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WEAK FORMULATION ANALYSIS OF AN OVERHEAD CRANE SYSTEM

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ABSTRACT. In this work, we consider a system of differential equations modeling an overhead crane consisting of a motorized platform moving along a horizontal beam using a flexible cable, supporting a load of mass M and subject to velocity and position control. For this system, we establish a stability result in the sense of Lyapunov, and by adapting different methods including the one presented in [11, 13], we establish for this system results of existence, uniqueness and regularity of a weak solution.

1. INTRODUCTION

Overhead cranes are essential industrial equipment, used for lifting and moving heavy loads in environments such as factories, shipyards, and warehouses. Their mathematical study is crucial to optimize their design, stability, and operation. In our study, we consider an overhead crane consisting of a motorized platform of mass m moving along a horizontal beam using a flexible cable, supporting a load of mass M and subject to velocity and position control. We assume that the cable is fully flexible and non-extensible, and that its length is constant (equal to unity). In addition, transverse and angular displacements are assumed to be small and

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friction is neglected. Furthermore, the masses m and M are considered point-like and the angle of the cable with respect to the vertical axis x is small everywhere. The mathematical model of this overhead crane can be described by the following hybrid PDE-ODE system:

$$y_{tt}(x,t) - (ay_x)_x(x,t) = 0, \quad 0 < x < 1, \ t > 0 \tag{1.1}$$

$$-a(0)y_x(0,t) + my_{tt}(0,t) = F(t), \quad t > 0 \tag{1.2}$$

$$a(1)y_x(1,t) + My_{tt}(1,t) = 0, \quad t > 0 \tag{1.3}$$

$$y(x,0) = y_0(x), y_t(x,0) = z_0(x), \quad 0 < x < 1 \tag{1.4}$$

where $y(x,t)$ is the transverse deflection of the cable at position x and time t. $F(t)$ is the driving force of the trolley on the rails. In this work, we take $F(t) = -\lambda y(0,t) - \mu y_t(0,t)$. Thus, the control commands at the edge $x = 0$ depend only on the position and velocity of the platform (i.e., $y_t(0,t), y(0,t)$). The control parameters λ and μ are strictly positive constants. Moreover, the coefficient of the spatial variable a(x) representing the tension modulus of the cable and satisfies the following standard conditions (see [4]): $a(x) \in H(0,1)$, $a(x) > a > 0$.

Note that this system was studied by Francis Conrad and Abdelkrim Mifdal [10]. In their work, the authors demonstrate the wellposedness of this system in the sense of semigroup theory. Then, using the Hille-Yosida theorem, the existence and uniqueness of a strong solution as well as a mild solution for the system under consideration is proven. Furthermore, by applying the Lasalle invariance principle, they obtained a strong stability result while proving that the system is not uniformly stable. In [7], the long-term behavior of an overhead crane with input delays in boundary control is studied. Still relying on semigroup theory, the authors showed that the closed-loop system is well-posed. Their analysis also established the convergence of the solutions to a stationary position (dependent on the initial data) using the LaSalle invariance principle, with an exponential convergence rate deduced by the frequency domain method. Finally, the asymptotic distribution of the system's eigenvalues and eigenfunctions was characterized, leading to the proof of the differentiability of the associated semigroup and the verification of the spectral growth condition.

More generally, numerous studies on variants of the overhead crane system exploit semigroup theory to establish the existence and uniqueness of a solution for these models (see, for example, [2, 4, 10]).

In this work, we investigate the existence, uniqueness, and higher regularity of a weak solution to the system (1.1)–(1.4), formulated in a variational framework. This study provides a rigorous mathematical framework for numerical simulations (finite elements, finite differences) and makes it possible to process irregular physical data. To prove existence, we use the intermediate spaces introduced in [13] and the Faedo-Galerkin method as in [17]. This approach, is adapted here to define the function spaces and scalar products appropriate for analyzing the system.

This article is organized as follows. Section 2 presents the formulation of the system (1.1)-(1.4) as an abstract Cauchy problem in an adapted Hilbert space, as well as a key result on the existence and uniqueness of solutions in the framework of semigroups. Also, an asymptotic stability analysis of the closed-loop system, based on the LaSalle invariance principle with an appropriately chosen Lyapunov functional, is presented. Section 3 defines the weak solution after presenting an appropriate spatial framework for the weak formulation of this system. In sections 4 and 5 , we respectively prove the existence, uniqueness and an additional regularity result for the solution of the problem under study, based on the fundamental results of existence, uniqueness and regularity of the solutions of the evolution equations, as presented in the works of Lions [13], Yosida [22] and Temam [20]. Finally, in Section 6, a conclusion summarizes the results obtained and opens up perspectives.

