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ON SEVERAL NEW CONVEX INTEGRAL THEOREMS

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ABSTRACT. This article presents several fundamental theorems relating to convex integral inequalities. Each theorem has the potential to serve as a valuable intermediary tool in a wide range of analytical applications. The two main categories of results considered are convex simple integral theorems, involving single integrals, and convex double integral theorems, based on double integrals. Full, detailed proofs are provided that are designed to be easily reproducible, requiring only minimal preliminary knowledge. Comprehensive examples illustrate the theory.

1. INTRODUCTION

Convexity plays a key role in mathematical analysis and optimization, forming the basis of many significant inequality results. For the sake of completeness, the formal definition of a convex function is provided below.

Definition 1.1. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $b > a$. A function $\varphi : [a, b] \to \mathbb{R}$ is said to be convex on $[a, b]$ if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, the following inequality holds:*

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y). \tag{1.1}$$

If the inequality is reversed, f is called concave on $[a, b]$.

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Geometrically, the graph of a convex function lies below the straight line segment joining any two points on its curve. Many classical inequalities, including the Jensen and Hermite-Hadamard integral inequalities, fundamentally rely on the convexity property of functions. Consequently, studying convex integral inequalities provides valuable insight into the structure and behavior of convex functions and their associated integral means. For comprehensive discussions and important references related to convex functions and inequalities, see [1–12].

This article presents and derives several theorems concerning convex integral inequalities. Seven such theorems are established in total, each of which has the potential to serve as a valuable intermediary tool in a wide range of analytical applications. The convex simple integral theorems are distinguished from the convex double integral theorems based on whether they use single or double integrals. Some of the obtained convex double integral theorems are related with the Hardy-Hilbert-type integral inequalities. See [13]. The proofs are presented in full detail and are designed to be easily reproducible, requiring only basic prior knowledge. Examples of basic exponential and power convex functions are given to illustrate the theory.

The remainder of this article is organized as follows: Section 2 presents the convex simple integral theorems, while Section 3 is devoted to the convex double integral theorems. Section 4 provides concluding remarks.

2. Simple integral theorems

This section contains four convex simple integral theorems, each with distinct features.

2.1. First theorem.

The first convex simple integral theorem is given below.

Theorem 2.1. *Let $\varphi : [0,1] \to [0,\infty)$ be a convex function. Then we have*

$$\int_0^1 (1-x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi(2x(1-x))dx \geq \frac{1}{2}\varphi\left(\frac{1}{3}\right).$$

Proof. Making the change of variables $y = 1-x$, $y \in (0,1)$, we get

$$\int_0^1 (1-x)\varphi(x)dx = \int_0^1 y\varphi(1-y)dy = \int_0^1 x\varphi(1-x)dx.$$

Therefore, we have

$$\int_0^1 (1-x)\varphi(x)dx = \frac{1}{2}\left(\int_0^1 (1-x)\varphi(x)dx + \int_0^1 (1-x)\varphi(x)dx\right)$$
$$= \frac{1}{2}\left(\int_0^1 (1-x)\varphi(x)dx + \int_0^1 x\varphi(1-x)dx\right)$$
$$= \frac{1}{2}\int_0^1 \left((1-x)\varphi(x) + x\varphi(1-x)\right)dx. \tag{2.1}$$

For any $x \in [0,1]$, the convexity of φ on $[0,1]$, especially the basic convexity inequality with $\lambda = x$, implies that

$$(1-x)\varphi(x) + x\varphi(1-x) \geq \varphi(x(1-x) + x(1-x)) = \varphi\left(2x(1-x)\right).$$

Therefore, we have

$$\frac{1}{2}\int_0^1 \left((1-x)\varphi(x) + x\varphi(1-x)\right)dx \geq \frac{1}{2}\int_0^1 \varphi\left(2x(1-x)\right)dx.$$

Combining this with equation (2.1), we get

$$\int_0^1 (1-x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi\left(2x(1-x)\right)dx.$$

The first inequality is demonstrated.

