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REVISITING AND GENERALIZING A CONVEXITY INTEGRAL INEQUALITY

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ABSTRACT. This article revisits a little-studied convexity integral inequality, presenting eleven new propositions and a detailed generalization. Several examples are provided to illustrate the theory. Together, these results expand the scope of convex integral inequalities and improve our theoretical understanding of them.

1. INTRODUCTION

The concept of a convex function is fundamental to the fields of optimization theory, the study of inequalities, and the analysis of functional properties. Before commenting further, let us present the mathematical basis of this concept. Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$. A function $\phi : [a, b] \to \mathbb{R}$ is said to be convex if, for any $x, y \in [a, b]$ and any $\lambda \in [0, 1]$, we have

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y).$$

Geometrically, this means that the graph of the function ϕ lies below the chord joining any two points on the graph. If the function is twice differentiable, a classical sufficient condition for convexity is that its second derivative satisfies the inequality $\phi''(x) \geq 0$ for any $x \in [a, b]$; this implies that the first derivative ϕ' is non-decreasing.

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The concept of a convex function has been extensively investigated in various areas of mathematics, as documented in the literature (see [1–4, 7–12]).

The inequality properties of convex functions when composed with integrals of functions are of particular interest. The most well-known example of this is the Jensen integral inequality. For the purposes of this article, we present a modern inequality, as described in [5, 6], which provides an alternative to the Jensen integral inequality. More precisely, it provides an upper bound for the value of a convex function evaluated at the integral of a bounded function, in terms of the integral involving the derivative of the convex function. The precise statement is given in the theorem below.

Theorem 1.1. *[5, 6] Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that, for any $x \in [0, 1]$, $0 \leq f(x) \leq 1$. Let $\phi : [0, 1] \to \mathbb{R}$ be a twice differentiable convex function with $\phi(0) = 0$. Then we have*

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx.$$

This inequality is of particular interest as it offers an alternative perspective on the traditional Jensen integral inequality. Including the derivative ϕ' within the integral introduces a subtle yet compelling analytical structure. However, aside from the original sources, this result appears to have received little attention in the existing literature.

The aim of this article is to explore and develop this inequality further. First, we present the new consequences that arise from it, and then we derive ten propositions. As an original example, the following integral inequality will be demonstrated under some assumptions on f:

$$\int_0^{\int_0^1 f(x)dx} f(x)dx \leq \int_0^1 f^2(x)dx.$$

Our results primarily demonstrate the usefulness and scope of this relatively unknown inequality, thus rehabilitating it to some extent. Next, we derive a more general version of the theorem by relaxing the boundedness assumption on the function f and the assumption $\phi(0) = 0$. This generalization significantly broadens the applicability of the inequality and deepens our theoretical understanding.

The rest of the article is organized as follows: Section 2 presents the ten propositions. The general version of the inequality is presented in Section 3, supplemented by another proposition. Section 4 gives a conclusion.

2. Consequences of Theorem 1.1

The proposition below offers an alternative perspective on Theorem 1.1, making use of the composition function $f(\phi^{-1})$.

Proposition 2.1. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that, for any $x \in [0,1]$, $0 \leq f(x) \leq 1$. Let $\phi : [0,1] \to \mathbb{R}$ be a twice differentiable convex function with $\phi(0) = 0$. Then we have*

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^{\phi(1)} f(\phi^{-1}(x))dx.$$

Proof. It follows from Theorem 1.1 and the change of variables $y = \phi(x)$ with $\phi(0) = 0$ that

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx = \int_{\phi(0)}^{\phi(1)} f(\phi^{-1}(y))dy = \int_0^{\phi(1)} f(\phi^{-1}(y))dy.$$

Standardizing the notation, we have

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^{\phi(1)} f(\phi^{-1}(x))dx.$$

This concludes the proof of Proposition 2.1. □

Some examples of this proposition are given below.

