ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **15**(1) (2026), 167–181
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.15.1.13

STUDY OF AN ALTERNATIVE INEQUALITY TO THE HERMITE-HADAMARD INTEGRAL INEQUALITY

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ABSTRACT. This article introduces new convex and concave integral inequalities. We provide a detailed proof and several examples to demonstrate their validity and application. We then consider the limitations of our findings and show that the new inequalities are not as sharp as the classical Hermite-Hadamard integral inequality. By introducing new techniques to the study of integral inequalities, our findings reaffirm the remarkable sharpness of the Hermite-Hadamard integral inequality.

1. INTRODUCTION

The concepts of convexity and concavity play a fundamental role in mathematics and its many applications, such as optimization, analysis and economics. For the sake of completeness, we provide the formal definitions of convex and concave functions below, on which the results established in this article are based. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $b > a$.

- A function $f : [a, b] \to \mathbb{R}$ is said to be convex if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

2020 *Mathematics Subject Classification.* 26D15.
Key words and phrases. convex integral inequalities, Hermite-Hadamard integral inequality, integration by parts.
Submitted: 23.02.2026; *Accepted:* 12.03.2026; *Published:* 25.03.2026.

$$f(\lambda x+(1-\lambda)y)\leq \lambda f(x)+(1-\lambda)f(y).$$

- A function $f:[a,b]\to\mathbb{R}$ is said to be concave if, for any $x,y\in[a,b]$ and $\lambda\in[0,1]$, we have

$$f(\lambda x+(1-\lambda)y)\geq \lambda f(x)+(1-\lambda)f(y).$$

As a key property, if f is twice differentiable and convex, then $f''(x)\geq 0$ for any $x\in[a,b]$, implying that f' is non-decreasing. Conversely, if f is twice differentiable and concave, then $f''(x)\leq 0$ for any $x\in[a,b]$, implying that f' is non-increasing. One of the most significant consequences of convexity and concavity is the emergence of a diverse range of integral inequalities, collectively referred to as 'convex' and 'concave' integral inequalities. The most well-known of these is the Hermite-Hadamard integral inequality, for which the convex and concave versions are given below. Let $a,b\in\mathbb{R}$ such that $b>a$.

- Assuming that $f:[a,b]\to\mathbb{R}$ is a convex function, we have

$$f\left(\frac{a+b}{2}\right)\leq \frac{1}{b-a}\int_a^b f(x)dx\leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

- Assuming that $f:[a,b]\to\mathbb{R}$ is a concave function, we have

$$\frac{f(a)+f(b)}{2}\leq \frac{1}{b-a}\int_a^b f(x)dx\leq f\left(\frac{a+b}{2}\right). \tag{1.2}$$

In this setting, the convex integral double inequality is thus reversed in the concave case. The Hermite-Hadamard integral inequality is well known for its sharpness, providing one of the most sharp bounds for convex functions in integral form.

Convex and concave integral inequalities continue to play a central role in many areas of mathematics. In recent decades, significant progress has been made, with many researchers uncovering intricate connections between convexity, functional inequalities and integral transforms. The latest developments in this area are presented in [1–16].

This article first introduces a new convex integral double inequality, providing a thorough proof of the result. Several examples are then presented to demonstrate its applicability. Next, we formulate and prove the concave counterpart of the inequality. Finally, we discuss the limitations of our findings and rigorously demonstrate that the proposed inequality is not as sharp as the classical Hermite-Hadamard integral inequality. This study therefore highlights new techniques for analyzing integral inequalities and demonstrates the enduring sharpness of the Hermite-Hadamard integral inequality.

The remainder of this article is organized as follows: Section 2 presents the convex version of the integral inequality, while Section 3 is devoted to its concave counterpart. The limitations of the obtained results are discussed in Section 4. Concluding remarks are provided in Section 5.

