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ON A SPECIAL HILBERT-TYPE INTEGRAL INEQUALITY DEMONSTRATED VIA A HYPERBOLIC TANGENT CHANGE OF VARIABLES

Christophe Chesneau

ABSTRACT. This article introduces a new Hilbert-type integral inequality that is defined on the unit square and involves a singular integrand. A sharp upper bound is established using a change of variables based on the hyperbolic tangent function. As with classical Hilbert-type integral inequalities, the constant factor π arises naturally. Furthermore, other inequalities are derived from the main result, including a new cosine-Hilbert-type integral inequality.

1. INTRODUCTION

The Hilbert integral inequality is a classical and fundamental result in analysis. Its usual formulation can be stated as follows. Let $f, g : (0,+\infty) \to (0,+\infty)$ be two functions. Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y}dxdy \leq \pi \left(\int_0^{+\infty} f^2(x)dx\right)^{1/2} \left(\int_0^{+\infty} g^2(y)dy\right)^{1/2}, \tag{1.1}$$

provided that the two integrals on the right-hand side converge. The constant factor π is optimal and cannot be replaced by a smaller value. The Hilbert integral inequality has deep connections to operator theory, harmonic analysis, and the

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theory of special functions. For further details, we refer the reader to the books [4, 9, 11, 12], the comprehensive survey [1], and the key papers [2, 5–8, 10].

A less well-known but related integral inequality is as follows. Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{1+xy} dxdy \leq \pi \left(\int_0^{+\infty} f^2(x)dx \right)^{1/2} \left(\int_0^{+\infty} g^2(y)dy \right)^{1/2}, \tag{1.2}$$

provided that the two integrals on the right-hand side converge. See [12, Theorem 2.2 with $p = 2$]. It is also proved that the constant factor π is optimal. This result is interesting because it provides a counterpart to the classical Hilbert integral inequality, where the integrand depends on $1/(1 + xy)$. This demonstrates the flexibility of Hilbert-type integral inequalities in handling different mathematical scenarios. Furthermore, it has applications in operator theory, harmonic analysis and related fields, where such integral operators arise naturally.

In this article, we continue the study of Hilbert-type integral inequalities, introducing a new variation in the definition of the central double integral. More precisely, we rigorously prove the following. Let $f, g : (0, 1) \to (0, +\infty)$ be two functions. Then we have

$$\int_0^1 \int_0^1 \frac{f(x)g(y)}{1-xy} dxdy \leq \pi \left(\int_0^1 f^2(x)dx \right)^{1/2} \left(\int_0^1 g^2(y)\, dy \right)^{1/2},$$

provided that the two integrals on the right-hand side converge. Here, the double integral is defined on the unit square, and the integrand depends on $1/(1 - xy)$, which is singular at $(x, y) = (1, 1)$. This modification distinguishes the result from the classical Hilbert-type integral inequalities. The proof also exhibits a degree of originality, relying on a change of variables using the hyperbolic tangent function, the Cauchy-Schwarz integral inequality, and a specific integral formula involving the hyperbolic cosine function. Based on this new result, we also derive an associated integral inequality depending on a single function. Further applications include deriving a series inequality and a new Hilbert-type integral inequality based on the cosine function.

The remainder of the article is organized as follows. The main theorem is developed in the next section. Applications are given in Section 3. A conclusion is presented in Section 4.

2. Main theorem

The formal statement of our main result is given in the following theorem, alongside its detailed proof.

Theorem 2.1. *Let* $f, g : (0,1) \to (0,+\infty)$ *be two functions. Then we have*

$$\int_0^1 \int_0^1 \frac{f(x)g(y)}{1-xy} dxdy \leq \pi \left(\int_0^1 f^2(x)dx \right)^{1/2} \left(\int_0^1 g^2(y)dy \right)^{1/2},$$

provided that the two integrals on the right-hand side converge.

Proof. First, let us set

$$I := \int_0^1 \int_0^1 \frac{f(x)g(y)}{1-xy} dxdy.$$

We note that, for any $x, y \in (0,1)$,

$$\frac{1}{1-xy} \geq 0.$$

Furthermore, we recall that, for any $z \in \mathbb{R}$,

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \operatorname{sech} z = \frac{1}{\cosh z}.$$

Considering the hyperbolic tangent change of variables

$$x = \tanh u, \quad y = \tanh v,$$

with $x = 0$ when $u = 0$, $x \to 1$ when $u \to +\infty$, $y = 0$ when $v = 0$, $y \to 1$ when $v \to +\infty$, and the derivatives

$$dx = \operatorname{sech}^2 u du, \quad dy = \operatorname{sech}^2 v dv,$$

we can write

$$I = \int_0^{+\infty} \int_0^{+\infty} \frac{f(\tanh u)g(\tanh v)}{1 - \tanh u \tanh v} \operatorname{sech}^2 u \operatorname{sech}^2 v dudv.$$

