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A (λ, V)-GENERALISED GAMMA FUNCTION AND ITS PROPERTIES

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ABSTRACT. This paper introduces a (λ, v) generalization of the Gamma function and establishes some of its properties. The Hölder's and Young's inequalities and analytical techniques were used to establish the results.

1. INTRODUCTION

The gamma function, an extension of factorial notation to non-integer values, appears in many areas of mathematics. It is arguably one of the most extensively studied special functions. The gamma function has a wide range of applications in mathematical analysis, number theory, combinatorics, mathematical modeling, statistics, probability theory, engineering, and physics, among others. It is typically defined by [1]

$$\Gamma(\delta) = \int_0^\infty t^{\delta-1} e^{-t}\, dt,$$

for $\delta > 0$ and

$$\Gamma(\delta) = \lim_{h\to\infty} \frac{h! h^\delta}{\delta(\delta+1)(\delta+2)\cdots(\delta+h)}.$$

The gamma function's extensive use across mathematics has motivated research into its generalizations, with several extensions already established. For example,

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see [2–7,9]. Two Key generalizations relevant to this work include the v -analogue and the λ- analogue of the gamma function.

The v-analogue (or v-extension/deformation) of the Gamma function, introduced by [2] for $v > 0$ and $t > 0$ is given as

$$\Gamma_v(\delta) = \int_0^\infty e^{-t} \left(\frac{t}{v}\right)^{\frac{\delta}{v}-1} dt.$$

The λ-analogue of the gamma function, introduced by [3], for $\lambda > 0$ and $x > 0$ is given as

$$\Gamma_\lambda(x) = \int_0^\infty e^{-\lambda t} t^{x-1}\, dt.$$

Inspired by the v -analogue and the λ- analogue of the gamma function, this paper unifies and extends these concepts by introducing a (λ, v)-generalized gamma function and establishing its properties and inequalities.

2. Preliminaries

This section provides essential background materials, including established definitions and lemmas, which are fundamental to the proofs of our main results. These foundational elements are drawn from existing literature.

Theorem 2.1. *[8] Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are continuous real valued functions defined on $[a, b]$ then the inequality*

$$\int_a^b |f(t)g(t)|dt \leq \left[\int_a^b |f(t)|^p dt\right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q dt\right]^{\frac{1}{q}},$$

holds. With equality holding if and only if $|f(x)|^p = c|g(x)|^{p-1}$. If $p = q = 2$ the inequality becomes Schwarz's inequality.

Theorem 2.2. *[8] If $m \geq 0$, $n \geq 0$ are nonnegative real numbers and if $p > 1$ and $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$mn \leq \frac{m^p}{p} + \frac{n^q}{q}.$$

Equality holds if and only if $m^p = n^q$.

Definition 2.1. *For $h > 0$ and $t \in \mathbf{R}$,*

$$\lim_{h\to\infty}\left(1-\frac{t}{h}\right)^h = e^{-t}.$$

3. Main Results

This section introduces a two-parameter generalization of the gamma function known as the (λ, v)-generalized gamma function, and explores some of its key properties.

Definition 3.1. *Suppose $\delta, \lambda, v \in \mathbb{R}^+$. A (λ, v)-generalized Gamma function is defined as*

$$\Gamma_{\lambda,v}(\delta) = \int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}-1} dt. \tag{3.1}$$

When both λ and v are equal to 1, the $\Gamma_{\lambda,v}(\delta)$ gamma function converges to the classical gamma function, $\Gamma(\delta)$.

Proof. Let $u = \lambda t$, then $dt = \frac{1}{\lambda}\, du$, and substituting into (3.1) gives

$$\Gamma_{\lambda,v}(\delta) = \int_0^\infty \left(\frac{u}{\lambda v}\right)^{\frac{\delta}{v}-1} e^{-u}\cdot\frac{1}{\lambda}\, du. \tag{3.2}$$

Simplifying yields

$$\Gamma_{\lambda,v}(\delta) = \lambda^{-\frac{\delta}{v}} v^{1-\frac{\delta}{v}} \int_0^\infty u^{\frac{\delta}{v}-1} e^{-u}\, du.$$