By the Jensen integral inequality for the convex function φ and the probability measure $\mathbf{1}_{[0,1]}(x)dx$, where $\mathbf{1}$ denotes the indicator function, we obtain

$$\frac{1}{2}\int_0^1 \varphi\left(2x(1-x)\right)dx \geq \frac{1}{2}\varphi\left(\int_0^1 2x(1-x)dx\right) = \frac{1}{2}\varphi\left(\frac{1}{3}\right).$$

This concludes the proof. □

Clearly, if φ is concave instead of convex, the two inequalities are reversed.

As an example of Theorem 2.1, let us consider the convex function $\varphi(x) = e^{-x}$. Then we have

$$0.367 \approx e^{-1} = \int_0^1 (1-x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi(2x(1-x))dx = \frac{1}{2}\int_0^1 e^{-2x(1-x)}dx$$

and

$$\frac{1}{2}\int_0^1 e^{-2x(1-x)}dx = \frac{1}{2}\int_0^1 \varphi(2x(1-x))dx \geq \frac{1}{2}\varphi\left(\frac{1}{3}\right) = \frac{1}{2}e^{-1/3} \approx 0.358.$$

We therefore derive simple lower and upper bounds for the complex integral $\int_0^1 e^{-2x(1-x)}dx$. These bounds are relatively sharp.

2.2. **Second theorem.** The theorem below completes Theorem 2.1 by considering $\int_0^1 x\varphi(x)dx$ instead of $\int_0^1 (1-x)\varphi(x)dx$.

Theorem 2.2. *Let $\varphi : [0,1] \to [0,\infty)$ be a convex function. Then we have*

$$\int_0^1 x\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi(x^2+(1-x)^2)dx \geq \frac{1}{2}\varphi\left(\frac{2}{3}\right).$$

Proof. Making the change of variables $y = 1-x$, $y \in (0,1)$, we get

$$\int_0^1 x\varphi(x)dx = \int_0^1 (1-y)\varphi(1-y)dy = \int_0^1 (1-x)\varphi(1-x)dx.$$

Therefore, we have

$$\begin{aligned}\int_0^1 x\varphi(x)dx &= \frac{1}{2}\left(\int_0^1 x\varphi(x)dx + \int_0^1 x\varphi(x)dx\right)\\ &= \frac{1}{2}\left(\int_0^1 x\varphi(x)dx + \int_0^1 (1-x)\varphi(1-x)dx\right)\\ &= \frac{1}{2}\int_0^1 \left(x\varphi(x) + (1-x)\varphi(1-x)\right)dx. \qquad (2.2)\end{aligned}$$

For any $x \in [0,1]$, the convexity of φ on $[0,1]$ gives

$$x\varphi(x) + (1-x)\varphi(1-x) \geq \varphi\left(x \times x + (1-x)\times(1-x)\right) = \varphi\left(x^2+(1-x)^2\right).$$

Therefore, we have

$$\frac{1}{2}\int_0^1 \left(x\varphi(x) + (1-x)\varphi(1-x)\right)dx \geq \frac{1}{2}\int_0^1 \varphi\left(x^2+(1-x)^2\right)dx.$$

Combining this with equation (2.2), we get

$$\int_0^1 x\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi\left(x^2+(1-x)^2\right)dx.$$

The first inequality is shown.

By the Jensen integral inequality for the convex function φ and the probability measure $\mathbf{1}_{[0,1]}(x)dx$, we find that

$$\frac{1}{2}\int_0^1 \varphi\left(x^2+(1-x)^2\right)dx \geq \frac{1}{2}\varphi\left(\int_0^1 \left(x^2+(1-x)^2\right)dx\right) = \frac{1}{2}\varphi\left(\frac{2}{3}\right).$$

This ends the proof. □

Clearly, if φ is concave instead of convex, the two inequalities are reversed.

As an example of Theorem 2.2, let us consider the convex function $\varphi(x) = e^{-x}$. Then we have

$$0.264 \approx 1 - 2e^{-1} = \int_0^1 x\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi\left(x^2 + (1-x)^2\right)dx = \frac{1}{2}\int_0^1 e^{-x^2-(1-x)^2}dx$$

and

$$\frac{1}{2}\int_0^1 e^{-x^2-(1-x)^2}dx = \frac{1}{2}\int_0^1 \varphi\left(x^2 + (1-x)^2\right)dx \geq \frac{1}{2}\varphi\left(\frac{2}{3}\right) = \frac{1}{2}e^{-2/3} \approx 0.256.$$

We can therefore derive simple lower and upper bounds for the complicated integral $\int_0^1 e^{-x^2-(1-x)^2}dx$. These bounds are relatively sharp.