- Taking $\phi(x) = x^p$ with $p > 1$, then ϕ is convex, and we have $\phi(0) = 0$ and $\phi(1) = 1$, so that

$$\left(\int_0^1 f(x)dx\right)^p \leq \int_0^1 f(x^{1/p})dx.$$

- Taking $\phi(x) = \exp(x) - 1$, then ϕ is convex, and we have $\phi(0) = 0$ and $\phi(1) = \exp(1) - 1$, so that

$$\exp\left(\int_0^1 f(x)dx\right) - 1 \leq \int_0^{\exp(1)-1} f(\log(1+x))dx.$$

We then deduce

$$\int_0^1 f(x)dx \leq \log\left(1+\int_0^{\exp(1)-1} f(\log(1+x))dx\right).$$

- Taking $\phi(x)=\sqrt{x^2+1}-1$, then ϕ is convex, and we have $\phi(0)=0$ and $\phi(1)=\sqrt{2}-1$, so that

$$\sqrt{\left(\int_0^1 f(x)dx\right)^2+1}-1 \leq \int_0^{\sqrt{2}-1} f(\sqrt{(x+1)^2-1})dx.$$

This is equivalent to

$$\sqrt{\left(\int_0^1 f(x)dx\right)^2+1} \leq 1+\int_0^{\sqrt{2}-1} f(\sqrt{(x+1)^2-1})dx.$$

- Taking $\phi(x)=1-\cos((\pi/2)x)$, then ϕ is convex, and we have $\phi(0)=0$ and $\phi(1)=1$, so that

$$1-\cos\left(\frac{\pi}{2}\int_0^1 f(x)dx\right) \leq \int_0^1 f\left(\frac{2}{\pi}\arccos(1-x)\right)dx.$$

We thus derive the following original inequality:

$$\int_0^1 f\left(\frac{2}{\pi}\arccos(1-x)\right)dx+\cos\left(\frac{\pi}{2}\int_0^1 f(x)dx\right) \geq 1.$$

- Taking $\phi(x)=\tan((\pi/4)x)$, then ϕ is convex, and we have $\phi(0)=0$ and $\phi(1)=1$, so that

$$\tan\left(\frac{\pi}{4}\int_0^1 f(x)dx\right) \leq \int_0^1 f\left(\frac{4}{\pi}\arctan(1-x)\right)dx.$$

- Taking $\phi(x)=-\log(1-x)$, then ϕ is convex, and we have $\phi(0)=0$ and $\phi(1)=\lim_{t\to 0}[-\log(t)]=+\infty$, so that

$$-\log\left(1-\int_0^1 f(x)dx\right) \leq \int_0^{+\infty} f\left(1-\exp(-x)\right)dx.$$

Composing both sides by the exponential function, we obtain

$$\frac{1}{1-\int_0^1 f(x)dx} \leq \exp\left[\int_0^{+\infty} f\left(1-\exp(-x)\right)dx\right].$$

In the rest of the study, for the sake of simplicity, we will only focus on power functions for the illustrative examples.

The proposition below provides a refined upper bound for the function ϕ composed by its own integral under specific assumptions.

Proposition 2.2. *Let $\phi : [0,1] \to \mathbb{R}$ be a twice differentiable convex function with $\phi(0) = 0$ and, for any $x \in [0,1]$, $0 \le \phi(x) \le 1$. Then we have*

$$\phi\left(\int_0^1 \phi(x)dx\right) \le \frac{1}{2}\phi^2(1).$$

Proof. Applying Proposition 2.1 to $f = \phi$, we get

$$\phi\left(\int_0^1 \phi(x)dx\right) \le \int_0^{\phi(1)} \phi(\phi^{-1}(x))dx = \int_0^{\phi(1)} xdx = \left[\frac{1}{2}x^2\right]_{x=0}^{x=\phi(1)} = \frac{1}{2}\phi^2(1).$$

This ends the proof of Proposition 2.2. □

Taking $\phi(x) = x^p$ with $p > 1$, this proposition yields

$$\frac{1}{(p+1)^p} = \left(\int_0^1 x^p dx\right)^p \le \frac{1}{2}\phi^2(1) = \frac{1}{2},$$

which can be easily shown.

An elegant inequality combining Theorem 1.1 and the Jensen integral inequality is derived in the proposition below.

