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ON LINEAR TRANSFORMATION OF REPRODUCING KERNEL HILBERT C^*-MODULES

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ABSTRACT. In this paper, we investigate the reproducing kernel theory in the framework of Hilbert C^*- modules and the linear transformation of Hilbert C^*- modules. We give an analog of the inversion formula and the theorems of approximation in a reproducing kernel space.

1. INTRODUCTION

Reproducing kernel originated with the works of S. Bergman and S.Szego (See [3, 16]). The theory has been developed by Nachman Aronszajn and plays a very important role in mathematics. We can deduce from that many applications in many fields like Deep and machine learning, statistics, signal processing, quantum mechanics, interpolation. Let E be any set. A reproducing kernel Hilbert space (RKHS) H on E is a Hilbert space of functions on E for which point evaluations are continuous. The point evaluation functional is defined on H defined by: for all $x \in E$,

$$\begin{array}{rcl} \epsilon_x : H & \to & \mathbb{C} \\ f & \mapsto & \epsilon_x(f) = f(x). \end{array}$$

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Thanks to Riesz-Fréchet theorem, we can deduce the existence of a kernel $K : E \times E \to \mathbb{C}$ such that for all $x \in E$,

$$f(x) = (f, K(., x))_H, \text{ for all } f \in H.$$

Nachman Aronszajn [2] established the fundamental correspondence between RKHSs and positive definite kernels: each positive definite kernel defines a unique RKHS, and each RKHS admits a unique reproducing kernel. RKHSs now play a central role in analysis and its applications, particularly in machine learning, statistics, signal processing, and quantum mechanics. Many examples of reproducing kernel Hilbert spaces can be found in ([1,5,12,15]). The theory has since been generalized in several directions. Indeed, a generalization of RKHS to non-Hilbert spaces has been proposed in Canu et al. (see [4]). In 2009, Haizhang Zhang, Yuesheng Xu and Jun Zhang in (see [18]) extended the theory of RKHS on Banach spaces with many applications in machine learning. Naimark (see [17]) introduced a reprodcing kernel space using a kernel defined on a group with many applications in probability, harmonic analysis. In [10,11], our works presented an extension of the theory of RKHS to the Cartan sub-algebra of a semi-simple Lie algebra. Among all those spaces, we have Hilbert C^*-modules. In fact, they are natural generalization of Hilbert spaces. Indeed in [8], Murphy introduced reproducing kernel Hilbert modules (RKHM) and explored relationships between positive definite kernels and Hilbert C^*-modules. In 2008, Jaeseong Heo in (see [9]) also worked in reproducing kernel Hilbert C^*-modules. He discussed about reproducing kernels whose ranges are contained in a C^*-algebra and gave reproducing Hilbert C^*-modules associated with the kernels, and he showed that reproducing kernels whose ranges are contained in Hilbert C^*-modules can be expressed in terms of operators on Hilbert C^*-modules using representations on Hilbert C^*-modules. More details about C^*-algebra can be found in [6,7]. The ongoing trend is to extend results from RKHSs to RKHMs. Motivated by this, the present paper investigates linear transformations in the setting of Hilbert C^*-modules. In particular, we establish analogues in RKHMs of the inversion formula for linear transformations and of the approximation theorem introduced by S. Saitoh ([14,15]), and we study some of their structural properties. The paper is organized as follows. Section 2 introduces the necessary preliminaries and definitions, Section 3 presents our main results.

