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A WEIGHTED GENERALIZATION OF AN INTEGRAL INEQUALITY WITH A PROBABILISTIC INTERPRETATION

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ABSTRACT. This article presents a weighted generalization of an integral inequality involving a function and its powers of order one, two, and four. Several examples are provided, each considering a specific type of weight function. Furthermore, we discuss how the main result can be naturally interpreted in terms of a moment inequality, offering a new perspective in probability.

1. INTRODUCTION

Integral inequalities play a central role in mathematical analysis. They are a powerful tool for deriving precise bounds in many analytical contexts, including functional equations, variational problems, and differential systems. These inequalities arise naturally in fields such as approximation theory, convex and functional analysis, and the study of special functions. Detailed expositions of classical results and foundational approaches can be found in [1, 2, 8, 11, 21, 22].

Over the past decade, there has been a surge of interest in refining and generalizing classical integral inequalities. This has involved introducing new kernel function structures and weight functions, and extending established results to broader classes of functions and spaces. Notable recent contributions to this field are presented in [3–7, 9, 12, 14–20].

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Formulating new integral inequalities strengthens the theoretical basis of the subject. A few years ago, an intriguing inequality involving a function and its powers of order one, two and four was posed and discussed on the specialized website *Math.StackExchange.com*. This inequality is presented as a theorem below and forms the basis of our article.

Theorem 1.1. *[10] Let $f : [0,1] \to \mathbb{R}$ be a function. Then we have*

$$\int_0^1 \sqrt{f^4(x) + \left(\int_0^1 f(t)dt\right)^4} dx \leq \sqrt{2} \int_0^1 f^2(x)dx,$$

provided that the integrals converge.

A detailed proof of this result was provided in [13], relying on an algebraic inequality combined with standard integration techniques. New investigations demonstrate that the inequality is particularly sharp, with equality achieved for both $f(x) = 0$ and $f(x) = 1$. In the latter case, both sides are equal to $\sqrt{2}$.

In this article, we extend the inequality by introducing a weight function, w, that satisfies the properties of a probability density function. Several examples are provided, each considering a specific type of weight function. Our main result naturally leads to a probabilistic interpretation, enabling the inequality to be expressed in terms of moments of random variables. This perspective paves the way for potential applications in probability theory, statistics, and related areas of mathematical analysis.

Section 2 presents the main theorem, its full proof, precise examples, and its probabilistic interpretation. Concluding remarks and directions for future research are discussed in Section 3.

2. Contributions

2.1. **Result.** The main integral inequality is stated in the theorem below, followed by its proof. We highlight the explicit role of the parameters a, b and the weight function w.

Theorem 2.1. *Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $b > a$, $f : [a,b] \to \mathbb{R}$ be a function and $w : [a,b] \to [0,\infty)$ be a (weight) function such that*

$$\int_a^b w(x)dx = 1.$$

Then we have

$$\int_a^b \sqrt{f^4(x) + \left(\int_a^b f(t)w(t)dt\right)^4}\, w(x)dx \leq \sqrt{2}\int_a^b f^2(x)w(x)dx,$$

provided that the integrals converge.

Proof. The first step of the proof is similar to that in [13]. For any $y, z \in \mathbb{R}$, we have

$$y^4 + z^4 = 2(y^2 + z^2 - yz)^2 - (y - z)^4 \leq 2(y^2 + z^2 - yz)^2,$$

which implies that

$$\sqrt{y^4 + z^4} \leq \sqrt{2}(y^2 + z^2 - yz).$$

We now activate the weight function w by applying this inequality to

$$y = f(x), \quad z = \int_a^b f(t)w(t)dt.$$

Therefore, we obtain

$$\sqrt{f^4(x) + \left(\int_a^b f(t)w(t)dt\right)^4}$$
$$\leq \sqrt{2}\left(f^2(x) + \left(\int_a^b f(t)w(t)dt\right)^2 - f(x)\left(\int_a^b f(t)w(t)dt\right)\right).$$

Multiplying both sides by $w(x)$, which does not change the sense of the inequality because of non-negativity, we get

$$\sqrt{f^4(x) + \left(\int_a^b f(t)w(t)dt\right)^4}\, w(x)$$
$$\leq \sqrt{2}\left(f^2(x)w(x) + \left(\int_a^b f(t)w(t)dt\right)^2 w(x) - f(x)w(x)\left(\int_a^b f(t)w(t)dt\right)\right).$$

Integrating both sides with respect to $x \in [a, b]$ and using $\int_a^b w(x)dx = 1$, we obtain

$$\int_a^b \sqrt{f^4(x) + \left(\int_a^b f(t)w(t)dt\right)^4}\, w(x)dx$$

$$\leq \sqrt{2}\left(\int_a^b f^2(x)w(x)dx + \left(\int_a^b f(t)w(t)dt\right)^2 \int_a^b w(x)dx - \left(\int_a^b f(t)w(t)dt\right)^2\right)$$
$$= \sqrt{2}\int_a^b f^2(x)w(x)dx.$$

