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A NEW EXTENSION OF THE STEFFENSEN INTEGRAL INEQUALITY

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ABSTRACT. In this article, we establish a new extension of the Steffensen integral inequality, which relaxes several traditional monotonicity and integrability assumptions. The proof relies on the Chebyshev integral inequality, an important yet relatively understudied tool in this context. It is presented in full detail to ensure clarity and reproducibility.

1. INTRODUCTION

Integral inequalities are fundamental tools for establishing explicit bounds for integrals that would otherwise be difficult or even impossible to compute exactly. Their influence extends far beyond pure analysis. They permeate various areas of mathematics, including functional analysis, probability theory, statistics, differential equation theory, optimization and numerical analysis. Well-known examples include the Cauchy-Schwarz, Chebyshev, Hilbert, Hölder, Jensen, Minkowski, Opial, and Steffensen integral inequalities. Each of them has played a pivotal role in shaping the theoretical foundations of modern analysis (see [1, 2, 5, 13, 14] for a comprehensive overview).

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In this article, a particular attention is given to the Steffensen integral inequality first established in [9]. This inequality is notable for its elegant formulation and broad applicability. Its classical form is presented in the theorem below.

Theorem 1.1. *(Classical version) [9] Let* $a, b \in \mathbb{R}$ *with* $b > a$ *and* $f, g : [a, b] \mapsto [0, +\infty)$ *be two functions. We suppose that* f *is non-increasing and that, for any* $t \in [a, b]$*,*

$$g(t) \leq 1.$$

We set

$$\lambda = \int_a^b g(t)dt.$$

Then we have

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^b f(t)g(t)dt.$$

Taking the definition of λ into account, it can be seen that the Steffensen integral inequality involves a composition of two integrals and can be written as follows:

$$\int_a^{a+\int_a^b g(t)dt} f(t)dt \geq \int_a^b f(t)g(t)dt.$$

As a crucial remark, the Steffensen integral integral inequality is often accompanied by the complementary bound

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt.$$

This follows immediately from the original statement. We thus have the following double inequality:

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt.$$

Over the past few decades, the Steffensen integral inequality has inspired a great deal of research, as well as numerous extensions, generalizations and refinements. Significant contributions can be found in [3, 4, 7, 8, 10–12]. See also the comprehensive monograph [6] for an in-depth treatment. In this article, we highlight an extension established by W.T. Sulaiman in [10]. It is stated formally in the theorem below.

Theorem 1.2. *(Extended Sulaiman version) [10, Theorem 2.1 with $p = 1$] Let $a, b \in \mathbb{R}$ with $b > a$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three functions with g and h integrable. We suppose that f is non-increasing and that, for any $t \in [a, b]$,*

$$g(t) \leq h(t).$$

We set

$$\lambda = \int_a^b g(t)dt.$$

We suppose that there exists a function of λ, say $k(\lambda)$, such that

- $k(\lambda) \in [a, b]$,
- $$\int_a^{k(\lambda)} h(t)dt \geq \lambda.$$

Then we have

$$\int_a^{k(\lambda)} f(t)h(t)dt \geq \int_a^b f(t)g(t)dt.$$

A proof of this extension is provided in the appendix of this article. This theorem is notable because it introduces the auxiliary function h. Compared to the classical version, the inclusion of this function makes it possible to relax the strict upper bound $g(t) \leq 1$ for any $t \in [a, b]$, and to generalize the upper limit of integration from the fixed value $a + \lambda$ to the more flexible threshold $k(\lambda)$. However, this raises several natural questions, including the following:

Q_1: Can the non-negativity assumptions on f and h be relaxed?
Q_2: Is it possible to weaken the assumption that f is non-increasing?
Q_3: Can the assumption that h dominates g be relaxed?

In this article, we provide affirmative answers to these questions by introducing a new extension of the Steffensen integral inequality. Our approach relies on the Chebyshev integral inequality as a central tool in the proof, which allows for greater flexibility in the underlying assumptions. This perspective opens the way to further refinements and potential applications in related areas of analysis. The proof is presented in full detail, ensuring transparency and reproducibility for future research.

The remainder of the article is organized as follows: Section 2 presents the new extension, while its natural counterpart is developed in Section 3. Finally, Section 4 provides concluding remarks and outlines potential directions for future research.