2. Preleminaries

Note that the total mechanical energy of the system (1.1)–(1.4) is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[y_t^2(x,t) + a(x)y_x^2(x,t)\right] dx + \frac{M}{2}y_t^2(1,t) + \frac{m}{2}y_t^2(0,t) + \frac{\lambda}{2}y^2(0,t). \tag{2.1}$$

The time derivative of the energy function E(t) is

$$\frac{d}{dt}E(t) = -\mu y_t^2(0,t). \tag{2.2}$$

Now let us introduce the following energy space:

$$\chi = H^1(0,1) \times L^2(0,1) \times \mathbb{R}^2 \tag{2.3}$$

such as for all $w_1, w_2 \in \chi$, with $w_i = (y_i, z_i, u_i, v_i)^T$, $i = 1, 2$. χ is a Hilbert space with the inner product

$$\langle w_1, w_2 \rangle_\chi = \int_0^1 [z_1 z_2 + a(x)(y_1)_x (y_2)_x]\, dx + M v_1 v_2 + m u_1 u_2 + \lambda y_1(0) y_2(0). \tag{2.4}$$

We denote by $\|.\|_\chi$ the corresponding norm. The superscript T stands for the transpose.

Consider the linear operator $A : D(A) \subset \chi \to \chi$ with the domain

$$D(A) = \left\{ (y, z, u, v) \in H^2(0,1) \times H^1(0,1) \times \mathbb{R}^2 : z(0) = u, z(1) = v \right\},$$

defined by

$$A \begin{pmatrix} y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} z \\ (a(x) y_x)_x \\ \frac{1}{m}(a(0) y_x(0) - \lambda y(0)) \\ -\frac{a(1)}{M} y_x(1) \end{pmatrix}. \tag{2.5}$$

And the linear operator B such that for all $(y, z, u, v) \in \mathcal{H}$,

$$B \begin{pmatrix} y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\mu}{m} u \\ 0 \end{pmatrix}. \tag{2.6}$$

Now we can writte (1.1)–(1.4) as a following Cauchy problem:

$$\begin{cases} \frac{d}{dt} w(t) = (A + B) w(t) \\ w(0) = w_0 \in \chi, \end{cases} \tag{2.7}$$

where $w(t) = (y(.,t), z(.,t), u(t), v(t))^T$, $w(0) = w_0$ for all $t > 0$.
Consider the following lemma

Lemma 2.1. *The operator $A+B$ is maximal monotone on χ and its domain $D(A+B)$ is dense in χ.*

Proof. For the proof of the Lemma 2.1 see [10]. □

Based on the theory of semigroups and the theorem of Hille-Yosida (see [9]), from Lemma 2.1, we obtain the following two results.

Theorem 2.1. *The operator* $A + B$ *generates a* C_0*-semigroup of contractions on* χ *denoted by* $\{S(t)\}_{t\geq 0}$.

Theorem 2.2.

(1) *For all initial data* $w_0 = (y_0, z_0, u_0, v_0) \in D(A)$*, there exists a unique function* $w(t) = (y(.,t), z(.,t), u(t), v(t)) \in D(A)$*, with the regularity*

$$y \in C^0\left(0,\infty; H^2(0,1)\right) \cap C^1\left(0,\infty; H^1(0,1)\right), \tag{2.8}$$

$$y(0,.), y(1,.) \in C^2\left(0,\infty;\mathbb{R}\right), \tag{2.9}$$

solution of the system (2.7).

(2) *For all initial data* $w_0 = (y_0, z_0, u_0, v_0) \in \chi$*, there exists a unique function* $w(t) = (y(.,t), z(.,t), u(t), v(t)) \in \chi$*, with the regularity:*

$$y \in C^0\left(0,\infty; H^1(0,1)\right) \cap C^1\left(0,\infty; L^2(0,1)\right). \tag{2.10}$$

solution of the system (2.7).

Remark 2.1. *The solutions defined respectively in first and second items of Theorem 2.2 are often called classical solution and mild solition (see [9, 17])*

2.1. **Stability in the sense of Lyapunov of** (2.7)**.** Let us recall the definition of a Lyapunov function in a Hilbert space.

Definition 2.1. *The functional* $\mathtt{L} : \chi \to \mathbb{R}$ *is called a Lyapunov functional of the evolution problem* (2.7) *if the following propositions are verified*

(1) $\mathtt{L}(w) > 0,\ \forall\, w \in \chi \setminus \{0\}$,
(2) $\mathtt{L}(0) = 0$,
(3) $\frac{d}{dt}\mathtt{L}(w) \leq 0,\ \forall\, w_0 \in \chi$.

Now, to analyze the stability of the system (2.7), consider the following Lyapunov candidate $\mathtt{L} : \chi \to \mathbb{R}$ defined by

$$\mathtt{L}(w) = \frac{1}{2}\int_0^1 \left[z^2 + a y_x^2\right] dx + \frac{M}{2}v^2 + \frac{m}{2}u^2 + \frac{\lambda}{2}y^2(0). \tag{2.11}$$

The time derivative of L along the classical solutions of (2.7) is

$$\frac{d}{dt}\mathtt{L}(w) = -\mu u^2 \leq 0. \tag{2.12}$$

Given the above and Theorem 2.1, the decay of the functional L extends to the mild solution. Thus, L is a Lyapunov functional for (2.7) and the system is therefore stable in the Lyapunov sense. Finally, the LaSalle invariance principle [14] provides the following stability result:

Theorem 2.3. *Assume that $w(t)$ is the mild solution of* (2.7) *for all $w_0 \in \chi$. Then $w(t) \to 0 \in \chi$ when $t \to \infty$.*

