is in S if and only if $|a| \leq \frac{1}{m+1}$.*

Corollary 1.5. *$f(z) = z + az^{m+1}$ is in ST if and only if $|a| \leq \frac{1}{m+1}$.*

Corollary 1.6. *$f(z) = z + az^{m+1}$ is in CV if and only if $|a| \leq \frac{1}{(m+1)^2}$.*

In this study, we will consider the $f(z) = z + az^2 + bz^3$ function. We will examine the conditions for this function to belong to the S, ST and CV classes.

2. MAIN RESULT

Theorem 2.1. *The functions $f(z) = z + az^2 + bz^3$ is in S for $a, b \in \mathbb{C}, 2|a| + 3|b| \leq 1$ and not univalent in $D = \{z \in C : |z| < 1\}$ for $2|a| + 3|b| > 1$.*

Proof. It is clear that $f(z)$ is a holomorphic function in D and normalized with the conditions $f(0) = 0, f'(0) = 1$. For $2|a| + 3|b| \leq 1$, observe that when $a = 0, b = 0, f(z) = z$ is clearly a univalent function is S. For $a \neq 0, b \neq 0$, let

$$f(z) = f(w)$$

where $z, w \in U$. Then we have

$$\begin{aligned} & z + az^2 + bz^3 = w + aw^2 + bw^3 \\ \Rightarrow \quad & (z - w) + a\left(z^2 - w^2\right) + b\left(z^3 - w^3\right) = 0 \\ \Rightarrow \quad & (z - w)\left\{1 + az + aw + bz^2 + bzw + bw^2\right\} = 0. \end{aligned}$$

We claim that $z = w$. If not, then

$$\begin{aligned} & 1 + az + aw + bz^2 + bzw + bw^2 = 0 \\ \Rightarrow \quad & az + aw + bz^2 + bzw + bw^2 = -1. \end{aligned}$$

If the absolute value of both sides of this last equality is taken and the triangle inequality is applied, we can write

$$
\begin{aligned}
& |a||z| + |a||w| + |b||z|^2 + |b||zw| + |b||w|^2 \geq 1 \\
\Rightarrow\ & |a||z| + |b||z|^2 \geq 1 - |a||w| - |b||zw| - |b||w|^2 \\
\Rightarrow\ & |a||z| + |b||z| + |b||z|^2 \geq 1 - |a| - |b| \\
\Rightarrow\ & |b||z|^2 + (|a| + |b|)|z| - (1 - |a| - |b|) \geq 0. \\
\Rightarrow\ & |b||z|^2 + (|a| + |b|)|z| - (1 - |a| - |b|) = 0 \\
\Rightarrow\ & |z| = \frac{-(|a| + |b|) \pm \sqrt{(|a| + |b|)^2 + 4|b|(1 - |a| - |b|)}}{2|b|} \\
& \qquad = \frac{-(|a| + |b|) \pm \sqrt{|a|^2 + 2|a||b| + |b|^2 + 4|b| - 4|ab| - 4|b|^2}}{2|b|} \\
& \qquad = \frac{-(|a| + |b|) \pm \sqrt{|a|^2 - 2|a||b| + 4|b| - 3|b|^2}}{2|b|}.
\end{aligned}
$$

In case of $|z| \geq 1$, it will be $z \notin U$. Thus, we obtain

$$
\begin{aligned}
|z| \geq 1 \Rightarrow\ & \frac{-(|a| + |b|) \pm \sqrt{|a|^2 - 2|a||b| + 4|b| - 3|b|^2}}{2|b|} \geq 1 \\
& \Rightarrow -(|a| + |b|) \pm \sqrt{|a|^2 - 2|a||b| + 4|b| - 3|b|^2} \geq 2|b| \\
& \Rightarrow -(|a| + 3|b|) \geq -\sqrt{|a|^2 - 2|a||b| + 4|b| - 3|b|^2} \\
& \Rightarrow (|a| + 3|b|)^2 \leq \sqrt{|a|^2 - 2|a||b| + 4|b| - 3|b|^2} \\
& \Rightarrow 8|ab| - 4|b| + 12|b|^2 \leq 0 \\
& \Rightarrow 2|a| + 3|b| \leq 1.
\end{aligned}
$$

Then, $f(z) = f(w)$ requires that $2|a| + 3|b| \leq 1$ if and only if $z = w$. As a result the functions $f(z) = z + az^2 + bz^3$ is in S.