This completes the proof of the theorem. □

Remark 2.1. *Another proof of Theorem 2.1, less self-contained but more direct, can be obtained by applying Theorem 1.1 to the composite function* $f(W^{-1})$*, where, for any* $x \in [a,b]$*,*

$$W(x) = \int_a^x w(t)dt.$$

This yields

$$\int_0^1 \sqrt{f^4(W^{-1}(x)) + \left(\int_0^1 f(W^{-1}(t))dt\right)^4} dx \leq \sqrt{2}\int_0^1 f^2(W^{-1}(x))dx.$$

Making the changes of variables $x = W(y)$ *and* $t = W(y)$*, and noting that* $dx = w(y)dy$*,* $dt = w(y)dy$*,* $W(a) = 0$ *and* $W(b) = \int_a^b w(x)dx = 1$*, we obtain*

$$\int_a^b \sqrt{f^4(x) + \left(\int_a^b f(t)w(t)dt\right)^4} w(x)dx \leq \sqrt{2}\int_a^b f^2(x)w(x)dx.$$

In a sense, this shows that Theorem 1.1 possesses a self-extending property.

To the best of our knowledge, Theorem 2.1 introduces an inequality that has not previously appeared in the literature.

Compared with Theorem 1.1, the inclusion of the weight function w provides a broader framework that enables the inequality to be applied in weighted contexts. This generalization provides additional flexibility, particularly in applications where certain regions of the domain $[a,b]$ are given more significance through the choice of w. The proposition below provides a direct and elegant application of Theorem 2.1 to illustrate this claim.

Proposition 2.1. *Let* $a \in \mathbb{R} \cup \{-\infty\}$*,* $b \in \mathbb{R} \cup \{+\infty\}$ *with* $b > a$*, and* $f : [a,b] \to \mathbb{R}$ *be a function such that*

$$\int_a^b f(x)dx = 1.$$

Then we have

$$\int_a^b \sqrt{f^4(x) + \left(\int_a^b f^2(t)dt\right)^4} f(x)dx \leq \sqrt{2}\int_a^b f^3(x)dx,$$

provided that the integrals converge.

Proof. Applying Theorem 2.1 with $w = f$ yields the desired inequality. □

2.2. **Examples.** Other applications of Theorem 2.1, each corresponding to a specific choice of the weight function w, are presented below.

- Setting $a = 0$, $b = 1$ and $w(x) = \alpha x^{\alpha-1}$, $x \in [0, 1]$, with $\alpha > 0$, satisfying the required assumptions, Theorem 2.1 gives

$$\int_0^1 \sqrt{f^4(x) + \alpha^4\left(\int_0^1 f(t)t^{\alpha-1}dt\right)^4} x^{\alpha-1}dx \leq \sqrt{2}\int_0^1 f^2(x)x^{\alpha-1}dx,$$

  provided that the integrals converge. If we take $\alpha = 1$, then this inequality reduces to Theorem 1.1.
- Setting $a = 0$, $b = 1$ and $w(x) = -\log(x)$, $x \in (0, 1]$, satisfying the required assumptions, Theorem 2.1 gives

$$\int_0^1 \sqrt{f^4(x) + \left(\int_0^1 f(t)\log(t)dt\right)^4}(-\log(x))dx$$
$$\leq \sqrt{2}\int_0^1 f^2(x)(-\log(x))dx,$$

  provided that the integrals converge.
- Setting $a = 0$, $b = 1$ and $w(x) = (\pi/2)\cos((\pi/2)x)$, $x \in [0, 1]$, satisfying the required assumptions, Theorem 2.1 gives

$$\int_0^1 \sqrt{f^4(x) + \left(\frac{\pi}{2}\right)^4\left(\int_0^1 f(t)\cos\left(\frac{\pi}{2}t\right)dt\right)^4}\cos\left(\frac{\pi}{2}x\right)dx$$
$$\leq \sqrt{2}\int_0^1 f^2(x)\cos\left(\frac{\pi}{2}x\right)dx,$$

  provided that the integrals converge.

- Setting $a = 0$, $b = +\infty$ and $w(x) = \lambda e^{-\lambda x}$, $x \in [0, +\infty)$, with $\lambda > 0$, satisfying the required assumptions, Theorem 2.1 gives

$$\int_0^{+\infty} \sqrt{f^4(x) + \lambda^4 \left(\int_0^{+\infty} f(t) e^{-\lambda t} dt \right)^4} e^{-\lambda x} dx$$
$$\leq \sqrt{2} \int_0^{+\infty} f^2(x) e^{-\lambda x} dx,$$

provided that the integrals converge. This inequality can also be expressed in terms of the Laplace transforms of carefully selected functions on both sides.