Proposition 2.3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that, for any $x \in [0,1]$, $0 \le f(x) \le 1$. Let $\phi : [0,1] \to \mathbb{R}$ be a twice differentiable convex function with $\phi(0) = 0$ and $\phi(1) = 1$. Then we have*

$$\phi\left(\int_0^1 f(x)dx\right) \le \min\left(\int_0^1 \phi(f(x))dx, \int_0^1 f(\phi^{-1}(x))dx\right) \le \int_0^1 f(x)dx.$$

Proof. Since $\int_0^1 dx = 1$, applying the Jensen integral inequality to the convex function ϕ, we get

$$\phi\left(\int_0^1 f(x)dx\right) \le \int_0^1 \phi(f(x))dx. \tag{2.1}$$

On the other hand, applying Proposition 2.1 with $\phi(1) = 1$, we obtain

$$\phi\left(\int_0^1 f(x)dx\right) \le \int_0^{\phi(1)} f(\phi^{-1}(x))dx = \int_0^1 f(\phi^{-1}(x))dx. \tag{2.2}$$

Combining Equations (2.1) and (2.2), we find that

$$\phi\left(\int_0^1 f(x)dx\right) \leq \min\left(\int_0^1 \phi(f(x))dx, \int_0^1 f(\phi^{-1}(x))dx\right).$$

Applying the basic convexity inequality with $\lambda = f(x) \in [0,1]$ for any $x \in [0,1]$, and using $\phi(0) = 0$ and $\phi(1) = 1$, we obtain

$$\begin{aligned}\phi\left(f(x)\right) &= \phi\left(f(x) \times 1 + (1 - f(x)) \times 0\right)\\ &\leq f(x)\phi(1) + (1 - f(x))\,\phi(0)\\ &= f(x) \times 1 + (1 - f(x)) \times 0 = f(x).\end{aligned}$$

So we have

$$\int_0^1 \phi(f(x))dx \leq \int_0^1 f(x)dx,$$

which implies that

$$\begin{aligned}&\phi\left(\int_0^1 f(x)dx\right) \leq \min\left(\int_0^1 \phi(f(x))dx, \int_0^1 f(\phi^{-1}(x))dx\right)\\ &\leq \int_0^1 \phi(f(x))dx \leq \int_0^1 f(x)dx.\end{aligned}$$

This concludes the proof of Proposition 2.3. □

In particular, taking $\phi(x) = x^p$ with $p > 1$, this proposition yields

$$\left(\int_0^1 f(x)dx\right)^p \leq \min\left(\int_0^1 (f(x))^p dx, \int_0^1 f(x^{1/p})dx\right) \leq \int_0^1 f(x)dx.$$

Note that we have imposed $\phi(1) = 1$ in order to harmonize with the notations of the Jensen integral inequality and the integral of f only. In full generality, we have

$$\begin{aligned}&\phi\left(\int_0^1 f(x)dx\right) \leq \min\left(\int_0^1 \phi(f(x))dx, \int_0^{\phi(1)} f(\phi^{-1}(x))dx\right)\\ &\leq \int_0^1 \phi(f(x))dx \leq \int_0^1 f(x)dx.\end{aligned}$$

The proposition below demonstrates how Theorem 1.1 can be extended by relaxing the bound of 1 in the assumption made on f.

Proposition 2.4. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that there exists $\kappa > 0$ satisfying, for any $x \in [0,1]$, $0 \leq f(x) \leq \kappa$. Let $\phi : [0,1] \to \mathbb{R}$ be a twice* *differentiable convex function with $\phi(0) = 0$. Then we have*

$$\kappa\phi\left(\frac{1}{\kappa}\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx.$$

Proof. It is sufficient to note that the function $f_\star = f/\kappa$ satisfies $0 \leq f(x)/\kappa \leq 1$ for any $x \in [0,1]$. Applying Theorem 1.1 to $f_\star$ instead of f yields

$$\phi\left(\frac{1}{\kappa}\int_0^1 f(x)dx\right) = \phi\left(\int_0^1 f_\star(x)dx\right) \leq \int_0^1 f_\star(x)\phi'(x)dx = \frac{1}{\kappa}\int_0^1 f(x)\phi'(x)dx,$$

which can be rewritten as

$$\kappa\phi\left(\frac{1}{\kappa}\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx.$$

This concludes the proof of Proposition 2.4. □

In particular, taking $\phi(x) = x^p$ with $p > 1$, this proposition yields

$$\kappa^{1-p}\left(\int_0^1 f(x)dx\right)^p \leq \int_0^1 f(x)\phi'(x)dx.$$

The proposition below presents an original upper bound for the case when f is assumed to be differentiable. The proof remains based on Theorem 1.1.