3. Weak solution

Let's start by multiplying (1.1) by $\phi \in H^1(0,1)$. After two integrations by parts over $(0,1)$ and using (1.2)–(1.4), we obtain the following equation:

$$\int_0^1 \left[y_{tt}\phi + a(x)y_x\phi_x\right] dx + My_{tt}(1)\phi(1) \tag{3.1}$$

$$+ my_{tt}(0)\phi(0) + \lambda y(0)\phi(0) + \mu y_t(0)\phi(0) = 0. \tag{3.2}$$

The first step is to define a spatial framework appropriate to the weak formulation, drawing on the work of H.T. Banks et al., [5] and Evans [11]. Let the spaces

$$\mathtt{Y} = \mathbb{R}^2 \times L^2(0,1) = \left\{\widehat{\varphi} = (\varphi(0), \varphi(1), \varphi), \varphi \in L^2(0,1)\right\}, \tag{3.3}$$

with the inner product

$$\langle \widehat{\varphi}, \widehat{\phi}\rangle_{\mathtt{Y}} = \langle \varphi, \phi\rangle_{L^2(0,1)} + M\varphi(1)\phi(1) + m\varphi(0)\phi(0) \tag{3.4}$$

and

$$\mathtt{X} = \mathbb{R}^2 \times H^1(0,1) = \left\{\widehat{\varphi} = (\varphi(0), \varphi(1), \varphi), \varphi \in H^1(0,1)\right\}, \tag{3.5}$$

with the inner product

$$\langle \widehat{\varphi}, \widehat{\phi}\rangle_{\mathtt{X}} = \langle a\varphi_x, \phi_x\rangle_{L^2(0,1)} + \lambda\varphi(0)\phi(0). \tag{3.6}$$

Notice that X′ is the dual of X and Y′ the dual of Y. Furthermore, the duality pairing between X and X′ denoted by $\langle ., .\rangle_{\mathtt{X},\mathtt{X}'}$ is a natural extension of the inner product in Y′.

Now consider the following bilinear form $\mathtt{b} : \mathtt{Y} \times \mathtt{Y} \to \mathbb{R}$ such as

$$\mathtt{b}\left(\widehat{\varphi}, \widehat{\phi}\right) = \mu\varphi(0)\phi(0). \tag{3.7}$$

We will consider the notation below $\widehat{\varphi} = (\varphi(0), \varphi(1), \varphi) = ({}^1\widehat{\varphi}, {}^2\widehat{\varphi}, {}^3\widehat{\varphi})$. Let us now give the definition of a weak solution.

Definition 3.1. *Let $T > 0$ be fixed. We say that $\widehat{y} = (y(0), y(1), y)$ is a weak solution of problem* (1.1) – (1.4) *on* $(0,1)$ *if*

$$\widehat{y} \in L^2(0,T;\mathtt{X}) \quad \textit{with} \quad \widehat{y}_t \in L^2(0,T;\mathtt{Y}), \quad \widehat{y}_{tt} \in L^2(0,T;\mathtt{X}')$$

and satisfies

$$\langle \widehat{y}_{tt}, \widehat{\phi}\rangle_{\mathtt{X},\mathtt{X}'} + \langle \widehat{\varphi}, \widehat{\phi}\rangle_{\mathtt{X}} + \mathtt{b}(\widehat{y}_t, \widehat{\phi}) = 0 \tag{3.8}$$

almost everywhere on $t \in (0,T)$ and for all $\widehat{\phi} \in H^1(0,1)$, with the following initial conditions

$$\widehat{y}(0) = \widehat{y}_0 = (y_0(0), y_0(1), y_0) \in \mathtt{X}, \tag{3.9}$$

$$\widehat{y}_t(0) = \widehat{z}_0 = (z_0(0), z_0(1), z_0) \in \mathtt{Y}. \tag{3.10}$$

Considering the Theorem 3.1 of [13], we have the following lemma wich gives meaning to the initial conditions (3.9)–(3.10).

Lemma 3.1. *Let $\mathtt{V}$ and $\mathtt{H}$ be two Hilbert spaces, such that $\mathtt{V}$ is dense and continuously embedded in $\mathtt{H}$. Assume that*

$$\widehat{y} \in L^2(0,T;\mathtt{V}), \tag{3.11}$$

$$\widehat{y}_t \in L^2(0,T;\mathtt{H}). \tag{3.12}$$

Then

$$\widehat{y} \in C\left([0,T], [\mathtt{V},\mathtt{H}]_{\frac{1}{2}}\right) \tag{3.13}$$

after, possibly, a modification on a set of measure zero.

Remark 3.1. *According to J.L. Lions et al [13], note that*

1. *Here, the space $[\mathtt{X},\mathtt{Y}]_{\frac{1}{2}}$ is call intermediate space and is defined as in [13].*
2. *$\mathtt{X}$ and $\mathtt{Y}$ being two Hilbert spaces such that $\mathtt{X}$ is dense and continuous in $\mathtt{Y}$, for all $\theta \in]0,1[$, $[\mathtt{X},\mathtt{Y}]'_\theta = [\mathtt{Y}',\mathtt{X}']_{1-\theta}$ with equivalent norms. (Here $\theta = \frac{1}{2}$.)*

4. Existence and uniqueness result of the weak solution

The main objective of this section is to show the following theorem.