2.3. **Third theorem.** The third convex simple integral theorem is given below.

Theorem 2.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a convex function with $\varphi(0) = 0$, and $f : [0,1] \to [0,\infty)$ be a function such that*

$$\int_0^1 (1-t)\varphi(f(t))\,dt < \infty.$$

For any $x \in [0,1]$, we define

$$F(x) = \int_0^x f(t)dt.$$

Then

$$\int_0^1 \varphi(F(x))\,dx \leq \int_0^1 (1-t)\varphi(f(t))\,dt.$$

Proof. For any $x \in (0,1]$, we set

$$m_x = \frac{1}{x}\int_0^x f(t)dt = \frac{1}{x}F(x).$$

Then $F(x) = xm_x$. Since φ is convex on $[0,\infty)$ and $\varphi(0) = 0$, we have, for any $x \in [0,1]$ and $u \geq 0$

$$\varphi(xu) = \varphi\left((1-x)\times 0 + x \times u\right) \leq (1-x)\varphi(0) + x\varphi(u) = x\varphi(u).$$

Applying this with $u = m_x$ gives

$$\varphi(F(x)) = \varphi(xm_x) \leq x\varphi(m_x). \tag{2.3}$$

By the Jensen integral inequality for the convex function φ and the probability measure $(1/x)\mathbf{1}_{[0,x]}(t)dt$, we obtain

$$\varphi(m_x) = \varphi\left(\frac{1}{x}\int_0^x f(t)dt\right) \leq \frac{1}{x}\int_0^x \varphi\left(f(t)\right)dt.$$

Combining this with equation (2.3), we find that

$$\varphi\left(F(x)\right) \leq \int_0^x \varphi\left(f(t)\right)dt.$$

Integrating in x over $[0,1]$ and using the Fubini-Tonelli integral theorem, we get

$$\int_0^1 \varphi\left(F(x)\right)dx \leq \int_0^1 \int_0^x \varphi\left(f(t)\right)dtdx = \int_0^1 \left(\int_t^1 dx\right)\varphi\left(f(t)\right)dt$$
$$= \int_0^1 (1-t)\varphi\left(f(t)\right)dt.$$

This completes the proof. □

Clearly, if φ is concave instead of convex, the inequality is reversed.

As an example of Theorem 2.3, let us consider the convex function $\varphi(x) = x^p$, with $p > 1$. Then we have

$$\int_0^1 F^p(x)dx \leq \int_0^1 (1-t)f^p(t)dt.$$

This bound can be used in the context of L_p integral norm inequalities. It can also be seen as a simple variant of the Hardy integral inequality.

2.4. **Fourth theorem.**

The fourth convex simple integral theorem is given below.

Theorem 2.4. *Let $\beta > 0$, $\varphi : [0,\infty) \to [0,\infty)$ be a convex function with $\varphi(0) = 0$, and $f : [0,\infty) \to [0,\infty)$ be a function such that*

$$\int_0^\infty \varphi\left(f(t)\right)t^{-\beta}dt < \infty.$$

Then, we have

$$\int_0^\infty \varphi\left(\frac{1}{x}\int_0^x f(t)dt\right)x^{-\beta}dx \leq \frac{1}{\beta}\int_0^\infty \varphi\left(f(t)\right)t^{-\beta}dt.$$

Proof. By the Jensen integral inequality for the convex function φ and the probability measure $(1/x)\mathbf{1}_{[0,x]}(t)dt$, we get

$$\varphi\left(\frac{1}{x}\int_0^x f(t)dt\right) \leq \frac{1}{x}\int_0^x \varphi\left(f(t)\right)dt.$$

Multiplying both sides by $x^{-\beta}$ and integrating over $(0,\infty)$, we obtain

$$\begin{aligned}\int_0^\infty \varphi\left(\frac{1}{x}\int_0^x f(t)dt\right)x^{-\beta}dx &\leq \int_0^\infty \frac{1}{x}\left(\int_0^x \varphi\left(f(t)\right)dt\right)x^{-\beta}dx\\ &= \int_0^\infty x^{-\beta-1}\int_0^x \varphi\left(f(t)\right)dtdx. \qquad (2.4)\end{aligned}$$