2. Convex version

2.1. **Theorem and proof.** The theorem below presents the new convex integral double inequality.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f : [a, b] \to \mathbb{R}$ be a twice differentiable and convex function. Then we have*

$$\max\left\{f(b) - \frac{b-a}{2}f'(b), f(a) + \frac{b-a}{2}f'(a)\right\} \leq \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq \min\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\}.$$

Proof. By an integration by parts with the weight function $x - a$, we get

$$\int_a^b f(x)dx = [(x-a)f(x)]_{x=a}^{x=b} - \int_a^b (x-a)f'(x)dx$$
$$= (b-a)f(b) - \int_a^b (x-a)f'(x)dx.$$

Since f is convex, f' is non-decreasing on $[a, b]$. Moreover, the function $x - a$ is non-negative on $[a, b]$, so

$$f'(a)\int_a^b (x-a)dx \leq \int_a^b (x-a)f'(x)dx \leq f'(b)\int_a^b (x-a)dx.$$

Basically, we calculate

$$\int_a^b (x-a)dx = \left[\frac{(x-a)^2}{2}\right]_{x=a}^{x=b} = \frac{(b-a)^2}{2}.$$

Plugging these equations gives

$$(b-a)f(b)-\frac{(b-a)^2}{2}f'(b)\le \int_a^b f(x)dx \le (b-a)f(b)-\frac{(b-a)^2}{2}f'(a).$$

Dividing through by $b-a$, we get

$$f(b)-\frac{b-a}{2}f'(b)\le \frac{1}{b-a}\int_a^b f(x)dx \le f(b)-\frac{b-a}{2}f'(a). \tag{2.1}$$

Similarly, by an integration by parts with the weight function $-(b-x)$, we get

$$\int_a^b f(x)dx = [-(b-x)f(x)]_{x=a}^{x=b}+\int_a^b (b-x)f'(x)dx$$
$$=(b-a)f(a)+\int_a^b (b-x)f'(x)dx.$$

Since f is convex, f' is non-decreasing on $[a,b]$. Moreover, the function $b-x$ is non-negative on $[a,b]$, so

$$f'(a)\int_a^b (b-x)dx \le \int_a^b (b-x)f'(x)dx \le f'(b)\int_a^b (b-x)dx.$$

Basically, we calculate

$$\int_a^b (b-x)dx = \left[-\frac{(b-x)^2}{2}\right]_{x=a}^{x=b}=\frac{(b-a)^2}{2}.$$

Plugging these equations gives

$$(b-a)f(a)+\frac{(b-a)^2}{2}f'(a)\le \int_a^b f(x)dx \le (b-a)f(a)+\frac{(b-a)^2}{2}f'(b).$$

Dividing through by $b-a$, we get

$$f(a)+\frac{b-a}{2}f'(a)\le \frac{1}{b-a}\int_a^b f(x)dx \le f(a)+\frac{b-a}{2}f'(b). \tag{2.2}$$

Combining Equations (2.1) and (2.2), we finally obtain

$$\max\left\{f(b)-\frac{b-a}{2}f'(b), f(a)+\frac{b-a}{2}f'(a)\right\}\le \frac{1}{b-a}\int_a^b f(x)dx$$
$$\le \min\left\{f(b)-\frac{b-a}{2}f'(a), f(a)+\frac{b-a}{2}f'(b)\right\}.$$

This completes the proof. □

The proof relies on an application of integration by parts and the monotonicity of f'. To the best of our knowledge, this is one of the few convex integral inequalities that explicitly involves the derivatives of f at the endpoints a and b. Several examples are presented in the subsection below to further illustrate the theory.

2.2. **Examples.** We now present some examples of Theorem 2.1, focusing on the case $a = 0$ and $b = 1$.

- Taking $f(x) = x^2$, $x \in [0, 1]$, which is twice differentiable and convex with $f'(x) = 2x$, we get

$$\begin{aligned}&\max\left\{f(b) - \frac{b-a}{2}f'(b), f(a) + \frac{b-a}{2}f'(a)\right\}\\&= \max\left\{1 - \frac{1}{2} \times 2, 0 + \frac{1}{2} \times 0\right\} = 0,\end{aligned}$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{3}$$

and

$$\begin{aligned}&\min\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\}\\&= \min\left\{1 - \frac{1}{2} \times 0, 0 + \frac{1}{2} \times 2\right\} = 1.\end{aligned}$$

It is obvious that $0 < 1/3 < 1$, illustrating the desired inequality. Clearly, the obtained inequality is not sharp.