Theorem 4.1. *The weak formulation* (3.8)–(3.10) *has a unique solution* $\widehat{y}$ *such that*

$$\widehat{y} \in L^{\infty}(0,T;\mathtt{X}), \tag{4.1}$$

$$\widehat{y}_t \in L^{\infty}(0,T;\mathtt{Y}), \tag{4.2}$$

$$\widehat{y} \in C\left([0,T],[\mathtt{X},\mathtt{Y}]_{\frac{1}{2}}\right), \tag{4.3}$$

$$\widehat{y}_t \in C\left([0,T],[\mathtt{X},\mathtt{Y}]'_{\frac{1}{2}}\right). \tag{4.4}$$

4.1. **Existence of the weak solution.** To show the existence of the weak solution, we first construct approximate solutions using a finite-dimensional Galerkin scheme, then derive uniform a priori estimates, and move to the limit to obtain a weak solution.

Let the following lemma be useful for the rest.

Lemma 4.1. *Let the space* $H^1(0,1)$*. Then there exists a infinite sequence of functions* $\{\phi_i\}_{i=1}^{\infty}$ *such that* $\{\phi_i\}_{i=1}^{\infty}$ *is an orthogonal basis of* $H^1(0,1)$ *and* $\{\phi_i\}_{i=1}^{\infty}$ *is an orthonormal basis of* $L^2(0,1)$.

Proof. The proof of this Lemma is easily obtained by adapting the operator L considered in Theorem A.1 of [17]. ☐

4.1.1. *Construction of approximate solutions.*

Lemma 4.2. *Let* $T > 0$ *be fixed. For all* $m \in \mathbb{N}^*$*, there exists a unique function* $\widehat{y}_m(t)$ *such that*

$$\widehat{y}_m(t) = \sum_{j=1}^{m} \gamma_{j,m}(t)\widehat{\phi}_j, \quad \textit{with } \gamma_{j,m}(t) \in \mathbb{R} \ (0 \le t \le T, j = 1,\ldots,m) \tag{4.5}$$

satisfying:

$$\langle (\widehat{y}_m)_{tt}, \widehat{\phi} \rangle_{\mathtt{Y}} + \langle \widehat{y}_m, \widehat{\phi} \rangle_{\mathtt{X}} + \mathtt{b}((\widehat{y}_m)_t, \widehat{\phi}) = 0, \ \forall\, \widehat{\phi} \in \widehat{K}_m \tag{4.6}$$

with the initial conditions:

$$\widehat{y}_m(0) = \widehat{y}_{m0}, \text{ and } \widehat{y}_{m0} = \sum_{i=1}^{m} \delta_{i,m}\widehat{\phi}_i \longrightarrow \widehat{y}_0 \text{ in } \mathtt{X} \text{ when } m \longrightarrow \infty, \tag{4.7}$$

with $\delta_{i,m} = \gamma_{i,m}(0)$ *and* $\alpha_{i,m} = (\gamma_{i,m})_t(0)$.

Proof. By Lemma 4.1, there exist finite dimensional spaces spanned by $\left\{\widehat{\phi}_i\right\}_{i=1}^{m}$ defined as

$$\forall m \in \mathbb{N}^*, \ \widehat{K}_m := \text{span}\left\{\widehat{\phi}_1, \ldots, \widehat{\phi}_m\right\}, \tag{4.8}$$

such that $\left\{\widehat{\phi}_i\right\}_{i=1}^{m}$ be a sequence of functions that is an orthonormal basis for $\mathtt{Y}$, and an orthogonal basis for $\mathtt{X}$. So for a fixed $m \in \mathbb{N}^*$, we consider the Galerkin approximation $\widehat{y}_m(t) \in \widehat{K}_m$:

$$\widehat{y}_m(t) = \sum_{j=1}^{m} \gamma_{j,m}(t)\widehat{\phi}_j, \tag{4.9}$$

with $\gamma_{j,m}(t) \in \mathbb{R}$, which are the solutions of the weak formulation (3.1) on $\widehat{K}_m$. And, we have:

$$\langle (\widehat{y}_m)_{tt}, \widehat{\phi}\rangle_{\mathtt{Y}} + \langle \widehat{y}_m, \widehat{\phi}\rangle_{\mathtt{X}} + \mathtt{b}((\widehat{y}_m)_t, \widehat{\phi}) = 0, \ \forall \ \widehat{\phi} \in \widehat{K}_m \tag{4.10}$$

with the initial conditions:

$$\widehat{y}_m(0) = \widehat{y}_{m0}, \text{ and } \widehat{y}_{m0} = \sum_{i=1}^{m} \delta_{i,m}\widehat{\phi}_i \longrightarrow \widehat{y}_0 \text{ in } \mathtt{X} \text{ when } m \longrightarrow \infty, \tag{4.11}$$

$$(\widehat{y}_m)_t(0) = \widehat{z}_m(0), \ \ \widehat{z}_m(0) = \widehat{z}_{m0}, \tag{4.12}$$

and

$$\text{and } \widehat{z}_{m0} = \sum_{i=1}^{m} \alpha_{i,m}\hat{\phi}_i \longrightarrow \widehat{z}_0 \text{ in } \mathtt{Y} \text{ when } m \longrightarrow \infty \tag{4.13}$$

with $\delta_{i,m} = \gamma_{i,m}(0)$ and $\alpha_{i,m} = (\gamma_{i,m})_t(0)$.