2. Preliminaries and definitions

Definition 2.1. *(See [13]) Let $\mathcal{A}$ be a C^*-algebra. A right $\mathcal{A}$-module X is called a (right) pre-Hilbert $\mathcal{A}$-module if there is an $\mathcal{A}$-valued mapping*

$$\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathcal{A}$$

which is sesquilinear and satisfies the following properties:

(1) $\langle x, x\rangle \geq 0$ *for any* $x \in X$*;*
(2) $\langle x, x\rangle = 0$ *implies* $x = 0$*;*
(3) $\langle x, y\rangle = \langle y, x\rangle^*$ *for any* $x, y \in X$*;*
(4) $\langle x, ya\rangle = \langle x, y\rangle a$ *for any* $x, y \in X$ *and* $a \in \mathcal{A}$*.*

Let X be a pre-Hilbert $\mathcal{A}$-module, $x \in X$. We set

$$\|x\|_X := \|\langle x, x\rangle\|^{\frac{1}{2}}.$$

Proposition 2.1. *(See [13]) The function $\|.\|_X$ is a norm on X and satisfies the following properties:*

i) $\|x.a\|_X \leq \|x.\|_X\|a\|$ *for any* $x \in X$*,* $a \in \mathcal{A}$*;*
ii) $\langle x, y\rangle\langle y, x\rangle \leq \|y\|_X^2\langle x, x\rangle$ *for any* $x, y \in X$*;*
iii) $\|\langle x, y\rangle\| \leq \|x\|_X\|y\|_X$ *for any* $x, y \in X$*.*

Definition 2.2. *(see [13]) A pre-Hilbert $\mathcal{A}$-module X is called a Hilbert C^*-module if it is complete with respect to the norm $\|.\|_X$.*

We now recall some important facts concerning operators on Hilbert modules. Let $\mathcal{M}, \mathcal{N}$ be Hilbert C^*-modules over a C^* algebra $\mathcal{A}$. A bounded $\mathbf{C}$-linear $\mathcal{A}$-homomorphism from $\mathcal{M}$ to $\mathcal{N}$ is called an operator from $\mathcal{M}$ to $\mathcal{N}$. Let $Hom_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ denote the set of all operators from $\mathcal{M}$ to $\mathcal{N}$. Let $T \in Hom_A(\mathcal{M}, \mathcal{N})$, we say that T is adjointable if there exixts an operator $T^* \in Hom_{\mathcal{A}}(\mathcal{N}, \mathcal{M})$ such that:

$$\langle x, Ty\rangle = \langle T^*x, y\rangle \text{ for all } x \in \mathcal{M}, y \in \mathcal{N}.$$

Lemma 2.1. *(See [13]). Let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module and let $T : \mathcal{M} \longrightarrow \mathcal{M}$ and $T^* : \mathcal{M} \longrightarrow \mathcal{M}$ be maps such that*

$$\langle x, Ty\rangle = \langle T^*x, y\rangle \textit{ for all } x \in \mathcal{M}, y \in \mathcal{N}.$$

Then, T is a bounded $\mathbb{C}$-linear $\mathcal{A}$-homomorphism (and T^* as well).

After we defined the basics concerning a Hilbert C^*-module for our work, we discuss the structure of reproducing kernel in such a space.

Let S and $\mathcal{A}$ denote a nonempty set and a C^*-algebra, respectively. We denote by X a self-dual Hilbert $\mathcal{A}$-module of $\mathcal{A}$-valued functions on S such that each valuation $\psi \mapsto \psi(s)$ is continuous and linear. Then, for each $s \in S$ and $\psi \in X$, there exists an element $\phi_s \in X$ such that $\psi(s) = (\phi_s, \psi)$. The corresponding reproducing kernel $K : S \times S \mapsto \mathcal{A}$ is given by

$$K(s,t) = (\phi_s, \phi_t).$$

Definition 2.3. *(see [9]) A kernel $K : S \times S \longrightarrow \mathcal{A}$ is positive definite if for every $n \in \mathbb{N}$ and s_1,..,$s_n \in S$ and $b_1, \ldots, b_n \in \mathcal{A}$, the sum*

$$\sum_{i,j=1}^{n} b_i^* K(s_i, s_j) b_j \text{ is positive in } \mathcal{A}.$$

Proposition 2.2. *(see [9]) Under the consideration of the notations from the above definition, the kernel K verifies:*