- Taking $f(x) = e^{-x}$, $x \in [0, 1]$, which is twice differentiable and convex with $f'(x) = -e^{-x}$, we get

$$\begin{aligned}&\max\left\{f(b) - \frac{b-a}{2}f'(b), f(a) + \frac{b-a}{2}f'(a)\right\}\\&= \max\left\{e^{-1} - \frac{1}{2}(-e^{-1}), 1 + \frac{1}{2}(-1)\right\}\\&\approx \max(0.5518, 0.5) = 0.5518,\end{aligned}$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 e^{-x}dx = \left[-e^{-x}\right]_{x=0}^{x=1} = 1 - e^{-1} \approx 0.6321$$

and

$$\begin{aligned}
&\min\left\{f(b)-\frac{b-a}{2}f'(a),f(a)+\frac{b-a}{2}f'(b)\right\}\\
&=\min\left\{e^{-1}-\frac{1}{2}(-1),1+\frac{1}{2}(-e^{-1})\right\}\\
&\approx \min(0.8678,0.8160)=0.8160
\end{aligned}$$

It is clear that $0.5518<0.6321<0.8160$, illustrating the desired inequality.

- Taking $f(x)=e^{x}$, $x\in[0,1]$, which is twice differentiable and convex with $f'(x)=e^{x}$, we get

$$\begin{aligned}
&\max\left\{f(b)-\frac{b-a}{2}f'(b),f(a)+\frac{b-a}{2}f'(a)\right\}\\
&=\max\left\{e-\frac{1}{2}e,1+\frac{1}{2}\times 1\right\}\\
&\approx \max(1.3591,1.5)=1.5,
\end{aligned}$$

$$\frac{1}{b-a}\int_a^b f(x)dx=\int_0^1 e^{x}dx=[e^{x}]_{x=0}^{x=1}=e-1\approx 1.7182$$

and

$$\begin{aligned}
&\min\left\{f(b)-\frac{b-a}{2}f'(a),f(a)+\frac{b-a}{2}f'(b)\right\}\\
&=\min\left\{e-\frac{1}{2}\times 1,1+\frac{1}{2}\times e\right\}\\
&\approx \min(2.2182,2.3591)=2.2182.
\end{aligned}$$

It is evident that $1.5<1.7182<2.2182$, illustrating the desired inequality.

- Taking $p=3$ and $f(x)=\sqrt{1+x^{2}}$, $x\in[0,1]$, which is twice differentiable and convex with $f'(x)=x/\sqrt{1+x^{2}}$, we get

$$\begin{aligned}
&\max\left\{f(b)-\frac{b-a}{2}f'(b),f(a)+\frac{b-a}{2}f'(a)\right\}\\
&=\max\left\{\sqrt{2}-\frac{1}{2}\times\frac{1}{\sqrt{2}},1+\frac{1}{2}\times 0\right\}\\
&\approx \max(1.0606,1)=1.0606,
\end{aligned}$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 \sqrt{1+x^2}dx \approx 1.1478$$

and

$$\min\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\}$$
$$= \min\left\{\sqrt{2} - \frac{1}{2}\times 0, 1 + \frac{1}{2}\times\frac{1}{\sqrt{2}}\right\}$$
$$\approx \min(1.4142, 1.3535) = 1.3535.$$

It is clear that $1.0606 < 1.1478 < 1.3535$, illustrating the desired inequality. These are just a few examples of the many possibilities available.

3. Concave counterpart

3.1. **Theorem and proof.** The concave counterpart of Theorem 2.1 is presented below.