Relations (4.10) – (4.12) can be rewritten as follows:

$$\sum_{j=1}^{m} \left[(\langle \widehat{\phi}_j, \widehat{\phi}_i\rangle_{\mathtt{Y}} \ (\gamma_{j,m})_{tt} + \langle \widehat{\phi}_j, \widehat{\phi}_i\rangle_{\mathtt{X}} \ \gamma_{j,m} + \mathtt{b}(\widehat{\phi}_j, \widehat{\phi}_i) \ (\gamma_{j,m})_t \right] = 0, \tag{4.14}$$

for all $\widehat{\phi}_j, \widehat{\phi}_i \in \widehat{K}_m$. Considering that $\left\{\widehat{\phi}_i\right\}_{i=1}^{m}$ forms an orthogonal basis for $\mathtt{X}$ and an orthonormal basis for $\mathtt{Y}$, equation (4.14) becomes

$$(\gamma_m)_{tt} + \mathbb{A}(\gamma_m) + \mathbb{B}(\gamma_m)_t = 0, \tag{4.15}$$

where $\mathbb{A} = diag\,(\|\phi_j\|^2_{\mathtt{X}})_{1\le j\le m}$, $\mathbb{B} = \left(\mathtt{b}(\widehat{\phi}_j, \widehat{\phi}_i)\right)_{1\le j,i\le m}$ and $\gamma_m = (\gamma_{1,m}, \ldots, \gamma_{m,m})$ with the initial conditions $\gamma_m(0) = (\delta_{1,m}, \ldots, \delta_{m,m})$, $(\gamma_m)_t(0) = (\alpha_{1,m}, \ldots, \alpha_{m,m})$.

We thus have a linear system of second-order differential equations that we rewrite as a first-order system. Then, by the Cauchy-Lipschitz-Picard theorem [12] , we obtain the existence of a unique solution for (4.10)–(4.12) satisfying $\widehat{y}_m \in C^2\left([0;T], \mathtt{X}\right)$, with $0 \le t \le T$. □

4.1.2. *A-priori estimates on approximate solutions.*

Lemma 4.3. *Let $T > 0$ be fixed. The sequence of Galerkin's approximations $\{\widehat{y}_m\}_{m\in\mathbb{N}^*}$ of the form* (4.5) *satisfies:*

$$\{\widehat{y}_m\}_{m\in\mathbb{N}^*} \quad \textit{is bounded in} \quad C\left([0,T]; \mathtt{X}\right), \tag{4.16}$$

$$\{(\widehat{y}_m)_t\}_{m\in\mathbb{N}^*} \quad \textit{is bounded in} \quad C\left([0,T]; \mathtt{Y}\right), \tag{4.17}$$

$$\{(\widehat{y}_m)_{tt}\}_{m\in\mathbb{N}^*} \quad \textit{is bounded in} \quad C\left([0,T]; \mathtt{X}'\right). \tag{4.18}$$

Proof. Let $\hat{E} : \mathbb{R} \times \mathtt{X} \to \mathbb{R}$ the energy functional for the trajectory $\widehat{y}$ analogous to that defined by the expression (2.1). We have

$$\hat{E}(t, \widehat{y}) = \frac{1}{2}\|\widehat{y}_t\|^2_{\mathtt{Y}} + \frac{1}{2}\|\widehat{y}\|^2_{\mathtt{X}}. \tag{4.19}$$

For a solution $\widehat{y}_m \in C^2\left([0;\tau], \hat{K}_m\right)$ of (4.10) on some interval $[0;\tau]$, let's take $(\widehat{y}_m)_t = \widehat{\phi}$ in (4.10). We obtain

$$\langle(\widehat{y}_m)_{tt}, (\widehat{y}_m)_t\rangle_{\mathtt{Y}} + \langle\widehat{y}, (\widehat{y}_m)_t\rangle_{\mathtt{X}} + \mathtt{b}(\widehat{y}_t, (\widehat{y}_m)_t) = 0. \tag{4.20}$$

Thus, the derivative of the energy functional along the trajectory $\widehat{y}_m$ is:

$$\frac{d}{dt}\hat{E}(t, \widehat{y}_m) = -\mu\left[{}^1(\widehat{y}_m)_t\right]^2 \le 0, \tag{4.21}$$

for all $t \in [0, \tau]$. So we have uniform boundedness of the solution on $[0, \tau]$:

$$\hat{E}(t, \widehat{y}_m) \le \hat{E}(0, \widehat{y}_{m0}),\; t \ge 0 \tag{4.22}$$

which implies (4.16) and (4.17).