Let us now simplify the corresponding integrand. Using the hyperbolic identity

$$1 - \tanh u \tanh v = \cosh(u-v) \operatorname{sech} u \operatorname{sech} v,$$

we have

$$\frac{1}{1 - \tanh u \tanh v} \operatorname{sech}^2 u \operatorname{sech}^2 v = \frac{\operatorname{sech} u \operatorname{sech} v}{\cosh(u-v)}.$$

Hence, the integral becomes

$$\begin{aligned} I &= \int_0^{+\infty}\int_0^{+\infty} \frac{f(\tanh u)g(\tanh v)\operatorname{sech} u \operatorname{sech} v}{\cosh(u-v)} dudv \\ &= \int_0^{+\infty}\int_0^{+\infty} \frac{f_\diamond(u)g_\diamond(v)}{\cosh(u-v)} dudv, \end{aligned}$$

where

$$f_\diamond(u) := f(\tanh u)\operatorname{sech} u, \quad g_\diamond(v) := g(\tanh v)\operatorname{sech} v.$$

Noticing that, for any $u, v \in \mathbb{R}$, $\cosh(u-v) \geq 0$, the Cauchy-Schwarz integral inequality gives

$$\begin{aligned} I &= \int_0^{+\infty}\int_0^{+\infty} \frac{f_\diamond(u)}{\cosh^{1/2}(u-v)} \times \frac{g_\diamond(v)}{\cosh^{1/2}(u-v)} dudv \\ &\leq J^{1/2}K^{1/2}, \end{aligned} \tag{2.1}$$

where

$$J := \int_0^{+\infty}\int_0^{+\infty} \frac{f_\diamond^2(u)}{\cosh(u-v)} dudv, \quad K := \int_0^{+\infty}\int_0^{+\infty} \frac{g_\diamond^2(v)}{\cosh(u-v)} dudv.$$

Applying the Fubini-Tonelli integral theorem, we get

$$J = \int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{\cosh(u-v)} dv \right) f_\diamond^2(u) du.$$

Performing the change of variables $w = v - u$, $dw = dv$, and using the known integral formula

$$\int_{-\infty}^{+\infty} \frac{1}{\cosh(x)} dz = \pi$$

(see [3, 3.511 1]), we have

$$\int_0^{+\infty} \frac{1}{\cosh(u-v)} dv = \int_{-u}^{+\infty} \frac{1}{\cosh w} dw \leq \int_{-\infty}^{+\infty} \frac{1}{\cosh w} dw = \pi.$$

This and the change of variables $x = \tanh u$ give

$$J \leq \pi \int_0^{+\infty} f_\diamond^2(u) du = \pi \int_0^{+\infty} f^2(\tanh u)\operatorname{sech}^2 u du = \pi \int_0^1 f^2(x) dx. \tag{2.2}$$

Proceeding in a similar way (with the change of variables $y = \tanh v$), we obtain

$$K = \int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{\cosh(u-v)} du \right) g_\diamond^2(v) dv$$
$$\leq \pi \int_0^{+\infty} g_\diamond^2(v) dv = \pi \int_0^1 g^2(y) dy. \tag{2.3}$$

It follows from Equations (2.1), (2.2) and (2.3) that

$$I \leq \left(\pi \int_0^1 f^2(x) dx \right)^{1/2} \left(\pi \int_0^1 g^2(y) dy \right)^{1/2}$$
$$= \pi \left(\int_0^1 f^2(x) dx \right)^{1/2} \left(\int_0^1 g^2(y) dy \right)^{1/2},$$

so that

$$\int_0^1 \int_0^1 \frac{f(x)g(y)}{1-xy} dx dy \leq \pi \left(\int_0^1 f^2(x) dx \right)^{1/2} \left(\int_0^1 g^2(y) dy \right)^{1/2}.$$

This completes the proof. □

To the best of our knowledge, this is the first time that this particular integral inequality has been established. In a sense, it complements Equations (1.1) and (1.2), yielding the same upper bound for a fundamentally different double integral. The proof is also innovative, notably in its use of the hyperbolic tangent function. The optimality of the constant factor π is not addressed here, leaving an avenue for future work, as discussed in Section 4.

3. Applications

3.1. Secondary theorem. The formal statement of our secondary result is given in the following theorem, with the detailed proof provided thereafter.