Theorem 3.1. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f : [a,b] \to \mathbb{R}$ be a twice differentiable and concave function. Then we have*

$$\max\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\} \leq \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq \min\left\{f(b) - \frac{b-a}{2}f'(b), f(a) + \frac{b-a}{2}f'(a)\right\}.$$

Proof. By an integration by parts with the weight function $x-a$, we get

$$\int_a^b f(x)dx = [(x-a)f(x)]_{x=a}^{x=b} - \int_a^b (x-a)f'(x)dx$$
$$= (b-a)f(b) - \int_a^b (x-a)f'(x)dx.$$

Since f is concave, f' is non-increasing on $[a,b]$. Moreover, the function $x-a$ is non-negative on $[a,b]$, so

$$f'(b)\int_a^b (x-a)dx \leq \int_a^b (x-a)f'(x)dx \leq f'(a)\int_a^b (x-a)dx.$$

Basically, we calculate

$$\int_a^b (x-a)dx = \left[\frac{(x-a)^2}{2}\right]_{x=a}^{x=b} = \frac{(b-a)^2}{2}.$$

Plugging these equations gives

$$(b-a)f(b) - \frac{(b-a)^2}{2}f'(a) \leq \int_a^b f(x)dx \leq (b-a)f(b) - \frac{(b-a)^2}{2}f'(b).$$

Dividing through by $b-a$, we get

$$f(b) - \frac{b-a}{2}f'(a) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(b) - \frac{b-a}{2}f'(b). \tag{3.1}$$

Similarly, by an integration by parts with the weight function $-(b-x)$, we get

$$\int_a^b f(x)dx = [-(b-x)f(x)]_{x=a}^{x=b} + \int_a^b (b-x)f'(x)dx$$
$$= (b-a)f(a) + \int_a^b (b-x)f'(x)dx.$$

Since f is concave, f' is non-increasing on $[a,b]$. Moreover, the function $b-x$ is non-negative on $[a,b]$, so

$$f'(b)\int_a^b (b-x)dx \leq \int_a^b (b-x)f'(x)dx \leq f'(a)\int_a^b (b-x)dx.$$

Basically, we calculate

$$\int_a^b (b-x)dx = \left[-\frac{(b-x)^2}{2}\right]_{x=a}^{x=b} = \frac{(b-a)^2}{2}.$$

Plugging these equations gives

$$(b-a)f(a) + \frac{(b-a)^2}{2}f'(b) \leq \int_a^b f(x)dx \leq (b-a)f(a) + \frac{(b-a)^2}{2}f'(a).$$

Dividing through by $b-a$, we get

$$f(a) + \frac{b-a}{2}f'(b) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(a) + \frac{b-a}{2}f'(a). \tag{3.2}$$

Combining Equations (3.1) and (3.2), we finally obtain

$$\max\left\{f(b)-\frac{b-a}{2}f'(a), f(a)+\frac{b-a}{2}f'(b)\right\} \leq \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq \min\left\{f(b)-\frac{b-a}{2}f'(b), f(a)+\frac{b-a}{2}f'(a)\right\}.$$

This completes the proof. □

The primary modification compared with Theorem 2.1 concerns the definition of the derivatives of f at the endpoints. Several examples of applications are provided in the subsection below.

3.2. **Examples.** We now present some examples of Theorem 2.1, focusing on the case $a = 0$ and $b = 1$.

- Taking $f(x) = \ln(1+x)$, $x \in [0,1]$, which is twice differentiable and concave with $f'(x) = 1/(1+x)$, we get

$$\max\left\{f(b)-\frac{b-a}{2}f'(a), f(a)+\frac{b-a}{2}f'(b)\right\}$$
$$= \max\left\{\ln(2)-\frac{1}{2}\times 1, 0+\frac{1}{2}\times\frac{1}{2}\right\}$$
$$\approx \max(0.1931, 0.25) = 0.25,$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 \ln(1+x)dx = [(1+x)\ln(1+x)-x]_{x=0}^{x=1}$$
$$= 2\ln(2) - 1 \approx 0.3862$$

and

$$\min\left\{f(b)-\frac{b-a}{2}f'(b), f(a)+\frac{b-a}{2}f'(a)\right\}$$
$$= \min\left\{\ln(2)-\frac{1}{2}\times\frac{1}{2}, 0+\frac{1}{2}\times 1\right\}$$
$$\approx \min(0.4431, 0.6) = 0.4431.$$

It is clear that $0.25 < 0.3862 < 0.4431$.

