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STUDY OF A HYPERBOLIC PROBLEM WITH POLYNOMIAL NONLINEARITY

Dieudonné Ampini[1], Challoum Mouanda, and Simeon Ndikoumana Ngouya

ABSTRACT. This article presents a rigorous mathematical analysis of a nonlinear hyperbolic problem with a continuous potential. Using the Faedo-Galerkin method, we establish the existence and uniqueness of the solution within an appropriate functional framework. Tools such as energy inequalities, compactness, and weak-* convergence are employed to demonstrate the main results.

1. INTRODUCTION AN PROBLEM PRESENTATION

Hyperbolic partial differential equations arise in numerous physical models, particularly in continuum mechanics and acoustics. Studying their behavior under complex nonlinearities is essential for understanding the stability and dynamics of solutions.

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ its boundary, which we assume is sufficiently regular. We consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + b(u) = f(x,t) \quad \text{in} \quad Q = \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u(x,0)}{\partial t} = u_1(x), \quad x \in \Omega \subset \mathbb{R}^n. \tag{1.2}$$

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[1] *corresponding author*

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Here, f, u_0, and u_1 are given functions. Let $\rho > 2$ and $b(u) = c(1 + |u|^{\rho-1})$ with $c > 0$. The goal of this study is to prove the existence and uniqueness of a real-valued solution $u = u(x, t)$ for $x \in \Omega$ and $t \in [0, T]$.

2. WEAK FORMULATION OF THE PROBLEM

Let φ be a test function, i.e., $\varphi \in \mathbb{D}(\Omega)$. Multiplying equation (1.1) by φ and integrating over Ω, we obtain:

$$\left(\frac{\partial^2 u}{\partial t^2}, \varphi\right) - (\Delta u, \varphi) + (b(u), \varphi) = (f, \varphi)\,. \tag{2.1}$$

Using Green's formula:

$$(\Delta u, \varphi) = -\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \varphi \, dx \tag{2.2}$$

$$\Rightarrow (\Delta u, \varphi) = -\int_\Omega \nabla u \cdot \nabla \varphi \, dx. \tag{2.3}$$

Substituting (2.3) into (2.1), we get:

$$\frac{d^2}{dt^2}(u, \varphi) + \int_\Omega \nabla u \cdot \nabla \varphi \, dx + (b(u), \varphi) = (f, \varphi)$$

$$\Rightarrow \frac{d^2}{dt^2}(u, \varphi) + (\nabla u, \nabla \varphi) + (b(u), \varphi) = (f, \varphi). \tag{2.4}$$

Equation (2.4) is meaningful if, and only if:

$$\varphi,\ u,\ \nabla u \in L^2(\Omega).$$

This implies that $u \in \dot{H}^1(\Omega)$, where:

$$\dot{H}^1(\Omega) = \left\{ u \in H^1(\Omega) \,\middle|\, \frac{\partial u}{\partial \vec{n}} = 0 \right\}.$$

2.1. **A Priori Estimate.** Let us multiply (1.1) by $\dfrac{\partial u}{\partial t}$ and integrate over Ω. We obtain:

$$\left(\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\right) - \left(\Delta u, \frac{\partial u}{\partial t}\right) + \left(b(u), \frac{\partial u}{\partial t}\right) = \left(f, \frac{\partial u}{\partial t}\right). \tag{2.5}$$

Let us examine each term in (2.5):

$$\begin{aligned}
\left(\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\right) &= \int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\, dx = \frac{1}{2}\int_\Omega 2\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\, dx \\
&= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx,
\end{aligned}$$

$$\left(\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\right) = \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2, \tag{2.6}$$

$$\begin{aligned}
\left(\Delta u, \frac{\partial u}{\partial t}\right) &= -\int_\Omega \nabla u \cdot \nabla\frac{\partial u}{\partial t}\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}}\frac{\partial u}{\partial t}\, dx \\
&= -\int_\Omega \nabla u \cdot \nabla\frac{\partial u}{\partial t}\, dx \quad \text{since } \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega \\
&= -\frac{1}{2}\int_\Omega 2\nabla u \cdot \nabla\frac{\partial u}{\partial t}\, dx \\
&= -\frac{1}{2}\int_\Omega 2\nabla u \cdot \frac{\partial}{\partial t}(\nabla u)\, dx \\
&= -\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}(\nabla u)^2\, dx \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla u)^2\, dx,
\end{aligned}$$

$$\left(\Delta u, \frac{\partial u}{\partial t}\right) = -\frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2, \tag{2.7}$$

$$|u|^{\rho-1} = \begin{cases} u^{\rho-1} & \text{if } u \geq 0, \\ (-1)^{\rho-1}u^{\rho-1} & \text{if } u < 0. \end{cases}$$

Case 1: If $u \geq 0$:

$$\begin{aligned}
\left(b(u), \frac{\partial u}{\partial t}\right) &= \left(c(1+u^{\rho-1}), \frac{\partial u}{\partial t}\right) \\
&= c\int_\Omega \frac{\partial u}{\partial t}\, dx + \int_\Omega u^{\rho-1}\frac{\partial u}{\partial t}\, dx \\
&= c\int_\Omega \frac{\partial u}{\partial t}\, dx + \frac{c}{\rho}\int_\Omega \rho u^{\rho-1}\frac{\partial u}{\partial t}\, dx \\
&= c\int_\Omega \frac{\partial u}{\partial t}\, dx + \frac{c}{\rho}\left\|u\right\|^{\rho}_{L^{\rho}(\Omega)}. \qquad (2.8)
\end{aligned}$$

Case 2: If $u < 0$.

If ρ is even, then $\rho - 1$ is odd. This implies that: $(-1)^{\rho-1} < 0$ and $(-1)^{\rho-1}u^{\rho-1} > 0$. Thus, we have:

$$\left(b(u), \frac{\partial u}{\partial t}\right) = c\int_\Omega \frac{\partial u}{\partial t}\,dx + \frac{c}{\rho}\int_\Omega (-1)^{\rho-1}u^{\rho-1}\frac{\partial u}{\partial t}\,dx,$$

$$\left(b(u), \frac{\partial u}{\partial t}\right) = c\int_\Omega \frac{\partial u}{\partial t}\,dx + \frac{c}{\rho}\left\|u\right\|^\rho_{L^\rho(\Omega)}.$$

By substituting (2.6), (2.7), and (2.8) into (2.5), we obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2 + c\int_\Omega \frac{\partial u}{\partial t}\,dx + c\int_\Omega u^{\rho-1}\frac{\partial u}{\partial t}\,dx = \left(f, \frac{\partial u}{\partial t}\right).$$

According to the Cauchy–Schwarz inequality, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2 + c\int_\Omega \frac{\partial u}{\partial t}\,dx + \frac{c}{\rho}\frac{d}{dt}\left\|u\right\|^\rho_{L^\rho(\Omega)} \le \left\|f\right\|\left\|\frac{\partial u}{\partial t}\right\|.$$