2. New extension of the Steffensen integral inequality

The theorem below offers a new extension of the Steffensen integral inequality, providing an alternative perspective to that in [10]. The main difference addresses the questions Q_1, Q_2 and Q_3, but comes at the cost of new assumptions regarding the monotonicity and integrability of the involved functions.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f, h : [a, b] \mapsto \mathbb{R}$ and $g : [a, b] \mapsto [0, +\infty)$ be three functions, with f, g and h integrable.*

We suppose that f and $h - g$ are monotonic, and of the same monotonicity; that is, either both are non-increasing, or both are non-decreasing.

We set

$$\lambda = \int_a^b g(t)dt.$$

We suppose that there exists a function of λ, say $k(\lambda)$, such that

- $k(\lambda) \in [a, b]$,
- *either*

$$\operatorname{sign}\left[\frac{1}{k(\lambda) - a} \int_a^{k(\lambda)} f(t)dt\right] = \operatorname{sign}\left[\int_a^{k(\lambda)} h(t)dt - \lambda\right],$$

where sign *is the sign operator defined by* $\operatorname{sign}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0, \end{cases}$ *or, under a more convenient alternative assumption independent of f,*

$$\int_a^{k(\lambda)} h(t)dt = \lambda,$$

- *for any $t \in [k(\lambda), b]$, we have*

$$\frac{1}{k(\lambda) - a} \int_a^{k(\lambda)} f(t)dt \geq f(t).$$

Then we have

$$\int_a^{k(\lambda)} f(t)h(t)dt \geq \int_a^b f(t)g(t)dt.$$

Proof. Introducing $k(\lambda)$, taking into account that $k(\lambda) \in [a,b]$, and using the Chasles integral relation, we get

$$\begin{aligned}
&\int_a^{k(\lambda)} f(t)h(t)dt - \int_a^b f(t)g(t)dt \\
&= \int_a^{k(\lambda)} f(t)h(t)dt - \int_a^{k(\lambda)} f(t)g(t)dt - \int_{k(\lambda)}^b f(t)g(t)dt \\
&= \int_a^{k(\lambda)} f(t)[h(t)-g(t)]dt - \int_{k(\lambda)}^b f(t)g(t)dt. \qquad (2.1)
\end{aligned}$$

Since f and $h-g$ are monotonic, and of the same monotonicity, the Chebyshev integral inequality gives

$$\begin{aligned}
&\int_a^{k(\lambda)} f(t)[h(t)-g(t)]dt - \int_{k(\lambda)}^b f(t)g(t)dt \\
&\geq \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} [h(t)-g(t)]dt\right] - \int_{k(\lambda)}^b f(t)g(t)dt. \qquad (2.2)
\end{aligned}$$

Using the Chasles integral relation and the definition of λ, we get

$$\begin{aligned}
&\left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} [h(t)-g(t)]dt\right] - \int_{k(\lambda)}^b f(t)g(t)dt \\
&= \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} h(t)dt - \int_a^{k(\lambda)} g(t)dt\right] - \int_{k(\lambda)}^b f(t)g(t)dt \\
&= \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} h(t)dt - \int_a^b g(t)dt + \int_{k(\lambda)}^b g(t)dt\right] \\
&\quad - \int_{k(\lambda)}^b f(t)g(t)dt \\
&= \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} h(t)dt - \lambda\right] \\
&\quad + \int_{k(\lambda)}^b \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt - f(t)\right] g(t)dt. \qquad (2.3)
\end{aligned}$$

Using the assumption $\operatorname{sign}\left\{\{1/[k(\lambda)-a]\}\int_a^{k(\lambda)} f(t)dt\right\} = \operatorname{sign}\left[\int_a^{k(\lambda)} h(t)dt-\lambda\right]$ or $\int_a^{k(\lambda)} h(t)dt = \lambda$ which is more direct, $\{1/[k(\lambda)-a]\}\int_a^{k(\lambda)} f(t)dt \geq f(t)$ for any $t\in[k(\lambda),b]$, and the non-negativity of g, we have

$$\left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} h(t)dt-\lambda\right] + \int_{k(\lambda)}^b \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt - f(t)\right] g(t)dt \geq 0. \tag{2.4}$$

Combining Equations (2.1), (2.2), (2.3) and (2.4), we derive

$$\int_a^{k(\lambda)} f(t)h(t)dt - \int_a^b f(t)g(t)dt \geq 0,$$

so that

$$\int_a^{k(\lambda)} f(t)h(t)dt \geq \int_a^b f(t)g(t)dt.$$

This completes the proof of Theorem 2.1. □

As a technical remark, based on the current proof, we can replace the assumption "We suppose that f and $h-g$ are monotonic, and of the same monotonicity" with the more sophisticated assumption involving $k(\lambda)$ "For any $t\in[a,k(\lambda)]$, we suppose that $f(t)$ and $h(t)-g(t)$ are monotonic, and of the same monotonicity". While this complicates the situation, it makes the functions $f(t)$ and $h(t)-g(t)$ of independent nature for $t\in[k(\lambda),b]$.

