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SOME NEW CONVEX INTEGRAL INEQUALITIES INVOLVING TWO FUNCTIONS

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ABSTRACT. This article investigates new integral inequalities involving two functions under convexity assumptions. Specifically, we derive upper and lower bounds for two different integral forms using techniques based on the convexity inequality, as well as the Jensen and Chebyshev integral inequalities. The results are presented in a self-contained manner, with comprehensive proofs provided.

1. INTRODUCTION

The concept of a convex function is central in mathematics. A formal definition is given below. Let $a, b \in \mathbb{R}$ with $a < b$. A function $\varphi : [a, b] \to \mathbb{R}$ is said to be convex if and only if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, the following inequality holds:

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y). \tag{1.1}$$

Furthermore, if φ is twice differentiable, then $\varphi''(x) \geq 0$ for any $x \in [a, b]$, and consequently, φ' is non-decreasing on $[a, b]$. Convex functions provide a fundamental basis for developing a wide range of mathematical inequalities. The most prominent of these are the Jensen and Hermite-Hadamard integral inequalities,

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along with their numerous weighted, refined, and generalized extensions. Comprehensive expositions and further results on convex functions and their associated inequalities can be found in [1–14].

In this article, we contribute to the subject by investigating new convex integral inequalities involving two functions. Specifically, let f and g denote these functions. Under certain convexity assumptions on f and g, our aim is twofold:

(1) To determine sharp upper and lower bounds for an integral of the form
$$\int_0^1 xf(g(x))dx,$$
taking inspiration from [3, Theorems 2.1 and 2.2].

(2) To determine sharp upper and lower bounds for an integral of the form
$$\int_a^b f(x)g''(x)dx.$$

These results are obtained using techniques closely connected with the convexity assumptions. In particular, we employ the convexity inequality in Equation (1.1) and the Jensen integral inequality. We also use changes of variables and the Chebyshev integral inequality. The proofs are presented in full detail and are self-contained, requiring no specialized background.

The remainder of the article is organized as follows: Sections 2 and 3 are devoted to establishing upper and lower bounds for the first and second integral forms, respectively. Finally, Section 4 presents concluding remarks and potential directions for future research.

2. Bounds for the first integral form

The theorem below focuses on a lower bound for the first integral form.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and $g : [0, 1] \to \mathbb{R}$ be a function. Then, we have*
$$\int_0^1 xf(g(x))dx \geq \frac{1}{2}\int_0^1 f\left(xg(x) + (1-x)g(1-x)\right)dx,$$
provided that the integrals exist.

Proof. Making the change of variables $x = 1 - y$, we get

$$\int_0^1 xf(g(x))dx = \int_0^1 (1-y)f(g(1-y))dy = \int_0^1 (1-x)f(g(1-x))dx.$$

Therefore, averaging the two expressions and using the convexity inequality of f, we get

$$\int_0^1 xf(g(x))dx = \frac{1}{2}\left(\int_0^1 xf(g(x))dx + \int_0^1 (1-x)f(g(1-x))dx\right)$$
$$= \frac{1}{2}\left(\int_0^1 \left(xf(g(x)) + (1-x)f(g(1-x))\right) dx\right)$$
$$\geq \frac{1}{2}\int_0^1 f\left(xg(x) + (1-x)g(1-x)\right) dx.$$

This completes the proof. □

Taking $g(x) = x$, this theorem reduces to [3, Theorem 2.2], i.e.,

$$\int_0^1 xf(x)dx \geq \frac{1}{2}\int_0^1 f\left(x^2 + (1-x)^2\right) dx.$$

As an example activating the function g, for a non-negative function g, considering the convex function $f(x) = x^\alpha$, $x \in [0,1]$, with $\alpha > 1$, we get

$$\int_0^1 xg^\alpha(x)dx \geq \frac{1}{2}\int_0^1 \left(xg(x) + (1-x)g(1-x)\right)^\alpha dx.$$

We can find a lower bound of the obtained lower bound as follows: Using the Jensen integral inequality and a change of variables, one can show that

$$\int_0^1 f\left(xg(x) + (1-x)g(1-x)\right) dx \geq f\left(\int_0^1 \left(xg(x) + (1-x)g(1-x)\right) dx\right)$$
$$= f\left(\int_0^1 xg(x)dx + \int_0^1 (1-x)g(1-x)dx\right) = f\left(2\int_0^1 xg(x)dx\right).$$

The theorem below focuses on an upper bound for the first integral form.