- *The kernel K is positive definite.*
- *For each $s \in S$, $K(s,s)$ is a positive element in $\mathcal{A}$.*
- *For all $s, t \in S$, $\|K(s,t\|^2 \leq \|K(s,s\|\|K(t,t\|$.*
- *The set $\{\phi_s : s \in S\}$: generates X as a Hilbert $\mathcal{A}$-module.*

Theorem 2.1. *(see [9]) If a kernel $K : S \times S \longrightarrow \mathcal{A}$ is positive definite, then there exists a Hilbert $\mathcal{A}$-module X of $\mathcal{A}$-valued functions on S such that K is the reproducing kernel of X.*

This theorem will lead us to introduce the basic points concerning a linear transformation of reproducing kernel Hilbert C^*- modules.

Let's consider X a Hilbert $\mathcal{A}$-module with the scalar product $(.,.)_X$ which is the $\mathcal{A}$-valued mapping defined on $S \times S$, $\mathcal{F}(S)$ the set of $\mathcal{A}$-valued functions defined on S, h a function on S with values in X defined by $h(p) = h_p$, L the map defined by:

$$\begin{aligned} L : X &\rightarrow \mathcal{F}(S) \\ f &\mapsto Lf = \tilde{f}. \end{aligned}$$

with

$$\tilde{f}(p) = (Lf)(p) = (f, h_p)_X.$$

Let's consider the kernel K defined by:

$$K(p,q) = (h_q, h_p)_X = L(h_q)(p) \text{ for all } p, q \in S.$$

Let $R(L)$ be the range of L. We introduce an inner product in $R(L)$ induced by the norm:

$$\|\tilde{f}\|_{R(L)} = \inf\{\|f\|_H; \tilde{f} = Lf\}.$$

Theorem 2.2. *(see [15], p.21) If we consider the kernel K defined above, the space $(R(L), \langle ., . \rangle_H)$ is a Hilbert space satisfying the following properties:*

1. *For all $q \in E$, $K(p,q) \in R(L)$ as a function in p.*
2. *For all $f \in R(L)$ and for all $q \in E$, we have*

$$\tilde{f} = \langle \tilde{f}, K(., p) \rangle_{R(L)}.$$

Note that, the mapping L is an isometry if and only if $\{h_p, p \in E\}$ is complete in H.

From this theorem, we see that the range of the linear transform is a reproducing kernel space that will be denoted by H_K and the theorem still holds in the case of Hilbert C^*- modules. that is, the range of a linear transform defined on a Hilbert C^*- module with values in $\mathcal{F}(S)$ is a reproducing kernel Hilbert C^*- module. The proof is parallel to the one on the previous theorem.

3. Main results

In this first part of our main results, we present the inversion formula.

Let us consider X the Hilbert $\mathcal{A}$-module with the scalar product $\langle ., . \rangle_X$, $\mathcal{F}(S)$ the set of $\mathcal{A}$-valued functions defined on S, h a function on S with values in X defined by $h(p) = h_p$, L the map defined by:

$$\begin{array}{rcl} L : X & \to & \mathcal{F}(S) \\ f & \mapsto & Lf = \tilde{f}. \end{array}$$

with

$$\tilde{f}(p) = (Lf)(p) = \langle f, h_p \rangle_X.$$

For the Hilbert $\mathcal{A}$-module, let L be a linear map from X into $\mathcal{F}(S)$, h the map from S into X defined by $h(p) = h_p$ for all $p \in S$.