In view of the statement of this theorem, neither the non-positivity of f nor that of h is excluded. Similarly, f does not have to be non-increasing, nor is there any domination assumption imposed between h and g. Therefore, although the underlying mathematical foundation and the resulting inequality remain consistent, the scenario is different to that of Theorem 1.2. The distinction lies in the strategic application of the Chebyshev integral inequality at pivotal stages of the proof. This approach allows us to relax several classical assumptions, thereby broadening the applicability of the Steffensen integral inequality. This paves the way for new avenues for research in the field.

3. Natural counterpart

We complement the approach of Theorem 2.1 by developing its natural counterpart, presented in the theorem below.

Theorem 3.1. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f, h : [a, b] \mapsto \mathbb{R}$ and $g : [a, b] \mapsto [0, +\infty)$ be three functions, with f, g and h integrable.*

We suppose that f and $h - g$ are monotonic, and of the opposite monotonicity.

We set

$$\lambda = \int_a^b g(t)dt.$$

We suppose that there exists a function of λ, say $k(\lambda)$, such that

- $k(\lambda) \in [a, b]$,
- *either*

$$\operatorname{sign}\left[\frac{1}{k(\lambda) - a}\int_a^{k(\lambda)} f(t)dt\right] = -\operatorname{sign}\left[\int_a^{k(\lambda)} h(t)dt - \lambda\right],$$

  or, under a more convenient alternative assumption independent of f,

$$\int_a^{k(\lambda)} h(t)dt = \lambda,$$

- *for any $t \in [k(\lambda), b]$, we have*

$$\frac{1}{k(\lambda) - a}\int_a^{k(\lambda)} f(t)dt \leq f(t).$$

Then we have

$$\int_a^{k(\lambda)} f(t)h(t)dt \leq \int_a^b f(t)g(t)dt.$$

Proof. The proof follows the same general line of reasoning as in Theorem 2.1, while adapting the argument carefully to accommodate the new assumptions and to reverse the direction of the main inequalities. The first step is the same: introducing $k(\lambda)$, taking into account that $k(\lambda) \in [a, b]$, and using the Chasles integral relation, we get

$$\begin{aligned}
&\int_a^{k(\lambda)} f(t)h(t)dt - \int_a^b f(t)g(t)dt \\
&= \int_a^{k(\lambda)} f(t)h(t)dt - \int_a^{k(\lambda)} f(t)g(t)dt - \int_{k(\lambda)}^b f(t)g(t)dt \\
&= \int_a^{k(\lambda)} f(t)[h(t)-g(t)]dt - \int_{k(\lambda)}^b f(t)g(t)dt. \qquad (3.1)
\end{aligned}$$

Since f and $h-g$ are monotonic, and of the opposite monotonicity, the (reversed) Chebyshev integral inequality gives

$$\begin{aligned}
&\int_a^{k(\lambda)} f(t)[h(t)-g(t)]dt - \int_{k(\lambda)}^b f(t)g(t)dt \\
&\leq \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} [h(t)-g(t)]dt\right] - \int_{k(\lambda)}^b f(t)g(t)dt. \qquad (3.2)
\end{aligned}$$

Using the Chasles integral relation and the definition of λ, we get

$$\begin{aligned}
&\left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} [h(t)-g(t)]dt\right] - \int_{k(\lambda)}^b f(t)g(t)dt \\
&= \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} h(t)dt - \int_a^{k(\lambda)} g(t)dt\right] - \int_{k(\lambda)}^b f(t)g(t)dt \\
&= \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} h(t)dt - \int_a^b g(t)dt + \int_{k(\lambda)}^b g(t)dt\right] \\
&\quad - \int_{k(\lambda)}^b f(t)g(t)dt \\
&= \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} h(t)dt - \lambda\right] \\
&\quad + \int_{k(\lambda)}^b \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt - f(t)\right] g(t)dt. \qquad (3.3)
\end{aligned}$$