Now according to (4.16) and (4.17), for all $\widehat{\psi} \in \mathtt{X}$, there exists a positive constant M which does not depend on m, such that

$$|\langle \widehat{y}_m, \widehat{\psi}\rangle_{\mathtt{X}} + \mathtt{b}((\widehat{y}_m)_t, \widehat{\psi})| \leq M\|\widehat{\psi}\|_{\mathtt{X}}, \ \forall t \geq 0. \tag{4.23}$$

Now let us consider $\widehat{\psi} = \widehat{\varphi}_1 + \widehat{\varphi}_2$ such that $\widehat{\varphi}_1 \in \widehat{K}_m$ and $\widehat{\varphi}_2$ orthogonal to $\widehat{K}_m$ in $\mathtt{Y}$. From (4.10), we have:

$$\begin{aligned}\langle (\widehat{y}_m)_{tt}, \widehat{\psi}\rangle_{\mathtt{Y}} &= \langle (\widehat{y}_m)_{tt}, \widehat{\varphi}_1\rangle_{\mathtt{Y}} \\ &= -\langle \widehat{y}_m, \widehat{\varphi}_1\rangle_{\mathtt{X}} - \mathtt{b}((\widehat{y}_m)_t, \widehat{\varphi}_1) \\ &\leq M\|\widehat{\varphi}_1\|_{\mathtt{X}} \leq M\|\widehat{\psi}\|_{\mathtt{X}},\end{aligned}$$

which implies (4.18). □

4.1.3. *Passage to the limit.*

Lemma 4.4. *There exists a weak solution to the weak formulation* (3.8)–(3.10).

Proof. Based on Lemma 4.3 and the Eberlein-Smulian Theorem [9, 22], we can extract weakly convergent subsequences $\{\widehat{y}_{m_l}\}_{l\in\mathbb{N}^*}$, $\{(\widehat{y}_{m_l})_t\}_{l\in\mathbb{N}^*}$ and $\{(\widehat{y}_{m_l})_{tt}\}_{l\in\mathbb{N}^*}$ with $\widehat{y} \in L^2(0,T;\mathtt{X})$, $\widehat{y}_t \in L^2(0,T;\mathtt{Y})$, $\widehat{y}_{tt} \in L^2\left(0,T;\mathtt{X}'\right)$ such that:

$$\{\widehat{y}_{m_l}\} \rightharpoonup \widehat{y} \text{ in } L^2(0,T;\mathtt{X}) \tag{4.24}$$

$$\{(\widehat{y}_{m_l})_t\} \rightharpoonup \widehat{y}_t \text{ in } L^2(0,T;\mathtt{Y}) \tag{4.25}$$

$$\{(\widehat{y}_{m_l})_{tt}\} \rightharpoonup \widehat{y}_{tt} \text{ in } L^2\left(0,T;\mathtt{X}'\right). \tag{4.26}$$

Consider now the functions $\widehat{\varphi} \in L^2\left(0,T;\widehat{K}_{m0}\right)$ such that

$$\widehat{\varphi}(x,t) = \sum_{j=1}^{m_0} \alpha_{j,m_0}(t)\phi_j(x) \tag{4.27}$$

where $\alpha_{j,m_0} \in L^2(0,T;\mathbb{R})$ and $m_0 \in \mathbb{N}^*$. For all $m_l \geq m_0$, the formulation (4.10) becomes:

$$\int_0^T [\langle (\widehat{y}_{m_l})_{tt}, \hat{\varphi}\rangle_{\mathtt{Y}} + \langle \widehat{y}_{m_l}, \hat{\varphi}\rangle_{\mathtt{X}} + \mathtt{b}((\widehat{y}_{m_l})_t, \hat{\varphi})]\, dt = 0. \tag{4.28}$$

Therefore, passing on to the limit in (4.28) for $m = m_l$ when $l \to \infty$ and using the convergence results (4.24)–(4.26), one obtains:

$$\int_0^T [\langle \widehat{y}_{tt}, \hat{\varphi} \rangle_{\mathtt{X},\mathtt{X}'} + \langle \widehat{y}, \hat{\varphi} \rangle_{\mathtt{X}} + \mathtt{b}(\widehat{y}_t, \hat{\varphi})] \, dt = 0. \tag{4.29}$$

Then $\langle (\widehat{y})_{tt}, \widehat{\varphi} \rangle + \langle \widehat{y}, \widehat{\varphi} \rangle_{\mathtt{X}} + \mathtt{b}(\widehat{y}_t, \widehat{\varphi}) = 0$ a.e on $[0, T]$ for all $\widehat{\varphi} \in L^2(0, T; \mathtt{X})$.

The functions $\widehat{\varphi}$ of the form (4.27) are dense in $L^2(0, T; \mathtt{X})$. So (4.29) is well defined for all $\widehat{\varphi} \in L^2(0, T; \mathtt{X})$. We thus obtain the expression of the weak formulation almost everywhere on $[0, T]$ and $\widehat{y}$ is well the solution of (3.8) . □

Remark 4.1.