Using Young's inequality, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2 + \frac{c}{\rho}\frac{d}{dt}\left\|u\right\|^\rho_{L^\rho(\Omega)} \le \frac{1}{2}\left\|f\right\|^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2 - c\int_\Omega \frac{\partial u}{\partial t}\,dx, \tag{2.9}$$

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2 + \frac{c}{\rho}\frac{d}{dt}\left\|u\right\|^\rho_{L^\rho(\Omega)} \le \frac{1}{2}\left\|f\right\|^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2 + c\int_\Omega \left|\frac{\partial u}{\partial t}\right|\,dx,$$

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2 + \frac{c}{\rho}\frac{d}{dt}\left\|u\right\|^\rho_{L^\rho(\Omega)} \le \frac{1}{2}\left\|f\right\|^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2 + c|\Omega|^{\frac{1}{2}}\left\|\frac{\partial u}{\partial t}\right\|.$$

Let us set: $K = c|\Omega|^{\frac{1}{2}}$. Then, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2 + \frac{c}{\rho}\frac{d}{dt}\left\|u\right\|^\rho_{L^\rho(\Omega)} \le \frac{1}{2}\left\|f\right\|^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2 + K\left\|\frac{\partial u}{\partial t}\right\|,$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla u\right\|^2 + \frac{c}{\rho}\frac{d}{dt}\left\|u\right\|^\rho_{L^\rho(\Omega)} &\le \frac{1}{2}\left\|f\right\|^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2 \\ &\quad + \frac{1}{2}K^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2.\end{aligned} \tag{2.10}$$

Multiplying (2.10) by 2 and integrating over $[0, t]$ with $t \le T$, we obtain:

$$\int_0^t \frac{d}{dt}\left[\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla u\right\|^2 + \frac{2c}{\rho}\left\|u\right\|^\rho_{L^\rho(\Omega)}\right]dt$$

$$\int_0^T \|f\|^2 dt + \int_0^T K^2 dt + 2\int_0^t \left\|\frac{\partial u}{\partial t}\right\|^2 dt.$$

We obtain:

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla u\|^2 + \frac{2c}{\rho}\|u\|^{\rho}_{L^{\rho}(\Omega)} \le \int_0^T \|f\|^2 dt + 2\int_0^t \left\|\frac{\partial u}{\partial t}\right\|^2 dt + \int_0^T K^2 dt$$
$$+ \|u_1\|^2 + \|\nabla u_0\|^2 + \frac{2c}{\rho}\|u_0\|^{\rho}_{L^{\rho}(\Omega)}.$$

Let us denote:

$$C = \int_0^T \|f\|^2 dt + \int_0^T K^2 dt + \|u_1\|^2 + \|\nabla u_0\|^2 + \frac{2c}{\rho}\|u_0\|^{\rho}_{L^{\rho}(\Omega)}.$$

Then we have:

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla u\|^2 + \frac{2c}{\rho}\|u\|^{\rho}_{L^{\rho}(\Omega)} \le C + 2\int_0^t \left\|\frac{\partial u}{\partial t}\right\|^2 dt. \tag{2.11}$$

Adding $\|u\|^2$ to both sides:

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla u\|^2 + \frac{2c}{\rho}\|u\|^{\rho}_{L^{\rho}(\Omega)} + \|u\|^2 \le C + 2\int_0^t \left\|\frac{\partial u}{\partial s}\right\|^2 ds + \|u\|^2.$$

Let's now estimate $\|u\|^2$.

Assume $u(x,t) = u(t)$, we know that:

$$u(x,t) = \int_0^t \frac{\partial u(x,s)}{\partial s} ds + u(0).$$

Hence:

$$|u(x,t)| = \left|\int_0^t \frac{\partial u(x,s)}{\partial s} ds + u(0)\right| \le \left|\int_0^t \frac{\partial u(x,s)}{\partial s} ds\right| + |u(0)|,$$
$$|u(x,t)| \le \int_0^t \left|\frac{\partial u(x,s)}{\partial s}\right| ds + |u(0)|.$$

By the Cauchy–Schwarz inequality, we obtain:

$$|u(x,t)| \le \sqrt{t}\left(\int_0^t \left|\frac{\partial u(x,s)}{\partial s}\right|^2 ds\right)^{\frac{1}{2}} + |u(0)|, \tag{2.12}$$

$$|u(x,t)|^2 \leq \left[\sqrt{t}\left(\int_0^t \left|\frac{\partial u(x,s)}{\partial s}\right|^2 ds\right)^{\frac{1}{2}} + |u(0)|\right]^2.$$

Since $(a+b)^2 \leq 2(a^2+b^2)$, we deduce:

$$|u(x,t)|^2 \leq 2\left(\sqrt{t}\left(\int_0^t \left|\frac{\partial u(x,s)}{\partial s}\right|^2 ds\right)^{\frac{1}{2}}\right)^2, +2|u(0)|^2$$

$$|u(x,t)|^2 \leq 2t\int_0^t \left|\frac{\partial u(x,s)}{\partial s}\right|^2 ds + 2|u(0)|^2. \tag{2.13}$$

Now, integrating (2.13) over Ω, we obtain:

$$\int_\Omega |u(x,t)|^2 dx \leq 2t\int_0^t \int_\Omega \left|\frac{\partial u(x,s)}{\partial s}\right|^2 dxds + 2\int_\Omega |u(0)|^2 dx. \tag{2.14}$$

Hence, we obtain:

$$\|u\|^2 \leq 2T\int_0^t \left\|\frac{\partial u}{\partial s}\right\|^2 ds + \|u_0\|^2. \tag{2.15}$$

After estimating, we obtain:

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla u\|^2 + \frac{2c}{\rho}\|u\|^\rho_{L^\rho(\Omega)} + \|u\|^2 \leq C_1 + 2(T+1)\int_0^t \left\|\frac{\partial u}{\partial s}\right\| ds, \tag{2.16}$$

where $C_1 = C + \|u_0\|^2$. We also have:

$$\begin{aligned} &\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{2c}{\rho}\|u\|^\rho_{L^\rho(\Omega)} + \|u\|^2_{H^1(\Omega)} \\ &\leq C_1 + 2(T+1)\int_0^t \left[\left\|\frac{\partial u}{\partial s}\right\| + \|u\|^2_{H^1(\Omega)} + \frac{2c}{\rho}\|u\|^\rho_{L^\rho(\Omega)}\right] ds. \end{aligned} \tag{2.17}$$

Let us define:

$$F(t) = \left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{2c}{\rho}\|u\|^\rho_{L^\rho(\Omega)} + \|u\|^2_{H^1(\Omega)}.$$

Then inequality (2.17) becomes:

$$F(t) \leq C_1 + 2(T+1)\int_0^t F(s)ds.$$

According to Grönwall's inequality:

$$F(t) \leq C_1 e^{2(T+1)t}, \tag{2.18}$$

$$F(t) \leq C_1 e^{2(T+1)T},$$

$$F(t) \leq C_2. \tag{2.19}$$

Therefore:

$$\begin{cases} u \in L^\infty(0,T;V), \\ \dfrac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)), \end{cases}$$

where:

$$V = \dot{H}^1(\Omega) \cap L^\rho(\Omega).$$

The space V, equipped with the norm $\|\cdot\|_{H^1(\Omega)} + \|\cdot\|_{L^\rho(\Omega)}$, is a Hilbert space.