Theorem 3.1. *Let $\{\phi_i\}$ be a complete orthonormal system of a the Hilbert $\mathcal{A}$-module X and suppose that L defined like above is an adjointable mapping between the $\mathcal{A}$-modules X and H_K, $\tilde{f} \in \mathrm{H}_K$ and $\Psi_i(p) = (\phi_i, h_p)_X$,. Then,*

1) *For* $p, q \in S$*,* $K(p,q) = \sum_i \Psi_i(p)(\Psi_i(q))^*$ *which is convergent in* $S \times S$ *and* $\|\tilde{f}\|_{\mathrm{H}_K} \leq \|\langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}\|_X$.
2) *Furthermore, if* $\{h_p, p \in S\}$ *is dense in* X *then,* $\|\tilde{f}\|_{\mathrm{H}_K} = \|\langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}\|_X$ *and there exists an unique* $f^\sharp$ *in* X *such that:*

$$f^\sharp = \langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K} = \textstyle\sum_i \langle \tilde{f}(.), \langle \phi_i, h_{(.)}\rangle_X\rangle_{\mathrm{H}_K} \phi_{s_i}.$$

Proof. 1) Let's consider $p, q \in S$. We have $\langle K_p, \Psi_i\rangle_{\mathrm{H}_K} = \langle \Psi_i, K_p\rangle^*_{\mathrm{H}_K} = (\Psi_i(p))^*$. Hence, from the Parseval identity:

$$\begin{aligned} K(p,q) = \langle K_q, K_p\rangle_{\mathrm{H}_K} &= \sum_{i=1}^{n} \langle K_q, \Psi_i\rangle_{\mathrm{H}_K} \langle K_p, \Psi_i\rangle^*_{\mathrm{H}_K} \\ &= \sum_{i=1}^{n} \Psi_i(p)(\Psi_i(q))^*. \end{aligned}$$

If $\Psi_i(p) = \langle \phi_i, h_p\rangle_X$, then $h_p = \sum_i \langle h_p, \phi_i\rangle_X \phi_i = \sum_i (\Psi_i(p))^* \phi_i$. Hence, by setting $h^*_p = \sum_{i=i}^{n} \Psi_i(p)\phi_{s_i}$, we have

$$h^*_{(.)} = \sum_{i=i}^{n} \Psi_i(.)\phi_{s_i} \tag{3.1}$$

For $\tilde{f} \in \mathrm{H}_K$, $\langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K} = \sum_i \langle \tilde{f}, \Psi_i(.)\rangle_{\mathrm{H}_K} \phi_i$ then $\langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K} \in X$.

For any $p \in X$, let's remark that since $\langle h_p, h_{(.)}\rangle_X = \langle \sum_i (\Psi_i(p))^* \phi_i, \sum_i (\Psi_i(.))^* \phi_{s_i}\rangle_X = \sum_i (\Psi_i(p))^* \Psi_i(.)$, then $\langle \tilde{f}(.), \langle h_p, h_{(.)}\rangle_X\rangle_{\mathrm{H}_K} = \langle \tilde{f}, \sum_i (\Psi_i(p))^* \Psi_i(.)\rangle_{\mathrm{H}_K} = \sum_i \Psi_i(p) \langle \tilde{f}, \Psi_i(.)\rangle_{\mathrm{H}_K}$, and

$$\begin{aligned} \langle \langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}, h_p\rangle_X &= \langle \sum_i \langle \tilde{f}, \Psi_i(.)\rangle_{\mathrm{H}_K} \phi_i, h_p\rangle_X \\ &= \langle \sum_i \langle \tilde{f}, \Psi_i(.)\rangle_{\mathrm{H}_K} \phi_i, \sum_i (\Psi_i(p))^* \phi_i\rangle_X \\ &= \sum_i \langle \tilde{f}, \Psi_i(.)\rangle_{\mathrm{H}_K} \Psi_i(p). \end{aligned}$$

Then, $\langle \tilde{f}(.), \langle h_p, h_{(.)}\rangle_X\rangle_{\mathrm{H}_K} = \langle \langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}, h_p\rangle_X$.