Proposition 2.5. *Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that, for any $x \in [0,1]$, $0 \leq f(x) \leq 1$. Let $\phi : [0,1] \to \mathbb{R}$ be a twice differentiable convex function with $\phi(0) = 0$. Then we have*

$$\phi\left(\int_0^1 f(x)dx\right) + \int_0^1 f'(x)\phi(x)dx \leq f(1)\phi(1).$$

Proof. It follows from an integration by parts, $\phi(0) = 0$ and Theorem 1.1, that

$$\begin{aligned}\int_0^1 f'(x)\phi(x)dx &= [f(x)\phi(x)]_{x=0}^{x=1} - \int_0^1 f(x)\phi'(x)dx \\ &= f(1)\phi(1) - f(0)\phi(0) - \int_0^1 f(x)\phi'(x)dx \\ &= f(1)\phi(1) - \int_0^1 f(x)\phi'(x)dx \\ &\leq f(1)\phi(1) - \phi\left(\int_0^1 f(x)dx\right).\end{aligned}$$

This can be arranged as follows:

$$\phi\left(\int_0^1 f(x)dx\right) + \int_0^1 f'(x)\phi(x)dx \le f(1)\phi(1).$$

This concludes the proof of Proposition 2.5. □

In particular, taking $\phi(x) = x^p$ with $p > 1$, this proposition yields

$$\left(\int_0^1 f(x)dx\right)^p + \int_0^1 f'(x)x^p dx \le f(1).$$

The proposition below highlights a composition integral inequality, in the spirit of the Steffensen integral inequality.

Proposition 2.6. *Let $f : [0,1] \to \mathbb{R}$ be a differentiable non-decreasing function such that, for any $x \in [0,1]$, $0 \le f(x) \le 1$. Then we have*

$$\int_0^{\int_0^1 f(x)dx} f(x)dx \le \int_0^1 f^2(x)dx.$$

Proof. Since f is a differentiable non-decreasing function, the function $\phi(x) = \int_0^x f(t)dt$ is convex, i.e., $\phi''(x) = f'(x) \ge 0$ for any $x \in [0,1]$, and we have $\phi(0) = \int_0^0 f(t)dt = 0$. Applying Theorem 1.1 to this function ϕ yields

$$\int_0^{\int_0^1 f(x)dx} f(x)dx = \phi\left(\int_0^1 f(x)dx\right) \le \int_0^1 f(x)\phi'(x)dx$$
$$= \int_0^1 f(x)f(x)dx = \int_0^1 f^2(x)dx.$$

This ends the proof of Proposition 2.6. □

In keeping with the spirit of the Steffensen integral inequality, the proposition below highlights another integral inequality.

Proposition 2.7. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that, for any $x \in [0,1]$, $0 \le f(x) \le 1$. Let $g : [0,1] \to \mathbb{R}$ be a differentiable non-decreasing function. Then we have*

$$\int_0^{\int_0^1 f(x)dx} g(x)dx \le \int_0^1 f(x)g(x)dx.$$

Proof. Since g is a differentiable non-decreasing function, the function $\phi(x) = \int_0^x g(t)dt$ is convex, i.e., $\phi''(x) = g'(x) \ge 0$ for any $x \in [0,1]$ and we have $\phi(0) =$

$\int_0^0 g(t)dt = 0$. Applying Theorem 1.1 to this function ϕ yields

$$\int_0^{\int_0^1 g(x)dx} f(x)dx = \phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx = \int_0^1 f(x)g(x)dx.$$

This ends the proof of Proposition 2.7. □

Taking $g = f$, Proposition 2.7 reduces to Proposition 2.6. Proposition 2.7 can also be viewed as part of the two-sided Steffensen inequality, with a proof based on convexity.

A generalization of this proposition is given below.