Using the assumption $\operatorname{sign}\left\{\{1/[k(\lambda)-a]\}\int_a^{k(\lambda)} f(t)dt\right\} = -\operatorname{sign}\left[\int_a^{k(\lambda)} h(t)dt - \lambda\right]$ or $\int_a^{k(\lambda)} h(t)dt = \lambda$ which is more direct, $\{1/[k(\lambda)-a]\}\int_a^{k(\lambda)} f(t)dt \leq f(t)$ for any

$t \in [k(\lambda), b]$, and the non-negativity of g, we have

$$\left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt\right]\left[\int_a^{k(\lambda)} h(t)dt-\lambda\right] + \int_{k(\lambda)}^b \left[\frac{1}{k(\lambda)-a}\int_a^{k(\lambda)} f(t)dt - f(t)\right] g(t)dt \leq 0. \tag{3.4}$$

Combining Equations (3.1), (3.2), (3.3) and (3.4), we derive

$$\int_a^{k(\lambda)} f(t)h(t)dt - \int_a^b f(t)g(t)dt \leq 0,$$

so that

$$\int_a^{k(\lambda)} f(t)h(t)dt \leq \int_a^b f(t)g(t)dt.$$

This completes the proof of Theorem 3.1. □

4. Concluding remarks

In conclusion, this article builds upon the traditional framework of the Steffensen integral inequality by relaxing restrictive assumptions and introducing new proof techniques. The results obtained demonstrate the potential of the Chebyshev integral inequality in advancing integral bound theory in particular. Future research could explore analogous relaxations for other classes of inequalities and examine their applications in analysis, probability, and numerical integration.

Appendix

For the sake of completeness, this appendix provides a proof of Theorem 1.2, which corresponds to [10, Theorem 2.1] in the restricted case $p = 1$. This development should be regarded as a minor clarification to inspire further research, rather than as a new contribution.

Proof of Theorem 1.2. Introducing $k(\lambda)$, taking into account that $k(\lambda) \in [a, b]$, and using the Chasles integral relation, we get

$$\int_a^{k(\lambda)} f(t)h(t)dt - \int_a^b f(t)g(t)dt$$
$$= \int_a^{k(\lambda)} f(t)h(t)dt - \int_a^{k(\lambda)} f(t)g(t)dt - \int_{k(\lambda)}^b f(t)g(t)dt$$
$$= \int_a^{k(\lambda)} f(t)[h(t) - g(t)]dt - \int_{k(\lambda)}^b f(t)g(t)dt. \tag{4.1}$$

Using the non-increasingness of f, $g(t) \leq h(t)$ for any $t \in [a, b]$, and the Chasles integral relation, we obtain

$$\int_a^{k(\lambda)} f(t)[h(t) - g(t)]dt - \int_{k(\lambda)}^b f(t)g(t)dt$$
$$\geq f(k(\lambda)) \int_a^{k(\lambda)} [h(t) - g(t)]dt - \int_{k(\lambda)}^b f(t)g(t)dt$$
$$= f(k(\lambda)) \left[\int_a^{k(\lambda)} h(t)dt - \int_a^{k(\lambda)} g(t)dt\right] - \int_{k(\lambda)}^b f(t)g(t)dt$$
$$= f(k(\lambda)) \left[\int_a^{k(\lambda)} h(t)dt - \int_a^b g(t)dt + \int_{k(\lambda)}^b g(t)dt\right] - \int_{k(\lambda)}^b f(t)g(t)dt$$
$$= f(k(\lambda)) \left[\int_a^{k(\lambda)} h(t)dt - \lambda\right] + \int_{k(\lambda)}^b \left[f(k(\lambda)) - f(t)\right] g(t)dt. \tag{4.2}$$

Using $\int_a^{k(\lambda)} h(t)dt \geq \lambda$, the non-negativity of f and g, and again the non-increasingness of f giving $f(k(\lambda)) \geq f(t)$ for any $t \in [k(\lambda), b]$, we obtain

$$f(k(\lambda)) \left[\int_a^{k(\lambda)} h(t)dt - \lambda\right] + \int_{k(\lambda)}^b \left[f(k(\lambda)) - f(t)\right] g(t)dt \geq 0. \tag{4.3}$$

Combining Equations (4.1), (4.2) and (4.3), we derive

$$\int_a^{k(\lambda)} f(t)h(t)dt - \int_a^b f(t)g(t)dt \geq 0,$$

so that

$$\int_a^{k(\lambda)} f(t)h(t)dt \geq \int_a^b f(t)g(t)dt.$$

This completes the proof of Theorem 1.2. □

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