By the Fubini-Tonelli integral theorem, we have

$$\begin{aligned}\int_0^\infty x^{-\beta-1}\int_0^x \varphi\left(f(t)\right)dtdx &= \int_0^\infty \varphi\left(f(t)\right)\left(\int_t^\infty x^{-\beta-1}dx\right)dt\\ &= \int_0^\infty \varphi\left(f(t)\right)\frac{1}{\beta}t^{-\beta}dt = \frac{1}{\beta}\int_0^\infty \varphi\left(f(t)\right)t^{-\beta}dt.\end{aligned}$$

Combining this with equation (2.4), we get

$$\int_0^\infty \varphi\left(\frac{1}{x}\int_0^x f(t)dt\right)x^{-\beta}dx \leq \frac{1}{\beta}\int_0^\infty \varphi\left(f(t)\right)t^{-\beta}dt.$$

This concludes the proof. □

As an example of Theorem 2.4, let us consider the convex function $\varphi(x) = x^p$, with $p > 1$. Then we have

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^{-\beta}dx \leq \frac{1}{\beta}\int_0^\infty f^p(t)t^{-\beta}dt.$$

This bound can be used in the context of L_p integral norm inequalities. It can also be seen as a simple variant of the Hardy integral inequality.

3. Double integral theorems

This section contains three convex double integral theorems, each with distinct features.

3.1. **First theorem.** The first convex double integral theorem is given below.

Theorem 3.1. *Let $p > 1$, q be its Hölder conjugate, i.e., $1/p + 1/q = 1$, $\varphi : [0,1] \to [0,\infty)$ be a convex function, and $f, g : [0,\infty) \to [0,\infty)$ be two functions such that*

$$\int_0^\infty f^p(x)dx < \infty, \quad \int_0^\infty g^q(y)dy < \infty.$$

Then we have

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{1}{x+y}\varphi\left(\frac{x}{x+y}\right) f(x)g(y)dxdy \\ \leq&\frac{\pi}{p\sin(\pi/p)}\big(\varphi(0) + (p-1)\varphi(1)\big)\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}.\end{aligned}$$

In particular, taking $\varphi = 1$ recovers the Hardy-Hilbert integral inequality, i.e.,

$$\int_0^\infty \int_0^\infty \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}.$$

Proof. Let us consider the change of variables

$$x = ut, \qquad y = u(1-t), \qquad u \in (0,\infty),\ t \in (0,1).$$

The Jacobian is $J(u,t) = u$, so $dxdy = ududt$. We also have $x + y = u$ and $x/(x+y) = t$. Therefore, we derive

$$\int_0^\infty \int_0^\infty \frac{1}{x+y}\varphi\left(\frac{x}{x+y}\right) f(x)g(y)dxdy = \int_0^1 \varphi(t)J(t)dt, \tag{3.1}$$

where

$$J(t) = \int_0^\infty f(ut)g(u(1-t))du.$$

Let us now bound this integral. Applying the Hölder integral inequality to $J(t)$ with exponents p and q, we get

$$J(t) \leq \left(\int_0^\infty f^p(ut)du\right)^{1/p}\left(\int_0^\infty g^q(u(1-t))du\right)^{1/q}.$$

Let us now perform changes of variables in each integral of this upper bound. For the first, setting $v = ut$, $v \in (0,\infty)$, so $du = dv/t$, we obtain

$$\int_0^\infty f^p(ut)du = t^{-1}\int_0^\infty f^p(v)dv.$$

For the second, setting $w = u(1-t)$, $w \in (0,\infty)$, so $du = dw/(1-t)$, we derive

$$\int_0^\infty g^q(u(1-t))du = (1-t)^{-1}\left(\int_0^\infty g^q(w)dw\right).$$

Hence, we have

$$J(t) \leq t^{-1/p}(1-t)^{-1/q}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}.$$

Substituting this in equation (3.1) gives

$$\int_0^\infty\int_0^\infty \frac{1}{x+y}\varphi\left(\frac{x}{x+y}\right)f(x)g(y)dxdy$$

$$\leq \left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}\int_0^1 \varphi(t)t^{-1/p}(1-t)^{-1/q}dt. \tag{3.2}$$