Using the assumptions and the equality above, we have: $\tilde{f}(p) = \langle \tilde{f}(.), K(.,p)\rangle_{\mathrm{H}_K} = \langle \tilde{f}(.), \langle h_p, h_{(.)}\rangle_X\rangle_{\mathrm{H}_K} = \langle \langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}, h_p\rangle_X$ which implies that:

$$\tilde{f} = L\langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}, \|\tilde{f}\|_{\mathrm{H}_K} \leq \|\langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}\|_X. \tag{3.2}$$

2) For $f_0 \in X$, and using (3.1),

$$\langle f_0, \langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}\rangle_X = \langle f_0, \sum_i \langle \tilde{f}, \Psi_i(.)\rangle_{\mathrm{H}_K}\phi_i\rangle_X$$
$$= \sum_i \langle \tilde{f}, \Psi_i(.)\rangle^*_{\mathrm{H}_K}\langle f_0, \phi_i\rangle_X.$$

We also have:

$$\langle f_0, h_{(.)}\rangle_X = \langle \sum_i \langle f_0, \phi_i\rangle_X \phi_i, h_{(.)}\rangle_X$$
$$= \langle \sum_i \langle f_0, \phi_i\rangle_X \phi_i, \sum_{i=1} (\Psi_i(p))^* \phi_i\rangle_X$$
$$= \sum_i \langle f_0, \phi_i\rangle_X \Psi_i(.).$$

Then, $\langle \langle f_0, h_{(.)}\rangle_X, \tilde{f}\rangle_{\mathrm{H}_K} = \sum_i \langle f_0, \phi_i\rangle_X \langle \Psi_i(.), \tilde{f}\rangle_{\mathrm{H}_K} = \sum_i \langle f_0, \phi_i\rangle_X \langle \tilde{f}, \Psi_i(.)\rangle^*_{\mathrm{H}_K}$.
We obtain finally:

$$\langle f_0, \langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}\rangle_X = \langle \langle f_0, h_{(.)}\rangle_X, \tilde{f}\rangle_{\mathrm{H}_K}. \tag{3.3}$$

If $f_0 \in Ker(L)$ then we obtain $\langle f_0, h_{(.)}\rangle_X = L(f_0)(.) = 0$. We get in (3.3) $\langle f_0, \langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}\rangle_X = 0$ and $\langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K} \in [Ker(L)]^\perp$.

If $\{h_p, p \in X\}$ is dense in X, then $[Ker(L)]^\perp = X$, which implies that L is an isometry between $[Ker(L)]^\perp$ and $R(L)$, then there exists an unique $f^\sharp \in [Ker(L)]^\perp$ such that, from (3.2):

$$f^\sharp = L^{-1}\tilde{f} = \langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K} \text{ and } \|\tilde{f}\|_{\mathrm{H}_K} = \|f^\sharp\|_X = \|\langle \tilde{f}, h^*_{(.)}\rangle_{\mathrm{H}_K}\|_X.$$

For the adjoint L^* of the isometry L between $[Ker(L)]^\perp$ and H_K, we have $L^* = L^{-1}$ hence, we obtain:

$$L^{-1}\tilde{f} = f^\sharp = \sum_i \langle f^\sharp, \phi_i\rangle_X \phi_i$$
$$= \sum_i \langle \tilde{f}, L\phi_i\rangle_{\mathrm{H}_K}\phi_i$$
$$= \sum_i \langle \tilde{f}, \langle \phi_i, h_{(.)}\rangle_X\rangle_{\mathrm{H}_K}\phi_i.$$

□

The following part of our main results presents the approximation theorems.