- Setting $a = 0$, $b = +\infty$ and $w(x) = (\alpha/\lambda)(1 + x/\lambda)^{-(\alpha+1)}$, $x \in [0, +\infty)$, with $\alpha, \lambda > 0$, satisfying the required assumptions, Theorem 2.1 gives

$$\int_0^{+\infty} \sqrt{f^4(x) + \left(\frac{\alpha}{\lambda}\right)^4 \left(\int_0^{+\infty} f(t) \left(1 + \frac{t}{\lambda}\right)^{-(\alpha+1)} dt \right)^4} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)} dx$$
$$\leq \sqrt{2} \int_0^{+\infty} f^2(x) \left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)} dx,$$

provided that the integrals converge.

- Setting $a = -\infty$, $b = +\infty$ and $w(x) = (1/\pi)(1/(1 + x^2))$, $x \in \mathbb{R}$, satisfying the required assumptions, Theorem 2.1 gives

$$\int_{-\infty}^{+\infty} \sqrt{f^4(x) + \frac{1}{\pi^4} \left(\int_{-\infty}^{+\infty} f(t) \frac{1}{1 + t^2} dt \right)^4} \frac{1}{1 + x^2} dx$$
$$\leq \sqrt{2} \int_{-\infty}^{+\infty} f^2(x) \frac{1}{1 + x^2} dx,$$

provided that the integrals converge.

- Setting $a = -\infty$, $b = +\infty$ and $w(x) = (1/\sqrt{2\pi\sigma^2}) e^{-x^2/(2\sigma^2)}$, $x \in \mathbb{R}$, with $\sigma > 0$, satisfying the required assumptions, Theorem 2.1 gives

$$\int_{-\infty}^{+\infty} \sqrt{f^4(x) + \frac{1}{(2\pi\sigma^2)^2} \left(\int_{-\infty}^{+\infty} f(t) e^{-t^2/(2\sigma^2)} dt \right)^4} e^{-x^2/(2\sigma^2)} dx$$
$$\leq \sqrt{2} \int_{-\infty}^{+\infty} f^2(x) e^{-x^2/(2\sigma^2)} dx,$$

provided that the integrals converge.

- Setting $a = -\infty$, $b = +\infty$ and $w(x) = (1/\nu)e^{-x/\nu}/(1 + e^{-x/\nu})^2$, $x \in \mathbb{R}$, with $\nu > 0$, satisfying the required assumptions, Theorem 2.1 gives

$$\int_{-\infty}^{+\infty} \sqrt{f^4(x) + \frac{1}{\nu^4}\left(\int_{-\infty}^{+\infty} f(t)\frac{e^{-t/\nu}}{(1+e^{-t/\nu})^2}dt\right)^4}\frac{e^{-x/\nu}}{(1+e^{-x/\nu})^2}dx$$
$$\leq \sqrt{2}\int_{-\infty}^{+\infty} f^2(x)\frac{e^{-x/\nu}}{(1+e^{-x/\nu})^2}dx,$$

provided that the integrals converge.

2.3. **Probabilistic interpretation.** A probability version of Theorem 2.1 is presented below, dealing with the moments of random variables.

Theorem 2.2. *Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $b > a$, and X be a random variable with the probability density function $w : [a, b] \to [0, \infty)$. Then we have*

$$\mathbb{E}\left(\sqrt{f^4(X) + (\mathbb{E}(f(X)))^4}\right) \leq \sqrt{2}\mathbb{E}\left(f^2(X)\right),$$

where $\mathbb{E}$ denotes the expectation operator, provided that the moments converge.

Proof. By the law of the unconscious statistician applied twice, we have

$$\mathbb{E}\left(\sqrt{f^4(X) + (\mathbb{E}(f(X)))^4}\right) = \int_a^b \sqrt{f^4(x) + (\mathbb{E}(f(X)))^4}w(x)dx$$
$$= \int_a^b \sqrt{f^4(x) + \left(\int_a^b f(t)w(t)dt\right)^4}w(x)dx$$

and

$$\mathbb{E}\left(f^2(X)\right) = \int_a^b f^2(x)w(x)dx.$$

Applying Theorem 2.1 and proceeding by identification, it is immediate that

$$\mathbb{E}\left(\sqrt{f^4(X) + (\mathbb{E}(f(X)))^4}\right) \leq \sqrt{2}\mathbb{E}\left(f^2(X)\right).$$

This concludes the proof. □

This result provides a probabilistic interpretation of Theorem 2.1, establishing a direct connection between integral inequalities and moment inequalities for random variables.

3. Concluding remarks

In conclusion, we have extended a specific integral inequality to a weighted framework, introducing a general weight function w that enhances flexibility and broadens the range of possible applications. The resulting formulation admits a natural probabilistic interpretation, connecting the inequality to the moments of random variables. Future work may focus on deriving sharper bounds, developing multidimensional extensions, and exploring applications in stochastic analysis and statistical modeling.

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Conflict of interest statement

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