- Taking $f(x) = \arctan(x)$, $x \in [0,1]$, which is twice differentiable and concave with $f(x) = 1/(1+x^2)$, we get

$$\begin{aligned}
&\max\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\} \\
&= \max\left\{\frac{\pi}{4} - \frac{1}{2} \times 1, 0 + \frac{1}{2} \times \frac{1}{2}\right\} \\
&\approx \max(0.2853, 0.25) = 0.2853,
\end{aligned}$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 \arctan(x)dx \approx 0.4388$$

and

$$\begin{aligned}
&\min\left\{f(b) - \frac{b-a}{2}f'(b), f(a) + \frac{b-a}{2}f'(a)\right\} \\
&= \min\left\{\frac{\pi}{4} - \frac{1}{2} \times \frac{1}{2}, 0 + \frac{1}{2} \times 1\right\} \\
&\approx \min(0.5353, 0.5) = 0.5.
\end{aligned}$$

It is obvious that $0.2853 < 0.4388 < 0.5$, illustrating the desired inequality.
- Taking $f(x) = \sin(x)$, $x \in [0,1]$, which is twice differentiable and concave with $f'(x) = \cos(x)$, we get

$$\begin{aligned}
&\max\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\} \\
&= \max\left\{\sin(1) - \frac{1}{2} \times 1, 0 + \frac{1}{2}\cos(1)\right\} \\
&\approx \max(0.3414, 0.2701) = 0.3414,
\end{aligned}$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 \sin(x)dx = [-\cos(x)]_{x=0}^{x=1} = 1 - \cos(1) \approx 0.4596$$

and

$$\begin{aligned}
&\min\left\{f(b)-\frac{b-a}{2}f'(b), f(a)+\frac{b-a}{2}f'(a)\right\}\\
&=\min\left\{\sin(1)-\frac{1}{2}\cos(1), 0+\frac{1}{2}\times 1\right\}\\
&\approx \min(0.5713, 0.5)=0.5.
\end{aligned}$$

It is clear that $0.3414 < 0.4596 < 0.5$, illustrating the desired inequality.

- Taking $f(x)=\cos(x)$, $x\in[0,1]$, which is twice differentiable and concave with $f'(x)=-\sin(x)$, we have

$$\begin{aligned}
&\max\left\{f(b)-\frac{b-a}{2}f'(a), f(a)+\frac{b-a}{2}f'(b)\right\}\\
&=\max\left\{\cos(1)-\frac{1}{2}\times 0, 1+\frac{1}{2}(-\sin(1))\right\}\\
&\approx \max(0.5403, 0.5792)=0.5792,
\end{aligned}$$

$$\frac{1}{b-a}\int_a^b f(x)dx=\int_0^1 \cos(x)dx=[\sin(x)]_{x=0}^{x=1}=\sin(1)\approx 0.8414$$

and

$$\begin{aligned}
&\min\left\{f(b)-\frac{b-a}{2}f'(b), f(a)+\frac{b-a}{2}f'(a)\right\}\\
&=\min\left\{\cos(1)-\frac{1}{2}(-\sin(1)), 1+\frac{1}{2}\times 0\right\}\\
&\approx \min(0.9610, 1)=0.9610.
\end{aligned}$$

It is obvious that $0.5792 < 0.8414 < 0.9610$, illustrating the desired inequality.

4. Limitation

Theorems 2.1 and 3.1 are of mathematical interest owing to their novelty. However, a closer analysis reveals a limitation: the derived inequalities are not as sharp as the classical Hermite-Hadamard integral inequality. This point is demonstrated in detail in the two propositions below.

Proposition 4.1. *Let* $a, b \in \mathbb{R}$ *with* $b > a$*, and* $f : [a, b] \to \mathbb{R}$ *be a twice differentiable and convex function. Then we have*

$$\max\left\{f(b) - \frac{b-a}{2}f'(b), f(a) + \frac{b-a}{2}f'(a)\right\} \leq f\left(\frac{a+b}{2}\right).$$

Proof. Applying the Taylor theorem to f with the Lagrange form of the remainder, for any $x, y \in [a, b]$, we have