By multiplying the first equation (1.1) by an element $v \in V$, integrating over Ω, and using Green's formula, we obtain the following variational formulation:

$$\frac{d^2}{dt^2}(u,v) + (\nabla u, \nabla v) + (b(u), v) = (f, v) \quad \forall v \in V \tag{2.20}$$

2.2. **Theorem of existence.**

Theorem 2.1. *Let $f \in L^2(Q)$, $u_0 \in V$, and $u_1 \in L^2(\Omega)$. Then problem* (1.1) *admits a solution u satisfying:*

$$u \in L^\infty(0,T;V), \tag{2.21}$$

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)). \tag{2.22}$$

2.2.1. *Proof of the Theorem.*

Remark 2.1. *The expressions $u(x,0) = u_0(x)$ and $u'(x,0) = u_1(x)$ are meaningful. Indeed, from* (2.21)*,* (2.22)*, and Lemma 2.1, it follows in particular that u is continuous from*

$$[0,T] \to L^2(\Omega),$$

so the expression $u(x,0) = u_0(x)$ is valid.

To verify that $u'(x,0) = u_1(x)$ makes sense, one must use equation (P.1), which is written as:

$$\frac{\partial^2 u}{\partial t^2} = f + \Delta u - c(1 + |u|^{\rho-1}).$$

This implies that

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega)) \cup L^\infty(0,T;H^{-1}(\Omega) \cup L^2(\Omega)).$$

In particular,

$$\frac{\partial^2 u}{\partial t^2} \in L^\infty(0,T;H^{-1}(\Omega) \cup L^2(\Omega)).$$

Hence, u_1 is indeed meaningful.

2.2.2. *Proof of the Theorem.*

The proof is based on the Faedo-Galerkin method, which consists of the following three steps:

Step 1. Construction of an approximate solution.

Step 2. Establishing a priori estimates on this approximate solution.

Step 3. Passing to the limit using compactness properties (especially in the nonlinear terms).

Step 1: Construction of an approximate solution. Since the space V is a separable Hilbert space, it admits a sequence $e_1, e_2, \dots, e_m$ with the following properties:

$$\begin{cases} e_i \in V,\ \forall i; \\ \forall m,\ e_1, e_2, \dots, e_m\, are\, linearly\, independent; \\ V_m = \text{span}(e_1, e_2, \dots, e_m)\, is\, dense\, in\, V; \\ (e_i, e_j) = \delta_{ij} \quad \forall i, j. \end{cases} \tag{2.23}$$

With the homogeneous Neumann boundary condition, the operator Δ admits a sequence of eigenvalues $(\lambda_i)_{i\geq 1}$, where each e_i is an eigenfunction associated with these eigenvalues, satisfying:

$$-\Delta e_i = \lambda_i e_i.$$

In particular: for every $u_0 \in V$, there exists a sequence $(u_{0m})_m$, with

$$u_{0m} = \sum_{k=1}^{m} \alpha_{km} e_k \longrightarrow u_0 \quad \text{as } m \to \infty.$$

For every $u_1 \in L^2(\Omega)$, there exists a sequence $(u_{1m})_m$, with

$$u_{1m} = \sum_{k=1}^{m} \beta_{km} e_k \longrightarrow u_1 \quad \text{as } m \to \infty.$$

We then seek an approximate solution of the form:

$$u_m(x,t) = \sum_{i=1}^{m} u_{im}(t) e_i,$$

as the candidate for the following problem:

$$(P_m) \begin{cases} \frac{\partial^2 u_m}{\partial t^2} - \Delta u_m + b(u_m) = f(x,t) \\ u_m(0,x) = u_{0m}, \quad u'_m(0,x) = u_{1m} \\ \frac{\partial u_m}{\partial \vec{n}}\big|_{\partial\Omega} = 0. \end{cases}$$

Multiply equation (2.23) by e_k and integrate over Ω. We obtain:

$$\left(\frac{\partial^2 u_m}{\partial t^2}, e_k\right) - (\Delta u_m, e_k) + (b(u_m), e_k) = (f(x,t), e_k)\,. \tag{2.24}$$

Replacing u_m in equation (2.24), we obtain:

$$\left(\sum_{i=1}^{m} \frac{\partial^2 u_{im}(t)}{\partial t^2} e_i, e_k\right) + \left(\sum_{i=1}^{m} u_{im}(-\Delta e_i), e_k\right) + (b(u_m), e_k) = (f(x,t), e_k)\,,$$

$$\sum_{i=1}^{m} u''_{im}(t)(e_i, e_k) + \sum_{i=1}^{m} u_{im}\lambda_i (e_i, e_k) + (b(u_m), e_k) = (f(x,t), e_k)\,.$$

For $i = k$, we have for all $i \in \mathbb{N}$:

$$u''_{im}+\lambda_i u_{im}+(b(u_m), e_i) = (f(x,t), e_i)\,, \quad \forall i \in \mathbb{N}^* u''_{im}+\lambda_i u_{im}+(b(u_m) - f(x,t), e_i) = 0$$

We then obtain the following system of equations for all $i \in \mathbb{N}^*$:

$$\begin{cases} u''_{1m} + \lambda_1 u_{1m} + (b(u_{1m}) - f(x,t), e_1) = 0, \\ u''_{2m} + \lambda_2 u_{2m} + (b(u_{2m}) - f(x,t), e_2) = 0, \\ \vdots \\ u''_{mm} + \lambda_m u_{mm} + (b(u_{mm}) - f(x,t), e_m) = 0. \end{cases} \tag{2.25}$$

$$\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\end{pmatrix}\begin{pmatrix}u''_{1m}(x,t)\\ u''_{2m}(x,t)\\ \vdots\\ u''_{mm}(x,t)\end{pmatrix}+\begin{pmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda_m\end{pmatrix}\begin{pmatrix}u_{1m}(x,t)\\ u_{2m}(x,t)\\ \vdots\\ u_{mm}(x,t)\end{pmatrix}+\begin{pmatrix}b(u_{1m}(x,t))-f(x,t)\\ b(u_{2m}(x,t))-f(x,t)\\ \vdots\\ b(u_{mm}(x,t))-f(x,t)\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ \vdots\\ e_m\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \vdots\\ 0\end{pmatrix}$$

Thus, the system (2.25) becomes the following matrix equation:

$$I_m X''_m + A_m X_m + B_m = 0,$$

where

$$I_m = \begin{pmatrix}1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\end{pmatrix}, \quad A_m = \begin{pmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda_m\end{pmatrix},$$

$$B_m = \begin{pmatrix}(b(u_{1m}(x,t)) - f(x,t), e_1)\\ (b(u_{2m}(x,t)) - f(x,t), e_2)\\ \vdots\\ (b(u_{mm}(x,t)) - f(x,t), e_m)\end{pmatrix}, \quad X_m = \begin{pmatrix}u_{1m}(x,t)\\ u_{2m}(x,t)\\ \vdots\\ u_{mm}(x,t)\end{pmatrix}.$$

Since $\det I_m \neq 0$, the matrix I_m is invertible. Therefore, the system admits a well-defined solution over the interval $]0; t_m[$.