Consider the linear operator $T : \mathrm{H}_K \longrightarrow X$. If we assume that T is adjointable, we consider its adjoint operator T^* and the following kernel:

$$k(p,q) = (T^*TK(.,q), T^*TK(.,p))_{\mathrm{H}_K} \text{ on } S \times S.$$

Then, we have

Theorem 3.2. *For $\varphi \in X$, there exists $\tilde{\zeta} \in \mathrm{H}_K$ such that:*

$$\inf_{\zeta \in \mathrm{H}_K} \|T(\zeta) - \varphi\|_X = \|T(\tilde{\zeta}) - \varphi\|_X \tag{3.4}$$

if and only if, for the reproducing kernel space h_k,

$$T^*\varphi \in h_k$$

Furthermore, if the existence of the best approximation $\tilde{\zeta}$ is ensured, then there exists an unique extremal function $\check{\zeta}$ with the minimum norm in H_K, and the function $\check{\zeta}$ is written in the form

$$\check{\zeta}(p) = (T^*\varphi, T^*TK(.,p))_{h_k}, p \in S. \tag{3.5}$$

Proof. For any $\zeta \in \mathrm{H}_K$ and using the reproducing kernel $K(p,q)$ in H_K, $T^*T\zeta$ is written in the form:

$$[T^*T\zeta](p) = (T^*T\zeta, K(.,p))_{\mathrm{H}_K} = (\zeta, T^*TK(.,p))_{\mathrm{H}_K}.$$

The range of T^*T coincides with the reproducing kernel h_k. Let P be the orthogonal projection of H_K onto $(\mathrm{H}_K \ominus Ker(T^*T)$. Then, we have:

$$\|T^*T\zeta\|_{h_k} = \|P\zeta\|_{\mathrm{H}_K}.$$

We assume that the best approximations $\tilde{\zeta}$ satisfying (3.4) exist. Then, we have:

$$\|T(\tilde{\zeta}) - \varphi\|_X \leq \|\varphi_0 - \varphi\|_X$$

for all φ_0 in $\overline{R(T)}$. Hence, $\varphi = T\tilde{\zeta} + \varphi'$ for some $\varphi' \in X \ominus \overline{R(T)}$. Since $Ker(T^*) = X \ominus \overline{R(T)}$, $T^*T\tilde{\zeta} = T^*\varphi$, and we have $T^*\varphi \in h_k$.

Conversely, let $\zeta_1 \in \mathrm{H}_K$ with $T^*T\zeta_1 = T^*\varphi$. We choose φ_1 in $\overline{R(T)}$ such that

$$\|\varphi_1 - \varphi\|_X \leq \|\varphi_0 - \varphi\|_X$$

for all φ_0 in $\overline{R(T)}$. Then, $T^*T\zeta_1 = T^*\varphi_1$ and $T\zeta_1 = \varphi_1$ because T^* is one-to-one on $\overline{R(T)}$. Hence, we have, from the previous inequality:

$$\|T(\zeta_1) - \varphi\|_X = \inf_{\zeta \in \mathrm{H}_K} \|T(\zeta) - \varphi\|_X.$$

By setting $\check{\zeta} = P\zeta_1$, we see that $\check{\zeta}$ is a unique element in H_K such that

$$\|T(\check{\zeta}) - \varphi\|_X = \inf_{\zeta \in \mathrm{H}_K} \|T(\zeta) - \varphi\|_X$$

and $\check{\zeta}$ has the minimum norm in H_K because the family of functions ζ_1 satisfying (3.4) is exactly $\check{\zeta} + Ker(T^*T)$.

Finally, we shall derive the expression (3.5). Since T^*T is an isometry of $\mathrm{H}_K \ominus Ker(T^*T)$ onto h_k, its adjoint S is the inversion of T^*T. Hence, we have

$$\check{\zeta}(p) = [ST^*l](p) = (ST^*\varphi, K(.,p))_{\mathrm{H}_K} = (T^*\varphi, T^*TK(.,p))_{h_k}.$$

□

For the next theorems, we assume that X is a Left-Hilbert $\mathcal{A}$-module.

Theorem 3.3. *Let h a Hilbert $\mathcal{A}$-module X- valued function from an abstract set S into a Hilbert $\mathcal{A}$-module X. If for some $\{s_j, j \in I\}$ of S, $\{h_{s_j}, j \in I\}$ is a complete orthonormal system in X, then for the RKHM H_K admitting the reproducing kernel*

$$K(p,q) = \langle h_q, h_p \rangle_X \text{ with } p, q \in S,$$

we have the sampling property

$$\tilde{f}(q) = \textstyle\sum_j K(q, s_j)\tilde{f}(s_j) \text{ on } X, \text{ for all } \tilde{f} \in \mathrm{H}_K.$$

Proof.