Proposition 2.8. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that, for any $x \in [0,1]$, $0 \leq f(x) \leq 1$. Let $g : [0,1] \to \mathbb{R}$ be a differentiable non-decreasing function. Let $h : \mathbb{R} \to \mathbb{R}$ be a twice differentiable non-decreasing convex function with $h(0) = 0$. Then we have*

$$h\left(\int_0^{\int_0^1 f(x)dx} g(x)dx\right) \leq \int_0^1 f(x)g(x)h'\left(\int_0^x g(t)dt\right)dx.$$

Proof. Since g is a differentiable non-decreasing function, the function $k(x) = \int_0^x g(t)dt$ is convex, i.e., $k''(x) = g'(x) \geq 0$ for any $x \in [0,1]$. Since h is non-decreasing and convex, the composed function $\phi = h(k)$ is also convex, and we have $\phi(0) = h(k(0)) = h\left(\int_0^0 g(t)dt\right) = h(0) = 0$. Applying Theorem 1.1 to this function ϕ, also satisfying $\phi'(x) = g(x)h'\left(\int_0^x g(t)dt\right)$, yields

$$h\left(\int_0^{\int_0^1 f(x)dx} g(x)dx\right) = \phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx$$
$$= \int_0^1 f(x)g(x)h'\left(\int_0^x g(t)dt\right)dx.$$

This concludes the proof of Proposition 2.8. □

Taking $h(x) = x$, Proposition 2.8 reduces to Proposition 2.7. Introducing a differentiable non-decreasing function $\ell : \mathbb{R} \to [0, +\infty)$ and taking $h(x) = \int_0^x \ell(t)dt$, Proposition 2.8 gives the following original inequality involving a triple integral composition:

$$\int_0^{\int_0^{\int_0^1 f(x)dx} g(x)dx} \ell(t)dt \leq \int_0^1 f(x)g(x)\ell\left(\int_0^x g(t)dt\right)dx.$$

The proposition below provides an alternative proof that Theorem 1.1 has the ability to extend itself. This is demonstrated by introducing a power parameter p.

Proposition 2.9. *Let $p > 1$. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that, for any $x \in [0,1]$, $0 \leq f(x) \leq 1$. Let $\phi : [0,1] \to \mathbb{R}$ be a twice differentiable convex function with $\phi(0) = 0$. Then we have*

$$\phi\left(\int_0^1 f(x)dx\right) \leq p^{1/p}\left(\int_0^1 f(x)\phi'(x)\phi^{p-1}(x)dx\right)^{1/p}.$$

Proof. Since ϕ is convex and $p > 1$, $\phi_\star = \phi^p$ is also convex and satisfies $\phi_\star(0) = \phi^p(0) = 0$. Applying Theorem 1.1 to $\phi_\star$ instead of ϕ yields

$$\phi^p\left(\int_0^1 f(x)dx\right) = \phi_\star\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi_\star'(x)dx = \int_0^1 f(x)\left(\phi^p(x)\right)' dx$$
$$= \int_0^1 f(x)\left(p\phi'(x)\phi^{p-1}(x)\right) dx = p\int_0^1 f(x)\phi'(x)\phi^{p-1}(x)dx.$$

Raising both sides to the power $1/p$ gives

$$\phi\left(\int_0^1 f(x)dx\right) \leq p^{1/p}\left(\int_0^1 f(x)\phi'(x)\phi^{p-1}(x)dx\right)^{1/p}.$$

This concludes the proof. □

In particular, taking $\phi(x) = x^q$ with $q > 1$, this proposition yields

$$\left(\int_0^1 f(x)dx\right)^q \leq p^{1/p}q^{1/p}\left(\int_0^1 f(x)x^{pq-1}dx\right)^{1/p}.$$

The proposition below concerns the standard situation in which the "log-convexity" of the function ϕ is considered.

Proposition 2.10. *Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that, for any $x \in [0,1]$, $0 \leq f(x) \leq 1$. Let $\phi : [0,1] \to (0,+\infty)$ be a differentiable function with $\phi(0) = 1$ such that $\log(\phi)$ is convex. Then we have*

$$\phi\left(\int_0^1 f(x)dx\right) \leq \exp\left(\int_0^1 f(x)\frac{\phi'(x)}{\phi(x)}dx\right).$$

Proof. Let us set $\phi_\dagger = \log(\phi)$. Then we know that $\phi_\dagger$ is convex. Furthermore, it satisfies $\phi_\dagger(1) = \log(\phi(0)) = \log(1) = 0$. Applying Theorem 1.1 to $\phi_\dagger$ instead of ϕ

yields

$$\log\left(\int_0^1 f(x)dx\right) = \phi_\dagger\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi_\dagger'(x)dx$$
$$= \int_0^1 f(x)\left(\log(\phi(x))\right)' dx = \int_0^1 f(x)\frac{\phi'(x)}{\phi(x)}dx.$$