We will show in what follows that $t_m = T$.

Step 2: A Priori Estimate. We multiply the first equation of (P_m) by $\dfrac{\partial u_m}{\partial t}$ and integrate over Ω. We obtain:

$$\left(\frac{\partial^2 u_m}{\partial t^2}, \frac{\partial u_m}{\partial t}\right) - \left(\Delta u_m, \frac{\partial u_m}{\partial t}\right) + \left(b(u_m), \frac{\partial u_m}{\partial t}\right) = \left(f(x,t), \frac{\partial u_m}{\partial t}\right), \tag{2.26}$$

$$\begin{aligned}\left(\frac{\partial^2 u_m}{\partial t^2}, \frac{\partial u_m}{\partial t}\right) &= \int_\Omega \frac{\partial^2 u_m}{\partial t^2}\frac{\partial u_m}{\partial t}dx = \frac{1}{2}\int_\Omega 2\frac{\partial^2 u_m}{\partial t^2}\frac{\partial u_m}{\partial t}dx\\ &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}(\frac{\partial u_m}{\partial t})^2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega (\frac{\partial u_m}{\partial t})^2 dx\end{aligned}$$

$$\left(\frac{\partial^2 u_m}{\partial t^2}, \frac{\partial u_m}{\partial t}\right) = \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}\right\|^2, \tag{2.27}$$

$$\begin{aligned}
\left(\Delta u_m, \frac{\partial u_m}{\partial t}\right) &= -\int_\Omega \nabla u_m \nabla \frac{\partial u_m}{\partial t} dx + \int_\Omega \frac{\partial u_m}{\partial \vec{n}} \frac{\partial u_m}{\partial t} dx \\
&= -\int_\Omega \nabla u_m \nabla \frac{\partial u_m}{\partial t} dx \\
&= -\frac{1}{2}\int_\Omega 2\nabla u_m \nabla \frac{\partial u_m}{\partial t} dx \\
&= -\frac{1}{2}\int_\Omega 2\nabla u_m \frac{\partial}{\partial t} \nabla u_m \\
&= -\frac{1}{2}\int_\Omega \frac{\partial}{\partial t} (\nabla u_m)^2 dx \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla u_m)^2 dx
\end{aligned}$$

$$\left(\Delta u_m, \frac{\partial u_m}{\partial t}\right) = -\frac{1}{2}\frac{d}{dt}||\nabla u_m||^2. \tag{2.28}$$

If $u_m \geq 0$:

$$\begin{aligned}
\left(b(u_m), \frac{\partial u_m}{\partial t}\right) &= \left(c(1 + u_m^{\rho-1}), \frac{\partial u_m}{\partial t}\right) \\
&= c\int_\Omega \frac{\partial u_m}{\partial t} dx + \int_\Omega u_m^{\rho-1} \frac{\partial u_m}{\partial t} dx \\
&= c\int_\Omega \frac{\partial u_m}{\partial t} dx + \frac{c}{\rho}\int_\Omega \rho u_m^{\rho-1} \frac{\partial u_m}{\partial t} dx
\end{aligned}$$

$$\left(b(u_m), \frac{\partial u_m}{\partial t}\right) = c\int_\Omega \frac{\partial u_m}{\partial t} dx + \frac{c}{\rho}||u_m||^\rho_{L^\rho(\Omega)}. \tag{2.29}$$

By substituting (2.27), (2.28), and (2.29) into (2.26), we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + c\int_\Omega \frac{\partial u_m}{\partial t}\, dx + c\int_\Omega u_m^{\rho-1}\frac{\partial u_m}{\partial t}\, dx \\
&= \left(f(x,t), \frac{\partial u_m}{\partial t}\right).
\end{aligned} \tag{2.30}$$

Using the Cauchy–Schwarz inequality, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + c\int_\Omega \frac{\partial u_m}{\partial t}\, dx + \frac{c}{\rho}\frac{d}{dt}\|u_m\|^\rho_{L^\rho(\Omega)} \leq \|f\|\left\|\frac{\partial u_m}{\partial t}\right\|.$$

According to Young's inequality, we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + \frac{c}{\rho}\frac{d}{dt}\|u_m\|^{\rho}_{L^{\rho}(\Omega)} \\
&\leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2 - c\int_{\Omega}\frac{\partial u_m}{\partial t}dx,
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + \frac{c}{\rho}\frac{d}{dt}\|u_m\|^{\rho}_{L^{\rho}(\Omega)} \\
&\leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + c\int_{\Omega}\left|\frac{\partial u_m}{\partial t}\right|dx,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + \frac{c}{\rho}\frac{d}{dt}\|u_m\|^{\rho}_{L^{\rho}(\Omega)} \\
&\leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + c|\Omega|^{\frac{1}{2}}\left\|\frac{\partial u_m}{\partial t}\right\|.
\end{aligned}$$

Let us set: $K_1 = c|\Omega|^{\frac{1}{2}}$. We have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + \frac{c}{\rho}\frac{d}{dt}\|u_m\|^{\rho}_{L^{\rho}(\Omega)} \\
&\leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + K_1\left\|\frac{\partial u_m}{\partial t}\right\|.
\end{aligned}$$

Applying Young's inequality to the right-hand side yields:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + \frac{c}{\rho}\frac{d}{dt}\|u_m\|^{\rho}_{L^{\rho}(\Omega)} \leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2 \\
+ \frac{1}{2}K_1^2 + \frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2.
\end{aligned} \tag{2.32}$$

We multiply (2.32) by 2 and integrate over the interval $[0, t]$ with $t \leq T$:

$$\begin{aligned}
&\int_0^t \frac{d}{dt}\left[\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \|\nabla u_m\|^2 + \frac{2c}{\rho}\|u_m\|^{\rho}_{L^{\rho}(\Omega)}\right]dt \\
&\leq \int_0^T \|f\|^2 dt + \int_0^T K_1^2 dt + 2\int_0^t \left\|\frac{\partial u_m}{\partial t}\right\|^2 dt.
\end{aligned}$$