Theorem 3.1. *Let $f : (0,1) \to (0,+\infty)$ be a function. Then we have*

$$\int_0^1 \left(\int_0^1 \frac{f(x)}{1-xy} dx \right)^2 dy \leq \pi^2 \int_0^1 f^2(x) dx,$$

provided that the integral on the right-hand side converges.

Proof. Let us set

$$L := \int_0^1 \left(\int_0^1 \frac{f(x)}{1-xy} dx \right)^2 dy.$$

By the Fubini-Tonelli integral theorem, we can write

$$L = \int_0^1 \left(\int_0^1 \frac{f(x)}{1-xy} dx \right) \left(\int_0^1 \frac{f(x)}{1-xy} dx \right) dy$$

$$= \int_0^1 \int_0^1 \frac{f(x) g_\dagger(y)}{1-xy} dx dy, \tag{3.1}$$

where

$$g_\dagger(y) = \int_0^1 \frac{f(x)}{1-xy} dx.$$

Applying Theorem 2.1 to the functions f and $g_\dagger$, we obtain

$$\int_0^1 \int_0^1 \frac{f(x) g_\dagger(y)}{1-xy} dx dy \leq \pi \left(\int_0^1 f^2(x) dx \right)^{1/2} \left(\int_0^1 g_\dagger^2(y) dy \right)^{1/2}. \tag{3.2}$$

We have

$$\int_0^1 g_\dagger^2(y) dy = \int_0^1 \left(\int_0^1 \frac{f(x)}{1-xy} dx \right)^2 dy = L. \tag{3.3}$$

It follows from Equations (3.1), (3.2) and (3.3) that

$$L \leq \pi \left(\int_0^1 f^2(x) dx \right)^{1/2} L^{1/2}.$$

This simplifies to

$$L \leq \pi^2 \int_0^1 f^2(x) dx,$$

so that

$$\int_0^1 \left(\int_0^1 \frac{f(x)}{1-xy} dx \right)^2 dy \leq \pi^2 \int_0^1 f^2(x) dx.$$

This concludes the proof. □

This result has important applications in operator theory. More precisely, it establishes the boundedness of the integral operator

$$T(f)(y) := \int_0^1 \frac{f(x)}{1-xy} dx$$

as a mapping from $L^2(0,1)$ to $L^2(0,1)$. This property can be useful for studying spectral properties, estimating the norms of related operators, and analyzing integral equations in applied and theoretical contexts.

3.2. **A proposition.** We present another elegant consequence of Theorem 2.1 in the following proposition. It involves a series and an integral.

Proposition 3.1. *Let* $f : (0,1) \to (0,+\infty)$ *be a function. Then we have*

$$\sum_{k=0}^{+\infty} \left(\int_0^1 x^k f(x) dx \right)^2 \leq \pi \int_0^1 f^2(x) dx,$$

provided that the integral on the right-hand side converges.

Proof. Applying Theorem 2.1 with $g = f$, we obtain

$$\begin{aligned} \int_0^1 \int_0^1 \frac{f(x)f(y)}{1-xy} dxdy &\leq \pi \left(\int_0^1 f^2(x) dx \right)^{1/2} \left(\int_0^1 f^2(y) dy \right)^{1/2} \\ &= \pi \int_0^1 f^2(x) dx. \end{aligned} \tag{3.4}$$

Let us now work on the double integral. Since $x, y \in (0,1)$, the geometric series expansion and the uniform convergence give

$$\begin{aligned} \int_0^1 \int_0^1 \frac{f(x)f(y)}{1-xy} dxdy &= \int_0^1 \int_0^1 f(x)f(y) \sum_{k=0}^{+\infty} (xy)^k dxdy \\ &= \sum_{k=0}^{+\infty} \int_0^1 \int_0^1 x^k f(x) y^k f(y) dxdy \\ &= \sum_{k=0}^{+\infty} \left(\int_0^1 x^k f(x) dx \right) \left(\int_0^1 y^k f(y) dy \right) \\ &= \sum_{k=0}^{+\infty} \left(\int_0^1 x^k f(x) dx \right)^2 . \end{aligned} \tag{3.5}$$

It follows from Equations (3.4) and (3.5) that

$$\sum_{k=0}^{+\infty} \left(\int_0^1 x^k f(x) dx \right)^2 \leq \pi \int_0^1 f^2(x) dx.$$

This ends the proof. □

This inequality is interesting because it gives a sharp bound on the sum of all moments of a function. This has applications in analysis, operator theory, and understanding the structure of functions through their moments.

3.3. **Another proposition.** To conclude this section, the following proposition presents a new Hilbert-type integral inequality involving the cosine function. This is also a non-trivial consequence of Theorem 2.1.