Let us now bound this last integral using the convexity of φ on $[0,1]$. For any $t \in [0,1]$, we have the linear upper bound

$$\varphi(t) = \varphi((1-t)\times 0 + 1\times t) \leq (1-t)\varphi(0) + t\varphi(1).$$

Therefore, we have

$$\int_0^1 \varphi(t)t^{-1/p}(1-t)^{-1/q}dt \leq \varphi(0)\int_0^1 (1-t)t^{-1/p}(1-t)^{-1/q}dt$$
$$+\varphi(1)\int_0^1 t\times t^{-1/p}(1-t)^{-1/q}dt.$$

Simplifying the integrands, identifying beta integrals and using $1-1/q = 1/p$, we get

$$\int_0^1 (1-t)t^{-1/p}(1-t)^{-1/q}dt = \int_0^1 t^{-1/p}(1-t)^{1/p}dt = B\left(1-\frac{1}{p}, 1+\frac{1}{p}\right)$$

and

$$\int_0^1 t\times t^{-1/p}(1-t)^{-1/q}dt = \int_0^1 t^{1-1/p}(1-t)^{-1/q}dt = B\left(2-\frac{1}{p}, \frac{1}{p}\right).$$

Using the classical gamma/beta identities and the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ for any $z \in (0,1)$, we get

$$B\left(1-\frac{1}{p}, 1+\frac{1}{p}\right) = \Gamma\left(1-\frac{1}{p}\right)\Gamma\left(1+\frac{1}{p}\right) = \frac{1}{p}\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right) = \frac{\pi}{p\sin(\pi/p)}$$

and

$$B\left(2-\frac{1}{p},\frac{1}{p}\right)=\Gamma\left(2-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)=\frac{p-1}{p}\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)=\frac{\pi(p-1)}{p\sin(\pi/p)}.$$

Hence, we have

$$\int_0^1 \varphi(t)t^{-1/p}(1-t)^{-1/q}dt \leq \frac{\pi}{p\sin(\pi/p)}\big(\varphi(0)+(p-1)\varphi(1)\big).$$

Combining this with equation (3.2) gives

$$\int_0^\infty \int_0^\infty \frac{1}{x+y}\varphi\left(\frac{x}{x+y}\right) f(x)g(y)dxdy$$
$$\leq \frac{\pi}{p\sin(\pi/p)}\big(\varphi(0)+(p-1)\varphi(1)\big)\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}.$$

This ends the proof. □

As an example of Theorem 3.1, let us consider the convex function $\varphi(x)=e^{-x}$. Then we have

$$\int_0^\infty \int_0^\infty \frac{1}{x+y}e^{-x/(x+y)} f(x)g(y)dxdy$$
$$\leq \frac{\pi}{p\sin(\pi/p)}\big(1+(p-1)e^{-1}\big)\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}.$$

This bound can be used in the context of Hardy-Hilbert-type integral inequalities.

3.2. **Second theorem.**

The second convex double integral theorem is given below.

Theorem 3.2. *Let $\varphi:[0,\infty)\to[0,\infty)$ be a convex convex function such that*

$$\int_0^\infty \varphi(t)dt<\infty,$$

and $k:[0,\infty)\to[0,\infty)$ be a function such that

$$\int_0^\infty k(t)dt<\infty.$$

Then we have

$$\int_0^\infty \int_0^\infty \varphi\left(\frac{x+y}{2}\right)k(x+y)dxdy \leq \left(\int_0^\infty k(t)dt\right)\int_0^\infty \varphi(x)dx.$$

Proof. By the convexity of φ on $[0,\infty)$, for any $x, y \geq 0$, we have

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(\varphi(x)+\varphi(y)\right).$$

Multiplying by $k(x+y)$, integrating and using a symmetry in the roles of the variables x and y, we get