After applying upper bounds, we obtain:

$$\begin{aligned}\left\|\frac{\partial u_m}{\partial t}\right\|^2+\|\nabla u_m\|^2+\frac{2c}{\rho}\|u_m\|^{\rho}_{L^{\rho}(\Omega)}\leq{}&\int_0^T\|f\|^2\,dt+2\int_0^t\left\|\frac{\partial u_m}{\partial t}\right\|^2\,dt\\&+\int_0^T K_1^2\,dt+\|u_{1m}\|^2+\|\nabla u_{0m}\|^2\\&+\frac{2c}{\rho}\|u_{0m}\|^{\rho}_{L^{\rho}(\Omega)}.\end{aligned}$$

Let us define:

$$C=\int_0^T\|f\|^2\,dt+\int_0^T K_1^2\,dt+\|u_{1m}\|^2+\|\nabla u_{0m}\|^2+\frac{2c}{\rho}\|u_{0m}\|^{\rho}_{L^{\rho}(\Omega)}.$$

We have:

$$\left\|\frac{\partial u_m}{\partial t}\right\|^2+\|\nabla u_m\|^2+\frac{2c}{\rho}\|u_m\|^{\rho}_{L^{\rho}(\Omega)}\leq C+2\int_0^t\left\|\frac{\partial u_m}{\partial t}\right\|^2\,dt$$

Let us add $\|u_m\|^2$ to both sides:

$$\left\|\frac{\partial u_m}{\partial t}\right\|^2+\|\nabla u_m\|^2+\frac{2c}{\rho}\|u_m\|^{\rho}_{L^{\rho}(\Omega)}+\|u_m\|^2\leq C+2\int_0^t\left\|\frac{\partial u_m}{\partial s}\right\|^2\,ds+\|u_m\|^2$$

Let us now seek an upper bound for $\|u_m\|^2$. Let us set $u_m(x,t)=u_m(t)$. We know that:

$$\begin{aligned}u_m(x,t)&=\int_0^t\frac{\partial u_m(x,s)}{\partial s}\,ds+u_m(0),\\|u_m(x,t)|&=\left|\int_0^t\frac{\partial u_m(x,s)}{\partial s}\,ds+u_m(0)\right|\leq\left|\int_0^t\frac{\partial u_m(x,s)}{\partial s}\,ds\right|+|u_m(0)|,\\|u_m(x,t)|&\leq\int_0^t\left|\frac{\partial u_m(x,s)}{\partial s}\right|\,ds+|u_m(0)|.\end{aligned}$$

According to the Cauchy–Schwarz inequality, we obtain:

$$|u_m(x,t)|\leq\sqrt{t}\left(\int_0^t\left|\frac{\partial u_m(x,s)}{\partial s}\right|^2ds\right)^{\frac{1}{2}}+|u_m(0)|,\tag{2.33}$$

$$|u_m(x,t)|^2\leq\left[\sqrt{t}\left(\int_0^t\left|\frac{\partial u_m(x,s)}{\partial s}\right|^2ds\right)^{\frac{1}{2}}+|u_m(0)|\right]^2.$$

Now, since $(a+b)^2 \leq 2(a^2+b^2)$, we deduce:

$$|u_m(x,t)|^2 \leq 2\left(\sqrt{t}\left(\int_0^t \left|\frac{\partial u_m(x,s)}{\partial s}\right|^2 ds\right)^{\frac{1}{2}}\right)^2 + 2|u_m(0)|^2,$$

$$|u_m(x,t)|^2 \leq 2t\int_0^t \left|\frac{\partial u_m(x,s)}{\partial s}\right|^2 ds + 2|u_m(0)|^2. \tag{2.34}$$

Integrating (2.34) over Ω, we obtain:

$$\int_\Omega |u_m(x,t)|^2 dx \leq 2t\int_0^t \int_\Omega \left|\frac{\partial u_m(x,s)}{\partial s}\right|^2 dx\, ds + 2\int_\Omega |u_m(0)|^2 dx.$$

We have the inequality:

$$\|u_m\|^2 \leq 2T\int_0^t \left\|\frac{\partial u_m}{\partial s}\right\|^2 ds + \|u_{0m}\|^2.$$

After further estimation, we obtain:

$$\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \|\nabla u_m\|^2 + \frac{2c}{\rho}\|u_m\|^\rho_{L^\rho(\Omega)} + \|u_m\|^2 \leq C_1 + 2(T+1)\int_0^t \left\|\frac{\partial u_m}{\partial s}\right\| ds,$$

where $C_1 = C + \|u_{0m}\|^2$. By regrouping terms:

$$\begin{aligned}\left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{2c}{\rho}\|u_m\|^\rho_{L^\rho(\Omega)} + \|u_m\|^2_{H^1(\Omega)} &\leq C_1 + 2(T+1)\int_0^t \left[\left\|\frac{\partial u_m}{\partial s}\right\| + \|u_m\|^2_{H^1(\Omega)}\right] ds \\ &\quad + 2(T+1)\cdot\frac{2c}{\rho}\int_0^t \|u_m\|^\rho_{L^\rho(\Omega)} ds.\end{aligned}$$

Let us define:

$$F(t) = \left\|\frac{\partial u_m}{\partial t}\right\|^2 + \frac{2c}{\rho}\|u_m\|^\rho_{L^\rho(\Omega)} + \|u_m\|^2_{H^1(\Omega)}.$$

Then the inequality becomes:

$$F(t) \leq C_1 + 2(T+1)\int_0^t F(s)ds.$$

According to Gronwall's lemma, we obtain:

$$\begin{aligned}&F(t) \leq C_1 \cdot \exp\left(2(T+1)t\right) \leq C_1 e^{2(T+1)T},\\ &F(t) \leq C_2.\end{aligned}$$

Hence,

$$\begin{cases} u_m \in L^\infty(0,T;\dot{H}^1(\Omega)\cap L^\rho(\Omega)), \\ \dfrac{\partial u_m}{\partial t} \in L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{2.35}$$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular open set. Then every bounded subset of $H^1(\Omega)$ is relatively compact.*

Step 3: Passing to the Limit. From (2.30), we deduce that we can extract two convergent subsequences (u_μ) and (u'_μ) from (u_m) and (u'_m), respectively, such that:

$$u_\mu \longrightarrow u \quad \text{in} \quad L^\infty(0,T;V) \quad \text{weak-*} \tag{2.36}$$

$$u'_m \longrightarrow u' \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{weak-*} \tag{2.37}$$

Moreover, it follows in particular from (2.30) that:

$$\begin{cases} (u_m) \text{ is bounded in } L^2(0,T;V), \\ (u'_m) \text{ is bounded in } L^2(0,T;L^2(\Omega)) = L^2(Q). \end{cases}$$