Thus,

$$\Gamma_{\lambda,v}(\delta) = \lambda^{-\frac{\delta}{v}} v^{1-\frac{\delta}{v}}\Gamma\left(\frac{\delta}{v}\right). \tag{3.3}$$

□

Lemma 3.1. *Let $\delta, \lambda, v \in \mathbb{R}^+$. Then the (λ, v)-generalized Gamma function satisfies the property*

$$\Gamma_{\lambda,v}(\delta) = \lim_{h\to\infty}\frac{h!\left(\frac{h}{\lambda v}\right)^{\frac{\delta}{v}} v^{2+h}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}.$$

Proof. Referring to Definition 2.1 and equation (3.3) we have

$$\Gamma\left(\frac{\delta}{v}\right) = \lim_{h\to\infty}\int_0^h t^{\frac{\delta}{v}-1}\left(1-\frac{t}{h}\right)^h dt. \tag{3.4}$$

By change of variables and letting $\omega = \frac{t}{h}$, $A_h = \int_0^h t^{\frac{\delta}{v}-1}\left(1-\frac{t}{h}\right)^h \ dt$ we obtain

$$A_h = h^{\frac{\delta}{v}} \int_0^1 (1-\omega)^h \omega^{\frac{\delta}{v}-1} \, d\omega.$$

Using Integration by parts, we obtain

$$\begin{aligned}\frac{A_h}{h^{\frac{\delta}{v}}} &= (1-\omega)^h \left.\frac{v\omega^{\frac{\delta}{v}}}{\delta}\right|_0^1 + \frac{vh}{\delta}\int_0^1 (1-\omega)^{h-1}\omega^{\frac{\delta}{v}} \, d\omega \\ &= \frac{vh}{\delta}\int_0^1 (1-\omega)^{h-1}\omega^{\frac{\delta}{v}} \, d\omega.\end{aligned}$$

Applying integration by parts repeatedly, we have

$$\begin{aligned}A_h &= h^{\frac{\delta}{v}} \frac{vh \times v(h-1) \times v(h-2) \times \cdots v\,(h-(h-1))}{\delta \times (\delta+v) \times (\delta+2) \times \cdots (\delta+(h-1)v)} \int_0^1 \omega^{\frac{\delta}{v}+h-1} \, d\omega \\ &= \frac{h^{\frac{\delta}{v}} h! v^{h+1}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we obtain

$$\Gamma\left(\frac{\delta}{v}\right) = \lim_{h\to\infty} A_h = \lim_{h\to\infty} \frac{h^{\frac{\delta}{v}} h! v^{h+1}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}.$$

Applying (3.3) gives

$$\Gamma_{\lambda,v}(\delta) = \lambda^{-\frac{\delta}{v}} v^{1-\frac{\delta}{v}} \lim_{h\to\infty} \frac{h^{\frac{\delta}{v}} h! v^{h+1}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}.$$

Simplifying yields

$$\Gamma_{\lambda,v}(\delta) = \lim_{h\to\infty} \frac{h!\left(\frac{h}{\lambda v}\right)^{\frac{\delta}{v}} v^{2+h}}{\delta(\delta+v)(\delta+2v)\cdots(\delta+hv)}.$$

□

Lemma 3.2. *For $\delta, \lambda, v \in \mathbb{R}^+$, the (λ, v)-generalized Gamma function satisfies the property*

$$\Gamma_{\lambda,v}(v) = \frac{1}{\lambda}.$$

Proof. By replacing δ with v in (3.3) yields

$$\Gamma_{\lambda,v}(v) = \lambda^{-\frac{v}{v}} v^{1-\frac{v}{v}} \Gamma\left(\frac{v}{v}\right) = \lambda^{-1} \times 1 \times \Gamma(1) = \frac{1}{\lambda}.$$

□

Lemma 3.3. *For $\delta, \lambda, v \in \mathbb{R}^+$, the (λ, v)-generalized Gamma function satisfies the property*

$$\Gamma_{\lambda,v}(\delta+v) = \frac{\delta}{\lambda v^2}\,\Gamma_{\lambda,v}(\delta).$$