In Theorem 2.1, if $b = 0$, Example 1 is obtained, and if $a = 0$, Corollary 1.1 is obtained. □

Theorem 2.2. *The functions* $f(z) = z + az^2 + bz^3$ *is in* ST *if and only if* $2|a| + 3|b| \leq 1$.

Proof. If $f \in ST$, then $f \in S$. From Theorem 2.1, we have $2|a| + 3|b| \leq 1$.

Conversely, we prove that if $2|a| + 3|b| \leq 1$, then f is starlike. For the proof, it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1$$

First, we can write

$$
\begin{aligned}
\left|\frac{zf'(z)}{f(z)}-1\right| &= \left|\frac{z\left(1+2az+3bz^2\right)}{z+az^2+bz^3}-1\right| = \left|\frac{3bz^2+2az+1}{bz^2+az+1}-1\right| \\
&= \left|\frac{2bz^2+az}{bz^2+az+1}\right| = \frac{|2bz^2+az|}{|1+(bz^2+az)|} \leq \frac{2|b||z|+|a||z|}{1-|bz^2+az|} \\
&\leq \frac{2|b|+|a|}{1-(|b|+|a|)}
\end{aligned} \tag{2.1}
$$

If $2|a|+3|b|\leq 1$ is written in the expression $|b|\leq \frac{1}{3}\{1-2|a|\}$ and this expression is used in (2.1),

$$
\begin{aligned}
\left|\frac{zf'(z)}{f(z)}-1\right| &\leq \frac{2|b|+|a|}{1-(|b|+|a|)} \leq \frac{\frac{2}{3}\{1-2|a|\}+|a|}{1-\left\{\frac{1}{3}(1-2|a|)+|a|\right\}} \\
&= \frac{2-4|a|+3|a|}{3-\{1-2|a|+3|a|\}} = \frac{2-|a|}{2-|a|} = 1
\end{aligned}
$$

is obtained. Hence, $\frac{zf'}{f}$ lies in a circle centered at 1 with radius $\mathrm{r}=1$ and thus $\frac{zf'(z)}{f(z)}\in P$ and hence f is starlike. □

Theorem 2.3. *The functions $f(z)=z+az^2+bz^3$ is in CV if and only if $4|a|+9|b|\leq 1$.*

Proof. We prove that if $4|a|+9|b|\leq 1$, then f is convex. For the proof, it is sufficient to show that

$$\left|1+\frac{zf'(z)}{f(z)}-1\right|\leq 1.$$

First, we can write

$$
\begin{aligned}
\left|1+\frac{zf''(z)}{f'(z)}-1\right| &= \left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{z(2a+6bz)}{1+2az+3bz^2}\right| \\
&= \left|\frac{6bz^2+2az}{3bz^2+2az+1}\right| \leq \frac{6|b|+2|a|}{1-\{3|b|+2|a|\}}.
\end{aligned} \tag{2.2}
$$

If $4|a|+9|b|\leq 1$ is written in the expression $|b|\leq \frac{1}{9}\{1-4|a|\}$ and this expression is used in (2.2),

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{6|b|+2|a|}{1-\{3|b|+2|a|\}} \leq \frac{6\left\{\frac{1}{9}(1-4|a|)\right\}+2|a|}{1-\left\{\frac{3}{9}(1-4|a|)+2|a|\right\}} = 1$$

is obtained. Hence, $1+\frac{zf''(z)}{f'(z)}$ lies in a circle centered at 1 with radius $\mathrm{r}=1$ and thus $1+\frac{zf''(z)}{f'(z)}\in P$ and thus $f\in CV$.Conversely, let us assume that $f\in CV$. We must prove that $4|a|+9|b|\leq 1$. It is known that for $\Psi(z)\in CV$ to exist, the necessary and sufficient condition is that $z\Psi'(z) \in ST$. Let $\Psi(z) = z + az^2 + bz^3$. In this case we write

$$z\Psi'(z) = z\left(1 + 2az + 3bz^2\right) = z + 2az^2 + 3bz^3.$$

Thus, if Theorem 2.2 is used,

$$2|2a| + 3|3b| \leq 1 \quad \Rightarrow \quad 4|a| + 9|b| \leq 1$$

is obtained.

In Theorem 2.2, if $a = 0$, Corollary 1.2 is obtained. In Theorem 2.3, if $a = 0$, Corollary 1.3 is obtained. □

3. Conclusion

In this paper, a special analytic function is taken. A condition for the univalence of this function is determined.Additionally, a criterion for its starlikeness and convexity is obtained.

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