Composing both sides with the exponential function gives

$$\phi\left(\int_0^1 f(x)dx\right) \leq \exp\left(\int_0^1 f(x)\frac{\phi'(x)}{\phi(x)}dx\right).$$

This ends the proof. □

These consequences of Theorem 1.1, along with the proposed examples, can be applied in diverse settings, including sharper integral-mean estimates in convex analysis, stability bounds in optimization, concentration results in probability theory, and error analysis in numerical integration. We leave these applied aspects for future work.

3. Generalizations of Theorem 1.1

After analyzing the proof of Theorem 1.1 in [5], it appears that it can be generalized in several ways. The theorem below formalizes this by introducing interval bounds a and b, a general function G, a new variable y, and a general integral inequality assumption relating f and G.

Theorem 3.1. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be an almost everywhere continuous function such that, for any $x \in [a, b]$, $f(x) \geq 0$. Let $G : [a, b] \to \mathbb{R}$ be a differentiable function such that, for any $x \in [a, b]$,*

$$\int_a^x f(t)dt \leq G(x). \tag{3.1}$$

Let $\phi : [0, +\infty) \to \mathbb{R}$ be a twice differentiable convex function. Then, for any $y \in [a, b]$, we have

$$\phi\left(\int_a^y f(x)dx\right) \leq \phi(0) + \int_0^{G(y)} f(G^{-1}(x))\frac{1}{G'(G^{-1}(x))}\phi'(x)dx.$$

Proof. For any $z \in [a,b]$, let us set

$$\psi(z) = \phi\left(\int_a^z f(x)dx\right).$$

Then, differentiating with respect to z and using standard differentiation rules, we have

$$\psi'(z) = \left(\int_a^z f(x)dx\right)' \phi'\left(\int_a^z f(x)dx\right) = f(z)\phi'\left(\int_a^z f(x)dx\right). \tag{3.2}$$

Since ϕ is convex, ϕ' is non-decreasing. This and Equation (3.1) imply that

$$\phi'\left(\int_a^z f(x)dx\right) \le \phi'(G(z)). \tag{3.3}$$

It follows from Equations (3.2) and (3.3), and the fact that $f(x) \ge 0$ for any $x \in [a,b]$, that, for any $z \in [a,b]$,

$$\psi'(z) \le f(z)\phi'\left(G(z)\right).$$

Integrating both sides for $z \in [a,y]$ and any $y \in [a,b]$, we get

$$\psi(y) - \psi(a) = \int_a^y \psi'(z)dz \le \int_a^y f(z)\phi'\left(G(z)\right)dz.$$

Using the definition of ψ and $\int_a^a f(x)dx = 0$, we obtain

$$\phi\left(\int_a^y f(x)dx\right) = \psi(y) \le \psi(a) + \int_a^y f(z)\phi'\left(G(z)\right)dz$$
$$= \phi(0) + \int_a^y f(z)\phi'\left(G(z)\right)dz.$$

Making the change of variables $w = G(z)$, i.e., $z = G^{-1}(w)$, in the last integral and using standard differentiation rules, we find that

$$\phi\left(\int_a^y f(x)dx\right) \le \phi(0) + \int_0^{G(y)} f(G^{-1}(w))\phi'(w)\left(\frac{1}{G'(G^{-1}(w))}dw\right).$$

Standardizing the notation, we have

$$\phi\left(\int_a^y f(x)dx\right) \le \phi(0) + \int_0^{G(y)} f(G^{-1}(x))\frac{1}{G'(G^{-1}(x))}\phi'(x)dx.$$

This concludes the proof. □

Note that f is assumed to be almost everywhere continuous on $[a, b]$, meaning that there can exist a finite number of points into $[a, b]$ such that it is not continuous. As a result, it is not necessarily bounded. We may think of the function $1/\sqrt{x}$ which is integrable on $[0, 1]$, but not continuous at $a = 0$.