Proof. By replacing δ with $(\delta+v)$ in (3.1) and integrating by parts, we obtain

$$\begin{aligned}
\Gamma_{\lambda,v}(\delta+v) &= \int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta+v}{v}-1} dt \\
&= \int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}} dt \\
&= \left.\frac{-e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}}}{\lambda}\right|_0^\infty + \frac{\delta}{\lambda v^2}\int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}-1} dt \\
&= \frac{\delta}{\lambda v^2}\int_0^\infty e^{-\lambda t}\left(\frac{t}{v}\right)^{\frac{\delta}{v}-1} dt \\
&= \frac{\delta}{\lambda v^2}\Gamma_{\lambda,v}(\delta).
\end{aligned}$$

□

Lemma 3.4. *For $\delta, \lambda, v \in \mathbb{R}^+$, the (λ, v)-generalized Gamma function satisfies the property*

$$\lim_{\lambda\to\infty}\Gamma_{\lambda,v}(\delta) = 0.$$

Proof. By using (3.3), we obtain

$$\lim_{\lambda\to\infty}\Gamma_{\lambda,v}(\delta) = \lim_{\lambda\to\infty}\lambda^{-\frac{\delta}{v}}v^{1-\frac{\delta}{v}}\Gamma\left(\frac{\delta}{v}\right) = 0.$$

□

Lemma 3.5. *If $\delta, \lambda, v \in \mathbb{R}^+$, the (λ, v)-generalized Gamma function satisfies the property*

$$\Gamma^{(n)}_{\lambda,v}(\delta) = \int_0^\infty \frac{e^{-\lambda t}}{v^n}\left(\frac{t}{v}\right)^{\frac{\delta}{v}-1}\left(\ln\left(\frac{t}{v}\right)\right)^n dt, \quad \delta, \lambda, v > 0.$$

where $n \in \mathbb{N}_0$.

Proof. By repeating the differentiation of the integrand of (3.1) with respect to δ, yields the desired result. □

Theorem 3.1. *Suppose* λ , $v \in \mathbb{R}^+$, $\tau > 1$, *and* $\frac{1}{\tau} + \frac{1}{\rho} = 1$. *Then the inequality*

$$\Gamma_{\lambda,v}\left(\frac{a}{\tau} + \frac{b}{\rho}\right) \leq \frac{1}{\tau}\,\Gamma_{\lambda,v}(a) + \frac{1}{\rho}\Gamma_{\lambda,v}(b)$$

holds for $a, b > 0$.

Proof. By replacing δ with $\left(\frac{a}{\tau} + \frac{b}{\rho}\right)$ in (3.1) and using Theorem 2.1, we have

$$\begin{aligned}
&\Gamma_{\lambda,v}\left(\frac{a}{\tau} + \frac{b}{\rho}\right) \\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{1}{v}\left(\frac{a}{\tau}+\frac{b}{\rho}\right)-\left(\frac{1}{\tau}+\frac{1}{\rho}\right)} e^{-\lambda t\left(\frac{1}{\tau}+\frac{1}{\rho}\right)}\,dt \\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{1}{\tau}(\frac{a}{v}-1)} \left(\frac{t}{v}\right)^{\frac{1}{\rho}(\frac{b}{v}-1)} e^{-\lambda t(\frac{1}{\tau})} e^{-\lambda t(\frac{1}{\rho})} \\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{1}{\tau}\left(\frac{a}{v}-1\right)} e^{-\lambda t(\frac{1}{\tau})} \left(\frac{t}{v}\right)^{\frac{1}{\rho}(\frac{b}{v}-1)} e^{-\lambda t(\frac{1}{\rho})}\,dt \\
&\leq \left(\int_0^\infty \left(\left(\frac{t}{v}\right)^{\frac{1}{\tau}(\frac{a}{v}-1)} e^{-\lambda t(\frac{1}{\tau})} dt\right)^\tau\right)^{\frac{1}{\tau}} \left(\int_0^\infty \left(\left(\frac{t}{v}\right)^{\frac{1}{\rho}(\frac{b}{v}-1)} e^{-\lambda t(\frac{1}{\rho})} dt\right)^\rho\right)^{\frac{1}{\rho}} \\
&= \left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{a}{v}-1} e^{-\lambda t} dt\right)^{\frac{1}{\tau}} \left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{b}{v}-1} e^{-\lambda t} dt\right)^{\frac{1}{\rho}} \\
&= \left(\Gamma_{\lambda,v}(a)\right)^{\frac{1}{\tau}} \left(\Gamma_{\lambda,v}(b)\right)^{\frac{1}{\rho}}.
\end{aligned}$$