(1) *According to definition of weak solution and the important results* (4.16)–(4.17) *we have that* $\widehat{y}$ *follows the regularity* (4.1)–(4.2).
(2) *Result* (4.3) *follows directly from Lemma 3.1 after possible modification on a set of measures zero.*
(3) *The regularity of* (4.4) *is deduced from Lemma 3.1 and Remark 3.1.*

4.2. **Uniqueness of the Weak Solution.**

Lemma 4.5. *A solution of the weak formulation* (3.8)–(3.10) *is unique.*

Proof. In order to show that the solution $\widehat{y}$ satisfies the conditions (3.9)–(3.10), let us consider $\widehat{\phi} \in C^2([0, T], \mathtt{X})$ such that $\widehat{\phi}(T) = 0$ and $\widehat{\phi}_t(T) = 0$. After two integrations by part of the identity (3.8) over $[0, T]$, we get:

$$\int_0^T \left[\langle \widehat{y}, \widehat{\phi}_{tt} \rangle_{\mathtt{Y}} + \langle \widehat{y}, \widehat{\phi} \rangle_{\mathtt{X}} + \mathtt{b}(\widehat{y}_t, \widehat{\phi}) \right] d\tau = \langle \widehat{y}_t(0), \widehat{\phi}(0) \rangle_{\mathtt{X},\mathtt{X}'} - \langle \widehat{y}(0), \widehat{\phi}_t(0) \rangle_{\mathtt{Y}}. \tag{4.30}$$

Similarly, for a fixed m, it follows by integrating twice by parts (4.10):

$$\int_0^T \left[\langle \widehat{y}_m, \widehat{\phi}_{tt} \rangle_{\mathtt{Y}} + \langle \widehat{y}_m, \widehat{\phi} \rangle_{\mathtt{X}} + \mathtt{b}((\widehat{y}_m)_t, \widehat{\phi}) \right] d\tau = \langle \widehat{z}_{m0}, \widehat{\phi}(0) \rangle_{\mathtt{Y}} - \langle \widehat{y}_{m0}, \widehat{\phi}_t(0) \rangle_{\mathtt{Y}}. \tag{4.31}$$

Using (4.11)–(4.12) with (4.24)–(4.26), and passing to the limit in (4.31) along the convergent subsequence $\{\widehat{y}_{m_l}\}$, this gives:

$$\int_0^T \left[\langle \widehat{y}, \widehat{\phi}_{tt} \rangle_{\mathtt{Y}} + \langle \widehat{y}, \widehat{\phi} \rangle_{\mathtt{X}} + \mathtt{b}((\widehat{y})_t, \widehat{\phi}) \right] d\tau = \langle \widehat{z}_0, \widehat{\phi}(0) \rangle_{\mathtt{Y}} - \langle \widehat{y}_0, \widehat{\phi}_t(0) \rangle_{\mathtt{Y}}. \tag{4.32}$$

According to (4.32) and (4.30), the initial conditions are therefore satisfied and we have $\widehat{y}(0) = \widehat{y}_0$ and $\widehat{z}_0 = \widehat{y}_t(0)$.

For the proof of the uniqueness of the weak solution, let us now consider $\widehat{y}$ a solution of (3.8) with zero initial conditions. Let $0 < \zeta < T$ be fixed and introduce an auxiliary function:

$$\widehat{\pi}(t) := \begin{cases} \int_t^s \widehat{y}(\tau)d\tau, & 0 < t < \zeta \\ 0, & t \geq \zeta. \end{cases}$$

Taking $\widehat{\pi}$ instead of $\widehat{\phi}$ in (3.8) and by integrating by parts on $[0, T]$, we obtain

$$\int_0^\zeta \left[\langle \widehat{y}_t(\tau), \widehat{y}(\tau)\rangle_{\mathtt{Y}} - \langle \widehat{\pi}_t(\tau), \widehat{\pi}(\tau)\rangle_{\mathtt{X}} + \mathtt{b}(\widehat{y}(\tau), \widehat{y}(\tau))\right] d\tau = 0 \tag{4.33}$$

$$\int_0^\zeta \frac{d}{dt}\left[\frac{1}{2}\langle \widehat{y}(\tau), \widehat{y}(\tau)\rangle_{\mathtt{Y}} - \frac{1}{2}\langle \widehat{\pi}(\tau), \widehat{\pi}(\tau)\rangle_{\mathtt{X}}\right] d\tau = -\int_0^\zeta \mathtt{b}(\widehat{y}(\tau), \widehat{y}(\tau))d\tau. \tag{4.34}$$

This is equivalent to

$$\left[\frac{1}{2}\langle \widehat{y}(\tau), \widehat{y}(\tau)\rangle_{\mathtt{Y}} - \frac{1}{2}\langle \widehat{\pi}(\tau), \widehat{\pi}(\tau)\rangle_{\mathtt{X}}\right]_0^\zeta = -\int_0^\zeta \mathtt{b}(\widehat{y}(\tau), \widehat{y}(\tau))d\tau.$$

Therefore,

$$\frac{1}{2}\|\widehat{y}(\zeta)\|_{\mathtt{Y}} + \frac{1}{2}\langle \widehat{\pi}(0), \widehat{\pi}(0)\rangle_{\mathtt{X}} \leq 0. \tag{4.35}$$

The bilinear form $\langle ., .\rangle_{\mathtt{X}}$ is coercive. Hence $\widehat{y}(\zeta) \equiv 0$ et $\widehat{\pi}(0) = 0$. Since $\zeta \in \,]0, T[$ was arbitrary then $\widehat{y} \equiv 0$. □

5. STRONG CONTINUITY RESULT OF THE WEAK SOLUTION

Before presenting the fundamental result of this subsection, let us recall the following lemma demonstrated in [13].