1) We know that $\tilde{f}(q) = \langle f(.), h(q) \rangle_X$. Since $h(q) = \sum_j \langle h(q), h_{s_j} \rangle_X h_{s_j}$,

$$\begin{aligned}
\tilde{f}(q) = \langle f(.), h(q) \rangle_X &= \langle f(.), \sum_j \langle h(q), h_{s_j} \rangle_X h_{s_j} \rangle_X \\
&= \sum_j \langle f(.), \langle h(q), h_{s_j} \rangle_X h_{s_j} \rangle_X \\
&= \sum_j \langle f(.), K(s_j, q) h_{s_j} \rangle_X \\
&= \sum_j (K(s_j, q))^* \langle f(.), h_{s_j} \rangle_X \\
&= \sum_j K(q, s_j) \langle f(.), h_{s_j} \rangle_X \\
&= \sum_j K(q, s_j) \tilde{f}(s_j).
\end{aligned}$$

□

Theorem 3.4. *For the Hilbert C^*-modules X, let J be a linear map on X with values in $\mathcal{A}$. Let h a Hilbert $\mathcal{A}$-module X- valued function from an abstract set S into a Hilbert $\mathcal{A}$-module X. If for some $\{s_j, j \in I\}$ of S, $\{h_{s_j}, j \in I\}$ is a complete orthonormal system in X, we have the following results.*

1) *Let x_0 an element of X with the minimum norm such that*

$$\langle x, h_{s_j}\rangle_X = b_j, j \in I^{'} \subset I \text{ and for } (X)'' = \{x \in X, \|x\|_X \leq B\}.$$

Then,

$$\|Jx - Jx_0\| \leq B \textstyle\sum_{I\setminus I'} \|Jh_{s_j}\|$$

2) *Let $\tilde{f} \in \mathrm{H}_K$ and consider some fixed $\{b_j \in \mathcal{A}, j \in I^{'}\}$ such that $I^{'} \subset I$ where $E_{I'}(q) = \sum_{I' \subset I} K(q, s_j)\tilde{f}(s_j)$ is called the truncation error. we have:*

$$\|E_{I'}(q)\|_X \leq B \sum_{I' \subset I} \|K(q, s_j)\|.$$

for any $\tilde{f} \in (\mathrm{H}_K)''$ where $(\mathrm{H}_K)'' = \{\tilde{f} \in \mathrm{H}_K; \|\tilde{f}\|_{\mathrm{H}_K} \leq B\}$.

Proof. 1) Let x be an element of X, $x = \sum_{j\in I} h_{s_j}\langle x, h_{s_j}\rangle_X = \sum_{j\in I'} h_{s_j} b_j + \sum_{j\in I\setminus I'} h_{s_j}\langle x, h_{s_j}\rangle_X$, we have $x_0 = \sum_{j\in I'} h_{s_j} b_j$ hence, $x - x_0 = \sum_{j\in I\setminus I'} h_{s_j}\langle x, h_{s_j}\rangle_X$ and, $Jx - Jx_0 = \sum_{j\in I\setminus I'} J(h_{s_j})\langle x, h_{s_j}\rangle_X$.

$$\begin{aligned}\|Jx - Jx_0\| = \|\sum_{j\in I\setminus I'} J(h_{s_j})\langle x, h_{s_j}\rangle_X\| &\leq \sum_{j\in I\setminus I'} \|J(h_{s_j}).\langle x, h_{s_j}\rangle_X\| \\ &\leq \sum_{j\in I\setminus I'} \|J(h_{s_j})\|\|\langle x, h_{s_j}\rangle\|_X. \\ &\leq \sum_{j\in I\setminus I'} \|J(h_{s_j})\|\|x\|_X\|h_{s_j}\|_X \\ &\leq B \sum_{j\in I\setminus I'} \|J(h_{s_j})\|\|h_{s_j}\|_X \\ &\leq B \sum_{j\in I\setminus I'} \|J(h_{s_j})\|.\end{aligned}$$

and we have the desired result.