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and $g : [0,1] \to \mathbb{R}$ be a function. Then, we have*

$$\int_0^1 xf(g(x))dx \leq \int_0^1 f(g(x))dx - \frac{1}{2}\int_0^1 f\left((1-x)g(x) + xg(1-x)\right) dx,$$

provided that the integrals exist.

Proof. Making the change of variables $x = 1 - y$, we get

$$\int_0^1 (1-x)f(g(x))dx = \int_0^1 yf(g(1-y))dy = \int_0^1 xf(g(1-x))dx.$$

Therefore, averaging the two expressions and using the convexity inequality of f, we get

$$\int_0^1 (1-x)f(g(x))dx = \frac{1}{2}\left(\int_0^1 (1-x)f(g(x))dx + \int_0^1 xf(g(1-x))dx\right)$$
$$= \frac{1}{2}\left(\int_0^1 \left((1-x)f(g(x)) + xf(g(1-x))\right) dx\right)$$
$$\geq \frac{1}{2}\int_0^1 f\left((1-x)g(x) + xg(1-x)\right) dx.$$

We thus have

$$\int_0^1 f(g(x))dx - \int_0^1 xf(g(x))dx$$
$$\geq \frac{1}{2}\int_0^1 f\left((1-x)g(x) + xg(1-x)\right) dx.$$

so that

$$\int_0^1 xf(g(x))dx \leq \int_0^1 f(g(x))dx - \frac{1}{2}\int_0^1 f\left((1-x)g(x) + xg(1-x)\right) dx.$$

This ends the proof. □

Taking $g(x) = x$, this theorem can be reduced to [3, Theorem 2.1], i.e.,

$$\int_0^1 (1-x)f(x)dx \geq \frac{1}{2}\int_0^1 f\left(2x(1-x)\right) dx.$$

As an example activating the function g, for a non-negative function g, considering the convex function $f(x) = x^\alpha$, $x \in [0,1]$, with $\alpha > 1$, we get

$$\int_0^1 xg^\alpha(x)dx \leq \int_0^1 g^\alpha(x)dx - \frac{1}{2}\int_0^1 \left((1-x)g(x) + xg(1-x)\right)^\alpha dx.$$

We can find an upper bound of the obtained upper bound using the following development: Using the Jensen integral inequality and a change of variables, one can show that

$$\int_0^1 f\left((1-x)g(x)+xg(1-x)\right)dx \geq f\left(\int_0^1 \left((1-x)g(x)+xg(1-x)\right)dx\right)$$
$$= f\left(\int_0^1 (1-x)g(x)dx + \int_0^1 xg(1-x)dx\right) = f\left(2\int_0^1 (1-x)g(x)dx\right).$$

We thus have

$$\int_0^1 xf(g(x))dx \leq \int_0^1 f(g(x))dx - \frac{1}{2}f\left(2\int_0^1 (1-x)g(x)dx\right).$$

Note that we impose only minimal integrability assumptions on g. The resulting inequalities are therefore particularly flexible.

3. Bounds for the second integral forms

The theorem below focuses on a lower bound for the second integral form.

Theorem 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, $f : \mathbb{R} \to \mathbb{R}$ be a convex function and $g : [a, b] \to \mathbb{R}$ be a twice differentiable convex function. Then, we have*

$$\int_a^b f(x)g''(x)dx$$
$$\geq (g'(b) - g'(a))\, f\left(\frac{1}{g'(b)-g'(a)}\left(bg'(b) - ag'(a) - g(b) + g(a)\right)\right),$$

provided that the integral exists.

Proof. Since f is a convex function on $\mathbb{R}$ and g is a twice differentiable convex function on $[a, b]$, we can apply the Jensen integral inequality to f and the probability measure $g''(x)dx/\left(g'(b) - g'(a)\right)$. This and an integration by parts give

$$\int_a^b f(x)g''(x)dx = (g'(b) - g'(a)) \int_a^b f(x)\frac{g''(x)}{g'(b)-g'(a)}dx$$
$$\geq (g'(b) - g'(a))\, f\left(\int_a^b x\frac{g''(x)}{g'(b)-g'(a)}dx\right)$$
$$= (g'(b) - g'(a))\, f\left(\frac{1}{g'(b)-g'(a)}\int_a^b xg''(x)dx\right)$$
$$= (g'(b) - g'(a))\, f\left(\frac{1}{g'(b)-g'(a)}\left(\left[xg'(x)\right]_a^b - \int_a^b g'(x)dx\right)\right)$$

$$= (g'(b) - g'(a))\, f\left(\frac{1}{g'(b) - g'(a)}\left(bg'(b) - ag'(a) - g(b) + g(a)\right)\right).$$