Lemma 5.1. *Let* $\mathtt{X}$ *and* $\mathtt{Y}$ *two Banach spaces,* $\mathtt{X} \subset \mathtt{Y}$ *with continuous injection, the space* $\mathtt{X}$ *being reflexive. We set:*

$$C_y([0,T]; \mathtt{Y}) = \left\{ y \in L^\infty(0,T; \mathtt{Y}) : t \longrightarrow \langle f, y(t)\rangle \text{ is continuous on } [0,T], \forall f \in \mathtt{Y}' \right\}$$

which denotes the space of weakly continuous functions with values in $\mathtt{Y}$*. Thus we get*

$$L^\infty(0,T; \mathtt{X}) \cap C_y([0,T]; \mathtt{Y}) = C_y([0,T]; \mathtt{X}).$$

Theorem 5.1. *After, possibly, a modification on a set of measure zero, the weak solution* $\widehat{y}$ *of* (3.8)-(3.10) *satisfies*

$$\widehat{y} \in C\left([0,T]\,;\mathtt{X}\right) \tag{5.1}$$

$$\widehat{y}_t \in C\left([0,T]\,;\mathtt{Y}\right). \tag{5.2}$$

Proof. For this proof we adapt standard strategies presented in section 8.4 of [13] pp. $297-301$ and section 2.4 of [20].

Considering Lemma 5.1, it follows from (4.1) and (4.3) that $\widehat{y} \in C_y\left([0,T]\,;\mathtt{X}\right)$. Similarly (4.2) and (4.4) imply $\widehat{y}_t \in C_y\left([0,T]\,;\mathtt{Y}\right)$.

Let $\xi \in C^\infty(\mathbb{R})$ a scalar cutoff function such that $\xi(x) = 1$ if $x \in J \subset\subset [0,T]$ and $\xi(x) = 0$ else. Then the function $\xi\widehat{y}$ is then compactly supported. Let η^ε be a standard mollifier in time. Using convolutional regulation of distributions, we define

$$\widehat{y}^\varepsilon = \eta^\varepsilon * \xi\widehat{y} \in C_c^\infty(\mathbb{R}, \mathtt{X}).$$

$\widehat{y}^\varepsilon$ converges to $\widehat{y}$ in $\mathtt{X}$ and $\widehat{y}^\varepsilon_t$ converges to $\widehat{y}_t$ in $\mathtt{X}$ almost everywhere on J. Hence, $\hat{E}(t,\widehat{y}^\varepsilon)$ converges to $\hat{E}(t,\widehat{y})$ almost everywhere on J. Since $\widehat{y}^\varepsilon$ is smooth, we obtain by a straightforward calculation on J :

$$\frac{d}{dt}\hat{E}(t,\widehat{y}^\varepsilon) = -\mu\left[{}^1(\widehat{y}^\varepsilon)_t\right]^2. \tag{5.3}$$

Passing to the limit in (5.3) as $\varepsilon \to 0$, we obtain:

$$\frac{d}{dt}\hat{E}(t,\widehat{y}) = -\mu\left[{}^1(\widehat{y}^\varepsilon)_t\right]^2 \tag{5.4}$$

in the sense of distributions on J. (5.4) holds on all compact subintervals of $[0,T]$, since J was arbitrary.

Let $t \in [0,\infty[$ be fixed and $(t_n)_{n\in\mathbb{N}}$ a sequence such that $\lim_{n\to\infty} t_n = t$.

Let $(\nu_n)_{n\in\mathbb{N}}$ be defined by

$$\nu_n = \frac{1}{2}\|\widehat{y}_t(t) - \widehat{y}_t(t_n)\|^2_{\mathtt{Y}} + \frac{1}{2}\|\widehat{y}(t) - \widehat{y}(t_n))\|^2_{\mathtt{X}}. \tag{5.5}$$

Then

$$\nu_n = \hat{E}(t,\widehat{y}) + \hat{E}(t_n,\widehat{y}) - \langle\widehat{y}_t(t), \widehat{y}_t(t_n)\rangle_{\mathtt{Y}} - \langle\widehat{y}(t), \widehat{y}(t_n)\rangle_{\mathtt{X}}. \tag{5.6}$$

Since $\hat{w}$, $\hat{w}_t$ are weakly continuous and $\hat{E}$ is continuous in t, we have, passing to the limit in (5.6):

$$\lim_{n\to\infty} \nu_n = 0.$$

Therefore, this implies that

(5.7) $$\lim_{n\to\infty} \|\widehat{y}_t(t) - \widehat{y}_t(t_n)\|_{\mathtt{Y}}^2 = 0 \text{ and } \lim_{n\to\infty} \|(\widehat{y}(t) - \widehat{y}(t_n))\|_{\mathtt{X}}^2 = 0.$$

Finally, $\widehat{y} \in C\left([0,T]\,;\mathtt{X}\right)$ and $\widehat{y}_t \in C\left([0,T]\,;\mathtt{Y}\right)$. □

6. Conclusion

In this paper, we have proved the Lyapunov stability, existence and uniqueness of the weak solution for a system of differential equations modeling an overhead crane consisting of a motorized platform moving along a horizontal beam using a flexible cable, supporting a load of mass M and subject to speed and position control. We also demonstrate a strong continuity result for the weak solution. In addition to its theoretical contribution, this work establishes the necessary foundation for the development of robust numerical simulations applied to overhead cranes.

Conflict of interest

The authors declare no competing interests.

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