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2}f''(\zeta_{x,y})(y-x)^2,$$

for a certain $\zeta_{x,y} \in [\min(x, y), \max(x, y)]$. Since f is convex, for any $x, y \in [a, b]$, we have $f''(\zeta_{x,y}) \geq 0$, which implies that

$$f(y) \geq f(x) + f'(x)(y-x). \tag{4.1}$$

Puting $m = (a + b)/2$, and taking $x = b$ and $y = m$, we get

$$f(m) \geq f(b) + f'(b)(m-b) = f(b) - \frac{b-a}{2}f'(b). \tag{4.2}$$

Similarly, taking $x = a$ and $y = m$, we obtain

$$f(m) \geq f(a) + f'(a)(m-a) = f(a) + \frac{b-a}{2}f'(a). \tag{4.3}$$

It follows from Equations (4.2) and (4.3) that

$$f\left(\frac{a+b}{2}\right) \geq \max\left\{f(b) - \frac{b-a}{2}f'(b), f(a) + \frac{b-a}{2}f'(a)\right\}.$$

This completes the proof. □

If f is a twice differentiable and convex function, then Proposition 4.1 and the left-hand side of the Hermite-Hadamard integral inequality imply that

$$\max\left\{f(b) - \frac{b-a}{2}f'(b), f(a) + \frac{b-a}{2}f'(a)\right\} \leq f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx.$$

The left-hand side of the Hermite-Hadamard integral inequality is thus sharper than the left-hand side of the inequality in Theorem 2.1.

The proposition below completes Proposition 4.1.

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f : [a, b] \to \mathbb{R}$ be a twice differentiable and convex function. Then we have*

$$\frac{f(a)+f(b)}{2} \le \min\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\}.$$

Proof. Since f is convex, it follows from Equation (4.1) that, for any $x, y \in [a, b]$,

$$f(y) \ge f(x) + f'(x)(y-x).$$

Taking $x = a$ and $y = b$, we get

$$f(b) \ge f(a) + f'(a)(b-a).$$

Therefore, we have

$$f(a) \le f(b) - (b-a)f'(a),$$

so that

$$\frac{f(a)+f(b)}{2} \le \frac{f(b)-(b-a)f'(a)+f(b)}{2} = f(b) - \frac{b-a}{2}f'(a). \tag{4.4}$$

Similarly, taking $x = b$ and $y = a$, we get

$$f(a) \ge f(b) + f'(b)(a-b),$$

so that

$$f(b) \le f(a) + f'(b)(b-a).$$

This implies that

$$\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(a)+f'(b)(b-a)}{2} = f(a) + \frac{b-a}{2}f'(b). \tag{4.5}$$

It follows from Equations (4.4) and (4.5) that

$$\frac{f(a)+f(b)}{2} \le \min\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\}.$$

This completes the proof. □

If f is a twice differentiable and convex function, then Proposition 4.1 and the right-hand side of the Hermite-Hadamard integral inequality imply that

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \le \min\left\{f(b) - \frac{b-a}{2}f'(a), f(a) + \frac{b-a}{2}f'(b)\right\}.$$

The right-hand side of the convex version of the Hermite-Hadamard integral inequality is thus sharper than the right-hand side of the inequality in Theorem 2.1.

Propositions 4.1 and 4.2 show the limitations of Theorem 2.1, demonstrating that the convex version of the Hermite-Hadamard integral inequality produces a sharper result.

Using Theorem 3.1, one can prove that the same conclusion holds in the concave case.

5. Conclusion

In this article, we have introduced new convex and concave integral inequalities, providing detailed proofs and illustrative examples. Our analysis emphasizes the importance of derivative information at the endpoints when formulating these inequalities. While these results are mathematically interesting and introduce novel techniques in the study of integral inequalities, we have also shown that they are not sharper than the classical Hermite-Hadamard integral inequality.

Several directions appear promising for future work. One possibility is to extend these inequalities to higher-order derivatives or to functions of several variables. Another is to investigate the conditions under which the inequalities could be sharpened, or combined with other functional inequalities, to obtain stronger bounds. Finally, applying these inequalities to optimization, probability and numerical analysis could provide further insights and demonstrate their practical relevance.

Conflict of interest

The author declare no competing interests.

Acknowledgment

The author would like to thank the reviewers for their constructive comments.

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