We know from the Rellich-Kondrachov theorem that the embedding of $H^1(Q)$ into $L^2(Q)$ is compact. Therefore,

$$u_\mu \longrightarrow u \tag{2.38}$$

strongly in $L^2(Q)$ and almost everywhere in Q. From (2.9), we have:

$$\frac{\partial^2 u_m}{\partial t^2} - \Delta u_m + b(u_m) = f(x,t),$$

$$\Rightarrow \frac{\partial^2 u_m}{\partial t^2} = f(x,t) + \Delta u_m - b(u_m)$$

$$\Delta : H^1(\Omega) \longrightarrow H^{-1}(\Omega) \quad \Rightarrow \quad \Delta \in \mathcal{L}(H^1(\Omega);H^{-1}(\Omega)).$$

And since:

$$u_m \in L^\infty(0,T;V), \quad \text{with } V \subset H^1(\Omega),$$

we have:

$$u_m \in L^\infty(0,T;H^1(\Omega)).$$

Moreover, since $H^1(\Omega) \subset H^{-1}(\Omega)$, it follows that there exists a constant C such that:

$$\|\cdot\|_{H^{-1}(\Omega)} \le C\|\cdot\|_{H^1(\Omega)}$$

$$\Rightarrow \Delta u_m \in L^\infty(0,T;H^{-1}(\Omega)).$$

As this space is bounded and separable, there exists a subsequence Δu_μ of Δu_m such that:

$$\Delta u_\mu \longrightarrow \Delta u \quad \text{in } L^\infty(0,T;H^{-1}(\Omega)) \quad \text{weak-*}.$$

We also know in particular that:

$$u_m \in L^\infty(0,T;L^\rho(\Omega)).$$

This implies that:

$$b(u_m) \in L^\infty(0,T;L^p(\Omega)), \tag{2.39}$$

with $p = \frac{\rho}{\rho-1}$, so that $\frac{1}{\rho} + \frac{1}{p} = 1$. From (2.9), we deduce that:

$$\frac{\partial^2 u_m}{\partial t^2} \in L^2(0,T;L^2(\Omega)) \cup L^\infty(0,T;H^{-1}(\Omega)) \cup L^\infty(0,T;L^p(\Omega)),$$

$$\Rightarrow \frac{\partial^2 u_m}{\partial t^2} \in L^2(0,T;L^2(\Omega)) \cup L^\infty(0,T;H^{-1}(\Omega)) \cup L^p(\Omega).$$

In particular:

$$\frac{\partial^2 u_m}{\partial t^2} \in L^\infty(0,T;H^{-1}(\Omega)) \cup L^p(\Omega). \tag{2.40}$$

Since this space is bounded and separable, it follows that $\frac{\partial^2 u_m}{\partial t^2}$ admits a subsequence $\frac{\partial^2 u_\mu}{\partial t^2}$ such that:

$$\frac{\partial^2 u_\mu}{\partial t^2} \longrightarrow \frac{\partial^2 u}{\partial t^2} \quad \text{in } L^\infty(0,T;H^{-1}(\Omega) \cup L^p(\Omega)) \quad \text{weak-*}.$$

We have:

$$b(u_m) \in L^\infty(0,T;L^p(\Omega)).$$

It follows that there exists a sequence g_μ such that:

$$g_\mu \longrightarrow \omega \quad \text{in } L^\infty(0,T;L^p(\Omega)) \quad \text{weak-*}, \tag{2.41}$$

with $g_\mu = c(1+|u_\mu|^{\rho-1})$. The key point is to show that:

$$\omega = g = c(1+|u|^{\rho-1}). \tag{2.42}$$

To do this, we need the following lemma:

Lemma 2.1. *Let Q be a bounded open subset of $\mathbb{R}^n_x \times \mathbb{R}_t$, and let g_μ and g be two functions in $L^q(Q)$, with $1<q<\infty$, such that:*

$$\|g_\mu\|_{L^q(Q)} \leq C, \quad g_\mu \longrightarrow g \quad \textit{almost everywhere in } Q.$$

Then $g_\mu \longrightarrow g$ in $L^q(Q)$ weakly.

In our case, we have:

$$g_\mu = c(1+|u_\mu|^{\rho-1}), \quad q = p = \frac{\rho}{\rho-1}.$$

Since $u \in C^1([0,T]\times\Omega)$, and from (2.38), we have:

$$g_\mu = c(1+|u_\mu|^{\rho-1}) \longrightarrow g = c(1+|u|^{\rho-1}) \quad \text{strongly in } L^2(Q),$$

and from (2.41), we also have:

$$g_\mu \longrightarrow \omega \quad \text{in } L^\infty(0,T;L^p(\Omega)) \quad \text{weak-*}.$$

Hence, by the previous lemma, $\omega = g = c(1+|u|^{\rho-1})$. Thus, we can pass to the limit in equation (2.23) for $m=\mu$.

For any fixed $k \in \mathbb{N}^*$ and $\mu > k$, we have:

$$\left(\frac{\partial^2 u_\mu}{\partial t^2}, e_k\right) - (\Delta u_\mu, e_k) + (b(u_\mu), e_k) = (f(x,t), e_k).$$

From (2.42) and the weak convergence, we deduce:

$$\frac{d^2}{dt^2}(u,e_k) - (\Delta u, e_k) + (b(u), e_k) = (f(x,t), e_k).$$

Since V_m is dense in V, for every $v \in V$, we have $e_k \longrightarrow v$ as $k \longrightarrow \infty$, hence:

$$\frac{d^2}{dt^2}(u,v) - (\Delta u, v) + (b(u), v) = (f(x,t), v),$$

$$\left(\frac{\partial^2 u}{\partial t^2}, v\right) - (\Delta u, v) + (b(u), v) = (f(x,t), v)\,.$$

From which it follows:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + b(u) = f(x,t).$$

It results that u satisfies (P.1) (and also equations (2.21) and (2.22)).

It remains to show that $u(x,0) = u_0$ and $\frac{\partial u(x,0)}{\partial t} = u_1$.

From (2.39), (2.40), and Lemma 1, we have in particular:

$$u_\mu(0) \longrightarrow u(0) \quad \text{weakly in } L^2(\Omega).$$

Moreover, since $u_{0\mu} \longrightarrow u_0$ in V, we conclude that $u(0) = u_0$. Using the same technique, we verify that $u'(0) = u_1$ holds.