Taking $a = 0$, $b = 1$, $y = 1$, $G(x) = x$ for any $x \in [0, 1]$ and $\phi(0) = 0$, Theorem 3.1 contains Theorem 1.1.

Taking $a = 0$, $b = 1$, $y = 1$, $G(x) = \kappa x$, we get

$$\phi\left(\int_0^1 f(x)dx\right) \leq \phi(0) + \frac{1}{\kappa}\int_0^\kappa f\left(\frac{x}{\kappa}\right)\phi'(x)dx.$$

In particular, this relaxes the special choice of $\kappa = 1$, as made in Theorem 1.1, as well as $\phi(0) = 0$.

Taking $f = h'$ for a differentiable function h satisfying the required assumptions, $G(x) = x$ for any $x \in [0, 1]$, $\phi(0) = 0$, we obtain, for any $y \in [a, b]$,

$$\phi\left(h(y) - h(a)\right) \leq \phi(0) + \int_0^y h'(x)\phi'(x)dx.$$

We can also use the function G in Equation (3.1) to overcome the restrictive boundedness assumption on f in Theorem 1.1. We support this claim in the proposition below, dealing with an integrability assumption.

Proposition 3.1. *Let $p > 1$, $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f : [a, b] \to \mathbb{R}$ be an almost everywhere continuous function such that, for any $x \in [a, b]$, $f(x) \geq 0$, and*

$$\int_a^b f^p(x)dx < \infty. \tag{3.4}$$

Let $\theta = \left(\int_a^b f^p(x)dx\right)^{1/p}$. Let $\phi : [0, +\infty) \to \mathbb{R}$ be a twice differentiable convex function. Then, for any $y \in [a, b]$, we have

$$\phi\left(\int_a^y f(x)dx\right) \leq \phi(0) + \frac{p}{\theta^{p/(p-1)}(p-1)}$$
$$\cdot \int_0^{\theta(y-a)^{1-1/p}} f\left(\left(\frac{x}{\theta}\right)^{p/(p-1)} + a\right) x^{1/(p-1)}\phi'(x)dx.$$

Proof. It follows from Equation (3.4) and the Hölder integral inequality that, setting $q = p/(p-1)$, for any $x \in [a, b]$,

$$\int_a^x f(t)dt \le \left(\int_a^x f^p(t)dt\right)^{1/p} \left(\int_a^x 1dt\right)^{1/q} = \left(\int_a^x f^p(t)dt\right)^{1/p} (x-a)^{1/q}$$
$$\le \theta(x-a)^{1/q}.$$

Therefore, we can apply Theorem 3.1 with

$$G(x) = \theta(x-a)^{1/q}.$$

Noting that $G^{-1}(x) = (x/\theta)^q + a$ and $G'(G^{-1}(x)) = (\theta/q)(x/\theta)^{1-q}$, this yields, for any $y \in [a, b]$,

$$\phi\left(\int_a^y f(x)dx\right) \le \phi(0) + \int_0^{G(y)} f(G^{-1}(x))\frac{1}{G'(G^{-1}(x))}\phi'(x)dx$$
$$= \phi(0) + \frac{q}{\theta}\int_0^{\theta(y-a)^{1/q}} f\left(\left(\frac{x}{\theta}\right)^q + a\right)\left(\frac{x}{\theta}\right)^{q-1}\phi'(x)dx$$
$$= \phi(0) + \frac{p}{\theta^{p/(p-1)}(p-1)}\int_0^{\theta(y-a)^{1-1/p}} f\left(\left(\frac{x}{\theta}\right)^{p/(p-1)} + a\right)x^{1/(p-1)}\phi'(x)dx.$$

This concludes the proof. □

Thus, the proposition holds for functions that are not necessarily bounded. This proposition demonstrates the versatility of Theorem 3.1 and illustrates how it extends the applicability of Theorem 1.1 to a broader context.

4. Conclusion

In conclusion, this work provides new information about a convexity integral inequality that has not received much attention. It offers a deeper theoretical framework through examples and generalization. Future research could explore applications in functional analysis, probability, and numerical methods, as well as analogous inequalities under weaker regularity conditions or in higher dimensions.

APPENDIX

As an appendix, the theorem below proposes a simple and general lower bound in relation to Theorem 3.1.