2) Let's put:

$$X = \mathrm{H}_K, (X)'' = (\mathrm{H}_K)'', J(x) = \tilde{f}(q),$$

and $b_j = \tilde{f}(s_j) = \langle \tilde{f}(.), K(., s_j)\rangle_{\mathrm{H}_K}$ Then, $J(x_0) = \tilde{f}_0(q) = \sum_{I'} K(q, s_j)\tilde{f}(s_j)$ and $J(x) = \tilde{f}(q) = \sum_{j\in I} K(q, s_j)\tilde{f}(s_j)$.

We thus obtain 2) from 1).

□

References

[1] D. Alpay: *Reproducing kernel spaces and applications.* Springer Basel AG, 2000.

[2] N. Aronszajn: *Theory of reproducing kernels.* Transactions of the American Mathematical Society, **68**(3) (1950), 337–404.

[3] S. Bergmann *Über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach orthogonalen Funktionen.* Mathematische Annalen, **86** (1922), 238–271.

[4] S. Canu, X. Mary, A. Rakotomamonjy: Functional learning through kernel. In *Advances in learning theory: Methods, models and applications* (NATO Science Series III: Computer and Systems Sciences, **190** (2003), 89–110.

[5] P.A.-F. Cyril: *Noyaux reproduisants d'Aronszajn et des mécaniques classique et quantique.* HAL Archives Ouvertes, 2018.

[6] J. Dixmier: *C^*-Algebras*, Uiversity of Paris VI, North-Holland publishing company Amsterdam-New York-Oxford, 1977.

[7] B.G. Folland: *A course in Abstract Harmonic Analysis*, University of Washington, Department of Mathematics. CRC Press, Boca Raton Ann Arbor LOndon Tokyo, 1977.

[8] G.J. Murphy: *Positive definite kernels and Hilbert C^*-modules.* Proceedings of the Edinburgh Mathematical Society **40** (1997), 367–374.

[9] J. Heo: *Reproducing kernel Hilbert C^*-modules and kernels associated with cocycles.* Journal of Mathematical Physics, **49**(10) (2008), 103507.

[10] A.Y. Kraidi, K. Kinvi: *Reproducing kernel Cartan subalgebra.* Moroccan Journal of Pure and Applied Analysis, **7**(1) (2021), 1–8.

[11] A.Y. Kraidi, K. Kinvi: *On a chain of reproducing kernel Cartan subalgebras.* Annales Mathématiques Africaines, **8** (2020) 7–14.

[12] Manton, J. H., & Amblard, P.-O. (2015). *A primer on reproducing kernel Hilbert spaces.* NOW Publishers, pp. 9–19, 34–44.

[13] Manuilov V. M. & Troitsky E.V. (2005). *Hilbert C^*-modules.* Translations of Mathematical Monographs, pp. 1–15.

[14] S. Saitoh: *Integral transforms, reproducing kernel and their applications.* Pitman Research Notes in Mathematics, 1997.

[15] S. Saitoh, Y. Sawano: *Theory of reproducing kernels and applications.* Springer, 2016.

[16] S. Szegö: *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören.* Mathematische Zeitschrift, **9** (1921), 218–270.

[17] I. Paulsen, M. Raghupathi *An introduction to the theory of reproducing kernel Hilbert spaces* (Vol. 152). Cambridge University Press, 2016.

[18] H. Zhang, Y. Xu, J. Zhang: *Reproducing kernel Banach spaces for machine learning.* Journal of Machine Learning Research, **10** (2009), 2741–2775.

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