3. UNIQUENESS OF THE SOLUTION

Theorem 3.1. *Let $f \in L^2(Q)$, $u_0(x) \in V$, and $u_1(x) \in L^2(\Omega)$ be given functions. Then the solution $u(x,t)$ obtained in Theorem 1 is unique.*

Proof. Let u and v be two solutions in the sense of Theorem 1. Define $\omega = u - v$, then ω satisfies:

$$\omega'' - \Delta\omega + c(|u|^{\rho-1} - |v|^{\rho-1}) = 0, \tag{3.1}$$

$$\omega(0) = 0, \quad \omega'(0) = 0, \tag{3.2}$$

$$\omega \in L^\infty(0,T;V), \tag{3.3}$$

$$\omega' \in L^\infty(0,T;L^2(\Omega)). \tag{3.4}$$

We cannot multiply equation (3.1) by $\dfrac{\partial\omega}{\partial t}$ because $\dfrac{\partial\omega}{\partial t} \notin V$. Therefore, we will construct an auxiliary function ψ such that for all $s \in [0,T]$, we have:

$$\psi(x,t) = \begin{cases} -\int_t^s \omega(\sigma)\,d\sigma & \text{if } 0 \le t \le s \\ 0 & \text{if } t \ge s \end{cases}, \tag{3.5}$$

This implies that:

$$\frac{\partial\psi(x,t)}{\partial t}=\omega(t)=\frac{d}{dt}\left(-\int_t^s \omega(\sigma)\,d\sigma\right). \tag{3.6}$$

On the other hand, $\psi(t)=\omega_1(t)-\omega_1(s)$ with $\omega_1(t)=\int_0^t\omega(\sigma)\,d\sigma$. We multiply equation (3.1) by $\psi(t)$ and integrate over $[0,s]$. We obtain:

$$\int_0^s \frac{\partial^2\omega}{\partial t^2}\psi\,dt-\int_0^s \Delta\omega\,\psi\,dt=c\int_0^s \left(|v|^{\rho-1}-|u|^{\rho-1}\right)\psi\,dt. \tag{3.7}$$

According to integration by parts:

$$\begin{aligned}\int_0^s \frac{\partial^2\omega}{\partial t^2}\psi\,dt&=\left[\frac{\partial\omega(x,s)}{\partial t}\psi(x,s)-\frac{\partial\omega(x,0)}{\partial t}\psi(x,0)\right]\\&\quad-\int_0^s \frac{\partial\omega}{\partial t}\frac{\partial}{\partial t}\left(-\int_t^s\omega(\sigma)\,d\sigma\right)dt.\end{aligned}$$

But:

$$\psi(x,s)=-\int_s^s\omega(\sigma)\,d\sigma=0\quad\text{and}\quad\frac{\partial\omega(x,0)}{\partial t}=0,$$

$$\int_0^s \frac{\partial^2\omega}{\partial t^2}\psi\,dt=-\int_0^s \frac{\partial\omega}{\partial t}\frac{\partial}{\partial t}\left(-\int_t^s\omega(\sigma)\,d\sigma\right)dt. \tag{3.8}$$

Now, integrate equation (3.8) over the domain Ω:

$$\begin{aligned}\int_\Omega\int_0^s \frac{\partial^2\omega}{\partial t^2}\psi\,dt\,dx&=-\int_\Omega\int_0^s \frac{\partial\omega}{\partial t}\frac{\partial}{\partial t}\left(-\int_t^s\omega(\sigma)\,d\sigma\right)dt\,dx\\&=-\int_\Omega\int_0^s\frac{\partial\omega}{\partial t}\omega\,dt\,dx=-\frac{1}{2}\int_\Omega\int_0^s 2\frac{\partial\omega}{\partial t}\omega\,dt\,dx\\&=-\frac{1}{2}\int_\Omega\int_0^s\frac{d}{dt}\left(\omega^2\right)\,dt\,dx=-\frac{1}{2}\int_\Omega\left[\omega^2(s)-\omega^2(0)\right]dx\\&=-\frac{1}{2}\int_\Omega\omega^2(s)dx.\end{aligned}$$

Hence:

$$\int_\Omega\int_0^s \frac{\partial^2\omega}{\partial t^2}\psi\,dt\,dx=-\frac{1}{2}\|\omega(s)\|^2. \tag{3.9}$$

Similarly:

$$\int_\Omega\int_0^s\Delta\omega\,\psi\,dt\,dx=-\int_0^s\int_\Omega\nabla\omega\cdot\nabla\psi\,dx\,dt+\int_0^s\int_{\partial\Omega}\frac{\partial\omega}{\partial\vec{n}}\psi\,dx\,dt.$$

Since $\dfrac{\partial \omega}{\partial \vec{n}} = 0$, we obtain:

$$\begin{aligned}
\int_\Omega \int_0^s \Delta\omega\, \psi \, dt\, dx &= -\int_0^s \int_\Omega \nabla\omega \cdot \nabla\psi \, dx\, dt \\
&= -\int_0^s \int_\Omega \nabla\omega \cdot \nabla \left(-\int_t^s \omega(\sigma)\, d\sigma \right) dx\, dt.
\end{aligned}$$

Now:

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}\left(-\int_t^s \omega(\sigma)\, d\sigma \right) = \omega(t).$$

This implies:

$$\nabla\omega = \nabla \left(\frac{\partial}{\partial t} \left(-\int_t^s \omega(\sigma)\, d\sigma \right) \right).$$

Therefore:

$$\begin{aligned}
\int_\Omega \int_0^s \Delta\omega\, \psi \, dt\, dx &= -\int_\Omega \int_0^s \frac{\partial}{\partial t} \left(-\int_t^s \nabla\omega(\sigma)\, d\sigma \right) \cdot \left(-\int_t^s \nabla\omega(\sigma)\, d\sigma \right) dt\, dx \\
&= -\frac{1}{2} \int_\Omega \int_0^s \frac{d}{dt} \left[\left(\int_t^s \nabla\omega(\sigma)\, d\sigma \right) \right]^2 dt\, dx \\
&= -\frac{1}{2} \int_\Omega [\nabla\omega_1(s)]^2 dx.
\end{aligned}$$

Hence:

$$\int_\Omega \int_0^s \Delta\omega\, \psi \, dt\, dx = -\frac{1}{2} \|\nabla\omega_1(x,s)\|^2. \tag{3.10}$$

Now integrate equation (3.7) over Ω:

$$\int_\Omega \int_0^s \frac{\partial^2 \omega}{\partial t^2} \psi \, dt\, dx - \int_\Omega \int_0^s \Delta\omega\, \psi \, dt\, dx = c \int_\Omega \int_0^s \left(|v|^{\rho-1} - |u|^{\rho-1} \right) \psi \, dt\, dx. \tag{3.11}$$

Substituting equations (3.9) and (3.10) into (3.11), we obtain:

$$-\frac{1}{2}\|\omega(s)\|^2 + \frac{1}{2}\|\nabla\omega_1(x,s)\|^2 = c \int_\Omega \int_0^s \left(|v|^{\rho-1} - |u|^{\rho-1} \right) \psi \, dt\, dx. \tag{3.12}$$