Applying Theorem 2.2 gives

$$\Gamma_{\lambda,v}\left(\frac{a}{\tau} + \frac{b}{\rho}\right) \leq \frac{1}{\tau}\,\Gamma_{\lambda,v}(a) + \frac{1}{\rho}\Gamma_{\lambda,v}(b).$$

□

Remark 3.1. *Putting* $\tau = \rho = 2$ *in* (3.1) *gives*

$$2\Gamma_{\lambda,v}\left(\frac{a+b}{2}\right) \leq \Gamma_{\lambda,v}(a) + \Gamma_{\lambda,v}(b).$$

Theorem 3.2. *Suppose* $\lambda, v \in \mathbb{R}^+$,$\tau > 1$, $\frac{1}{\tau} + \frac{1}{\rho} = 1$. *Then the inequality*

$$\tau\rho\Gamma_{\lambda,v}(a+b) \leq \rho\Gamma_{\lambda,v}(a\tau) + \tau\Gamma_{\lambda,v}(b\rho)$$

holds for $a, b > 0$.

Proof. By replacing δ with $(a+b)$ in (3.1) and using Theorem 2.1 , we have

$$\begin{aligned}
&\Gamma_{\lambda,v}(a+b)\\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{a+b}{v}\right)-\left(\frac{1}{\tau}+\frac{1}{\rho}\right)} e^{-\lambda t\left(\frac{1}{\tau}+\frac{1}{\rho}\right)}\, dt\\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{a}{v}-\frac{1}{\tau}} \left(\frac{t}{v}\right)^{\frac{b}{v}-\frac{1}{\rho}} e^{-\lambda t(\frac{1}{\tau})} e^{-\lambda t(\frac{1}{\rho})}\, dt\\
&= \int_0^\infty \left(\frac{t}{v}\right)^{\frac{a}{v}-\frac{1}{\tau}} e^{-\lambda t(\frac{1}{\tau})} \left(\frac{t}{v}\right)^{\frac{b}{v}-\frac{1}{\rho}} e^{-\lambda t(\frac{1}{\rho})}.\, dt\\
&\leq \left(\int_0^\infty \left(\left(\frac{t}{v}\right)^{\frac{a}{v}-\frac{1}{\tau}} e^{-\lambda t(\frac{1}{\tau})}\right)^\tau\right)^{\frac{1}{\tau}} \left(\int_0^\infty \left(\left(\frac{t}{v}\right)^{\frac{b}{v}-\frac{1}{\rho}} e^{-\lambda t(\frac{1}{\rho})}\right)^\rho\right)^{\frac{1}{\rho}}\\
&= \left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{a\tau}{v}-1} e^{-\lambda t} dt\right)^{\frac{1}{\tau}} \left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{b\rho}{v}-1} e^{-\lambda t}\right)^{\frac{1}{\rho}}\\
&= \left(\Gamma_{\lambda,v}(a\tau)\right)^{\frac{1}{\tau}} \left(\Gamma_{\lambda,}(b\rho)\right)^{\frac{1}{\rho}}.
\end{aligned}$$

Applying Theorem 2.2 yields

$$\tau\rho\Gamma_{\lambda,v}(a+b) \leq \rho\Gamma_{\lambda,v}(a\tau) + \tau\Gamma_{\lambda,v}(b\rho).$$

□

Remark 3.2. *Putting* $\tau = \rho = 2$ *in* (3.2) *gives*

$$2\Gamma_{\lambda,v}(a+b) \leq \Gamma_{\lambda,v}(2a) + \Gamma_{\lambda,v}(2b).$$

Conflict of Interest

The authors declare no competing interests.

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