Theorem 4.1. *Let* $a, b \in \mathbb{R} \cup \{\pm\infty\}$ *with* $a < b$*, and* $f : [a, b] \to \mathbb{R}$ *be an almost everywhere continuous function such that, for any* $x \in [a, b]$*,* $f(x) \geq 0$*. Let* $\phi : [0, +\infty) \to \mathbb{R}$ *be a twice differentiable convex function. Then, for any* $y \in [a, b]$*, we have*

$$\phi\left(\int_a^y f(x)dx\right) \geq \phi(0) + \phi'(0)\int_a^y f(x)dx.$$

Proof. For any $z \in [a, b]$, let us set

$$\psi(z) = \phi\left(\int_a^z f(x)dx\right).$$

Then, differentiating with respect to z and using standard differentiation rules, we have

$$\psi'(z) = \left(\int_a^z f(x)dx\right)' \phi'\left(\int_a^z f(x)dx\right) = f(z)\phi'\left(\int_a^z f(x)dx\right). \tag{4.1}$$

Since ϕ is convex, ϕ' is non-decreasing. This and the fact that $f(x) \geq 0$ for any $x \in [a, b]$, we have

$$\phi'\left(\int_a^z f(x)dx\right) \geq \phi'\left(\int_a^z 0dx\right) = \phi'(0). \tag{4.2}$$

It follows from Equations (4.1) and (4.2), and again the fact that $f(x) \geq 0$ for any $x \in [a, b]$, that, for any $z \in [a, b]$,

$$\psi'(z) \geq f(z)\phi'(0).$$

Integrating both sides for $z \in [a, y]$ and any $y \in [a, b]$, we get

$$\psi(y) - \psi(a) = \int_a^y \psi'(z)dz \geq \int_a^y f(z)\phi'(0)dz = \phi'(0)\int_a^y f(z)dz.$$

Using the definition of ψ and $\int_a^a f(x)dx = 0$, we obtain

$$\phi\left(\int_a^y f(x)dx\right) = \psi(y) \geq \psi(a) + \phi'(0)\int_a^y f(z)dz = \phi(0) + \phi'(0)\int_a^y f(z)dz.$$

Standardizing the notation, we get

$$\phi\left(\int_a^y f(x)dx\right) \geq \phi(0) + \phi'(0)\int_a^y f(x)dx.$$

This concludes the proof. □

In particular, for $a = 0$, $b = 1$, $\phi(0) = 0$ and $y = 1$, we establish that

$$\phi\left(\int_0^1 f(x)dx\right) \geq \phi'(0)\int_0^1 f(x)dx.$$

This provides a simple lower bound counterpart to Theorem 1.1. Theorem 4.1 can be used as a starting point for further work in this area.

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Conflict of interest statement

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References

[1] E.F. Beckenbach: *Convex functions*, Bull. Amer. Math. Soc., **54** (1948), 439–460.

[2] R. Bellman: *On the approximation of curves by line segments using dynamic programming*, Commun. ACM, **4**(6) (1961), 284.

[3] J. Hadamard: *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171–215.

[4] C. Hermite: *Sur deux limites d'une intégrale définie*, Mathesis, **3** (1883), 82.

[5] M.M. Iddrisu, C.A. Okpoti, K.A. Gbolagade: *A proof of Jensen's inequality through a new Steffensen's inequality*, Adv. Inequal. Appl., **2014** (2014), 1–7.

[6] M.M. Iddrisu, C.A. Okpoti, K.A. Gbolagade: *Geometrical proof of new Steffensen's inequality and applications*, Adv. Inequal. Appl., **2014** (2014), 1–10.

[7] J.L.W.V. Jensen: *Om konvekse Funktioner og Uligheder mellem Middelvaerdier*, Nyt Tidsskr. Math. B., **16** (1905), 49–68.

[8] J.L.W.V. Jensen: *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math., **30** (1906), 175–193.

[9] C.P. NICULESCU: *Convexity according to the geometric mean*, Math. Ineq. Appl., **3**(2) (2000), 155–167.

[10] D.S. MITRINOVIĆ: *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[11] D.S. MITRINOVIĆ, J.E. PEČARIĆ, A.M. FINK: *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[12] A.W. ROBERTS, P.E. VARBERG: *Convex Functions*, Academic Press, New York, 1973.

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