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \varphi\left(\frac{x+y}{2}\right) k(x+y)dxdy \\
\leq&\frac{1}{2}\int_0^\infty \int_0^\infty \left(\varphi(x)+\varphi(y)\right) k(x+y)dxdy \\
=&\frac{1}{2}\int_0^\infty \int_0^\infty \varphi(x)k(x+y)dxdy + \frac{1}{2}\int_0^\infty \int_0^\infty \varphi(y)k(x+y)dxdy \\
=&\frac{1}{2}\int_0^\infty \int_0^\infty \varphi(x)k(x+y)dxdy + \frac{1}{2}\int_0^\infty \int_0^\infty \varphi(x)k(x+y)dxdy \\
=&\int_0^\infty \int_0^\infty \varphi(x)k(x+y)dxdy. \qquad (3.3)
\end{aligned}$$

Making the change of variables $t = x+y$, $t \in (x,\infty)$, and majorizing, we get

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \varphi(x)k(x+y)dxdy = \int_0^\infty \varphi(x)\left(\int_0^\infty k(x+y)dy\right)dx \\
=&\int_0^\infty \varphi(x)\left(\int_x^\infty k(t)dt\right)dx \leq \left(\int_0^\infty k(t)dt\right)\int_0^\infty \varphi(x)dx.
\end{aligned}$$

Substituting this in equation (3.3), we obtain

$$\int_0^\infty \int_0^\infty \varphi\left(\frac{x+y}{2}\right) k(x+y)dxdy \leq \left(\int_0^\infty k(t)dt\right)\int_0^\infty \varphi(x)dx.$$

This completes the proof. □

As an example of Theorem 3.2, let us consider the convex function $\varphi(x) = e^{-x}$. Then we have

$$\int_0^\infty \int_0^\infty e^{-(x+y)/2}k(x+y)dxdy \leq \left(\int_0^\infty k(t)dt\right)\int_0^\infty \varphi(x)dx = \int_0^\infty k(t)dt.$$

This bound can be applied to diverse kinds of double integral inequalities.

3.3. **Third theorem.** The third convex double integral theorem is given below.

Theorem 3.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a convex function, and $f, g : [0,\infty) \to [0,\infty)$ be two functions such that*

$$\int_0^\infty f(x)dx < \infty, \quad \int_0^\infty g(y)dy < \infty.$$

Then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\varphi(x)+\varphi(y)}dxdy \leq \frac{1}{2}\int_0^\infty (f*g)(u)\frac{1}{\varphi(u/2)}du,$$

where

$$(f*g)(u) = \int_0^u f(t)g(u-t)dt$$

is the standard convolution product on $[0,\infty)$.

Proof. By the convexity of φ on $[0,\infty)$, for any $x, y \geq 0$, we have

$$\varphi(x)+\varphi(y) \geq 2\varphi\left(\frac{x+y}{2}\right),$$

so that

$$\frac{1}{\varphi(x)+\varphi(y)} \leq \frac{1}{2\varphi\left((x+y)/2\right)}.$$

Therefore, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\varphi(x)+\varphi(y)}dxdy \leq \frac{1}{2}\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\varphi\left((x+y)/2\right)}dxdy. \tag{3.4}$$

Making the change of variables $u = x+y$ and $v = x$, $u \in (0,\infty)$, $v \in (0,u)$, and using the Tonelli-Fubini integral theorem, we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\varphi\left((x+y)/2\right)}dxdy = \int_0^\infty \frac{1}{\varphi(u/2)}\left(\int_0^u f(v)g(u-v)dv\right)du$$
$$= \int_0^\infty (f*g)(u)\frac{1}{\varphi(u/2)}du.$$

Substituting this in equation (3.4) gives

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\varphi(x)+\varphi(y)}dxdy \leq \frac{1}{2}\int_0^\infty (f*g)(u)\frac{1}{\varphi(u/2)}du.$$

This ends the proof. □

As an example of Theorem 3.3, let us consider the convex function $\varphi(x) = e^{-x}$. Then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{e^{-x}+e^{-y}} dxdy \leq \frac{1}{2}\int_0^\infty (f*g)(u)\frac{1}{e^{-u/2}}du.$$

This bound can be used in the context of Hardy-Hilbert-type integral inequalities.

4. Conclusion

In this article, we present seven theorems that offer new perspectives on applications of convex functions in the context of both simple and double integral inequalities. Several examples are provided to illustrate the theory. Potential directions for future work include extending these results to higher-dimensional integrals, exploring applications in optimization and probability theory, and investigating connections with other classes of inequalities.

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