This completes the proof. □

In particular, considering the convex function $f(x) = \exp(-\beta x)$, $x \in [a, b]$, with $\beta > 0$, we get

$$\begin{aligned}&\int_a^b \exp(-\beta x) g''(x)dx \\ &\geq (g'(b) - g'(a)) \exp\left(-\beta\left(\frac{1}{g'(b) - g'(a)}\left(bg'(b) - ag'(a) - g(b) + g(a)\right)\right)\right).\end{aligned}$$

We can establish a link between this result and the lower bound of the Laplace transform of g''.

For the case $a = 0$ and $b = 1$, the inequality reduces to

$$\int_0^1 f(x)g''(x)dx \geq (g'(1) - g'(0))\, f\left(\frac{1}{g'(1) - g'(0)}\left(g'(1) - g(1) + g(0)\right)\right).$$

The theorem below focuses on an upper bound for the second integral form.

Theorem 3.2. *Let $a, b \in \mathbb{R}$ with $a < b$, and $f, g : [a, b] \to \mathbb{R}$ be two twice differentiable convex functions. Then, we have*

$$\begin{aligned}&\int_a^b f(x)g''(x)dx \\ \leq &f(b)\left(g'(b) - \frac{1}{b-a}(g(b) - g(a))\right) - f(a)\left(g'(a) - \frac{1}{b-a}(g(b) - g(a))\right),\end{aligned}$$

provided that the integral exists.

Proof. Since f and g are twice differentiable convex functions on $[a, b]$, f' and g' are non-decreasing on $[a, b]$. Applying an integration by parts and the Chebyshev integral inequality to f' and g', i.e.,

$$\int_a^b f'(x)g'(x)dx \geq \frac{1}{b-a}\left(\int_a^b f'(x)dx\right)\left(\int_a^b g'(x)dx\right),$$

we obtain

$$\begin{aligned}
&\int_a^b f(x)g''(x)dx = [f(x)g'(x)]_a^b - \int_a^b f'(x)g'(x)dx \\
&= f(b)g'(b) - f(a)g'(a) - \int_a^b f'(x)g'(x)dx \\
&\leq f(b)g'(b) - f(a)g'(a) - \frac{1}{b-a}\left(\int_a^b f'(x)dx\right)\left(\int_a^b g'(x)dx\right) \\
&= f(b)g'(b) - f(a)g'(a) - \frac{1}{b-a}(f(b)-f(a))(g(b)-g(a)) \\
&= f(b)\left(g'(b) - \frac{1}{b-a}(g(b)-g(a))\right) - f(a)\left(g'(a) - \frac{1}{b-a}(g(b)-g(a))\right).
\end{aligned}$$

This completes the proof. □

In particular, considering the convex function $f(x) = x^\alpha$, $x \in [a,b]$, with $\alpha > 1$ and $0 \leq a < b$, we get

$$\begin{aligned}
&\int_a^b x^\alpha g''(x)dx \\
\leq & b^\alpha\left(g'(b) - \frac{1}{b-a}(g(b)-g(a))\right) - a^\alpha\left(g'(a) - \frac{1}{b-a}(g(b)-g(a))\right).
\end{aligned}$$

Note that if f' and g' are assumed to be of opposite monotonicity instead of f and g assumed to be convex, then the Chebyshev integral inequality is reversed, implying that the final inequality is reversed, i.e.,

$$\begin{aligned}
&\int_a^b f(x)g''(x)dx \\
\geq & f(b)\left(g'(b) - \frac{1}{b-a}(g(b)-g(a))\right) - f(a)\left(g'(a) - \frac{1}{b-a}(g(b)-g(a))\right).
\end{aligned}$$

4. Conclusion

In this article, we have established new integral inequalities involving convex functions, providing upper and lower bounds for two distinct integral forms. The results extend classical inequalities and highlight the versatility of convexity-based techniques in deriving bounds. Future research may focus on extending these results to broader classes of functions, such as s-convex, quasi-convex, or harmonically convex functions, as well as exploring applications in numerical analysis and optimization theory.

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