Proposition 3.2. *Let* $f, g : (0, \pi/4) \to (0, +\infty)$ *be two functions. Then we have*

$$\int_0^{\pi/4} \int_0^{\pi/4} \frac{f(x)g(y)}{\cos(x+y)} dxdy \leq \pi \left(\int_0^{\pi/4} f^2(x)dx \right)^{1/2} \left(\int_0^{\pi/4} g^2(y)dy \right)^{1/2},$$

provided that the two integrals on the right-hand side converge.

Proof. First, let us set

$$M := \int_0^{\pi/4} \int_0^{\pi/4} \frac{f(x)g(y)}{\cos(x+y)} dxdy.$$

We note that, for any $x, y \in (0, \pi/4)$,

$$\frac{1}{\cos(x+y)} \geq 0.$$

Considering the arctangent change of variables

$$x = \arctan u, \quad y = \arctan v,$$

with $x = 0$ when $u = 0$, $x = \pi/4$ when $u = 1$, $y = 0$ when $v = 0$, $y = \pi/4$ when $v = 1$, and the derivatives

$$dx = \frac{1}{1+u^2} du, \quad dy = \frac{1}{1+v^2} dv,$$

we can write

$$M = \int_0^1 \int_0^1 \frac{f(\arctan u)g(\arctan v)}{\cos(\arctan u + \arctan v)} \frac{1}{(1+u^2)(1+v^2)} dudv.$$

Let us now simplify the corresponding integrand. Using the trigonometric identity

$$\cos(\arctan u + \arctan v) = \frac{1 - uv}{\sqrt{(1+u^2)(1+v^2)}},$$

we have

$$\frac{1}{\cos(\arctan u + \arctan v)} \frac{1}{(1+u^2)(1+v^2)} = \frac{1}{(1-uv)\sqrt{(1+u^2)(1+v^2)}}.$$

Hence, the integral becomes

$$M = \int_0^1 \int_0^1 \frac{f(\arctan u)g(\arctan v)}{(1-uv)\sqrt{(1+u^2)(1+v^2)}} dudv$$

$$= \int_0^1 \int_0^1 \frac{f_\triangle(u)g_\triangle(v)}{1-uv} dudv, \tag{3.6}$$

where

$$f_\triangle(u) := \frac{f(\arctan u)}{\sqrt{1+u^2}}, \quad g_\triangle(v) := \frac{g(\arctan v)}{\sqrt{1+v^2}}.$$

It follows from Theorem 2.1 that

$$\int_0^1 \int_0^1 \frac{f_\triangle(u)g_\triangle(v)}{1-uv} dudv \leq \pi \left(\int_0^1 f_\triangle^2(u)du \right)^{1/2} \left(\int_0^1 g_\triangle^2(v)dv \right)^{1/2}. \tag{3.7}$$

The change of variables $x = \arctan u$ gives

$$\int_0^1 f_\triangle^2(u)du = \int_0^1 \frac{f^2(\arctan u)}{1+u^2} du = \int_0^{\pi/4} f^2(x)dx. \tag{3.8}$$

Proceeding in a similar way (with the change of variables $y = \arctan v$), we obtain

$$\int_0^1 g_\triangle^2(v)dv = \int_0^1 \frac{g^2(\arctan v)}{1+v^2} dv = \int_0^{\pi/4} g^2(y)dy. \tag{3.9}$$

It follows from Equations (3.6), (3.7), (3.8) and (3.9) that

$$M \leq \pi \left(\int_0^{\pi/4} f^2(x)dx \right)^{1/2} \left(\int_0^{\pi/4} g^2(y)dy \right)^{1/2},$$

so that

$$\int_0^{\pi/4} \int_0^{\pi/4} \frac{f(x)g(y)}{\cos(x+y)} dxdy \leq \pi \left(\int_0^{\pi/4} f^2(x)dx \right)^{1/2} \left(\int_0^{\pi/4} g^2(y)dy \right)^{1/2}.$$

This completes the proof. □

To the best of our knowledge, this is the first time that a Hilbert-type integral inequality with a cosine ratio feature has been established. This opens the door to further generalization and applications in harmonic analysis.

4. Conclusion

In this article, we have contributed to the theory of Hilbert-type integral inequalities by introducing a new variation. The main features of this result are the double integral defined over the unit square and the singular integrand depending on $1/(1-xy)$. The obtained upper bound coincides with that of the classical Hilbert integral inequality. Our proof techniques, which are based on the hyperbolic tangent function, are also original and may inspire further developments in this area. The applicability of this result has been demonstrated in various ways.

Possible directions for future work include establishing the optimality of the obtained constant factor, extending the inequality to the L^p setting, and generalizing it to higher-dimensional domains.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
Email address: christophe.chesneau@gmail.com