We also have:

$$\begin{aligned}
&c \int_\Omega \int_0^s \left(|v|^{\rho-1} - |u|^{\rho-1} \right) \psi \, dt\, dx \\
&= c \int_\Omega \int_0^s (|v| - |u|)\big(|v|^{\rho-2} + |v|^{\rho-3}|u| + |v|^{\rho-4}|u|^2 \\
&\qquad + \cdots + |v||u|^{\rho-3} + |u|^{\rho-2} \big) dt\, dx.
\end{aligned}$$

Assume that $|u| \leq |v|$. Then we have:

$$|v|^{\rho-2} + |v|^{\rho-3}|u| + |v|^{\rho-4}|u|^2 + \cdots + |v||u|^{\rho-3} + |u|^{\rho-2} \leq (\rho-1)|v|^{\rho-2}.$$

This implies:

$$-\frac{1}{2}\|\omega(s)\|^2 + \frac{1}{2}\|\nabla\omega_1(x,s)\|^2 \leq c(\rho-1)\int_\Omega\int_0^s |v|^{\rho-2}|w||\psi|\,dx\,dt. \tag{3.13}$$

According to Hölder's inequality:

$$-\frac{1}{2}\|\omega(s)\|^2 + \frac{1}{2}\|\nabla\omega_1(x,s)\|^2 \leq c(\rho-1)\int_0^s \||v|^{\rho-2}\|_n\|\omega\|\|\psi\|_q\,dt. \tag{3.14}$$

Such that:

$$\frac{1}{2} + \frac{1}{n} + \frac{1}{q} = 1,$$

with the following regularity assumptions:

$$|v|^{\rho-2} \in L^\infty(0,T;L^n(\Omega)), \quad \omega \in L^\infty(0,T;L^2(\Omega)), \quad \psi \in L^\infty(0,T;L^q(\Omega)).$$

Let:

$$\psi = \omega_1(x,t) - \omega_1(x,s).$$

Then:

$$\|\psi\|_q \leq \|\omega_1(x,t)\|_q + \|\omega_1(x,s)\|_q.$$

Since $H^1(\Omega) \subset L^q(\Omega)$, it follows that there exists a constant K_1 such that:

$$\|\cdot\|_q \leq K_1\|\cdot\|_1.$$

It follows that:

$$\begin{aligned} &-\frac{1}{2}\|\omega(s)\|^2 + \frac{1}{2}\|\nabla\omega_1(x,s)\|^2 \\ &\leq K_2\int_0^s \||v|^{\rho-2}\|_n\|\omega\|\left(\|\omega_1(x,t)\|_1 + \|\omega_1(x,s)\|_1\right)dt, \end{aligned} \tag{3.15}$$

with $K_2 = cK_1(\rho-1)$. Moreover,

$$|v|^{\rho-2} \in L^\infty\Big(0,T;L^{\frac{\rho}{\rho-2}}(\Omega)\Big).$$

This implies that there exists a constant C_1 such that:

$$\|\cdot\|_{L^n(\Omega)} \leq C_1, \tag{3.16}$$

$$-\frac{1}{2}\|\omega(s)\|^2+\frac{1}{2}\|\nabla\omega_1(x,s)\|^2\leq K\int_0^s\|\omega\|\left(\|\omega_1(x,t)\|_1+\|\omega_1(x,s)\|_1\right)dt, \tag{3.17}$$

with $K=K_2C_1$. It follows that:

$$-\frac{1}{2}\|\omega(s)\|^2+\frac{1}{2}\|\nabla\omega_1(x,s)\|^2\leq K\left(\int_0^s\|\omega\|_2\|\omega_1(x,t)\|_1dt+\int_0^s\|\omega\|\|\omega_1(x,s)\|_1dt\right).$$

By Young's inequality:

$$\begin{aligned}-\frac{1}{2}\|\omega(s)\|^2+\frac{1}{2}\|\nabla\omega_1(x,s)\|^2\leq K\int_0^s\left(\frac{\varepsilon^2}{2}\|\omega\|^2+\frac{1}{2\varepsilon^2}\|\omega_1(x,t)\|_1^2\right)dt\\+K\frac{\varepsilon^2s}{2}\|\omega_1(x,s)\|_1^2+K\frac{1}{2\varepsilon^2}\int_0^s\|\omega\|^2dt,\end{aligned} \tag{3.18}$$

$$\begin{aligned}-\frac{1}{2}\|\omega(s)\|^2+\frac{1}{2}\|\nabla\omega_1(x,s)\|^2\leq K\int_0^s\left(\frac{\varepsilon^2}{2}+\frac{1}{2\varepsilon^2}\right)\|\omega\|^2dt\\+K\frac{1}{2\varepsilon^2}\int_0^s\|\omega_1(x,t)\|_1^2dt+K\frac{\varepsilon^2s}{2}\|\omega_1(x,s)\|_1^2.\end{aligned} \tag{3.19}$$

Multiplying equation (3.19) by 2, we obtain the following inequality:

$$\begin{aligned}-\|\omega(s)\|^2+\|\nabla\omega_1(x,s)\|^2\leq K\int_0^s\left(\varepsilon^2+\frac{1}{\varepsilon^2}\right)\|\omega\|^2dt\\+K\frac{1}{\varepsilon^2}\int_0^s\|\omega_1(x,t)\|_1^2dt+K\varepsilon^2s\|\omega_1(x,s)\|_1^2.\end{aligned} \tag{3.20}$$

From inequality (3.20), we deduce:

$$\begin{aligned}-\|\omega(s)\|^2\leq K\left(\varepsilon^2+\frac{1}{\varepsilon^2}\right)\int_0^s\|\omega\|^2dt\\+K\frac{1}{\varepsilon^2}\int_0^s\|\omega_1(x,t)\|_1^2dt+K\varepsilon^2s\|\omega_1(x,s)\|_1^2.\end{aligned} \tag{3.21}$$

Adding $2\|\omega_1(x,s)\|_1^2$ to both sides of inequality (3.21), we get:

$$\begin{aligned}&-\|\omega(s)\|^2+2\|\omega_1(x,s)\|_1^2\\&\leq K\left(\varepsilon^2+\frac{1}{\varepsilon^2}\right)\int_0^s\|\omega\|^2dt+K\frac{1}{\varepsilon^2}\int_0^s\|\omega_1(x,t)\|_1^2dt+\left(K\varepsilon^2s+2\right)\|\omega_1(x,s)\|_1^2,\end{aligned} \tag{3.22}$$

$$\begin{aligned}&-\|\omega(s)\|^2+2\|\omega_1(x,s)\|_1^2\\ &\le C_2\int_0^s\|\omega\|^2dt+C_3\int_0^s\|\omega_1(x,t)\|_1^2dt+C_4\|\omega_1(x,s)\|_1^2,\end{aligned}\tag{3.23}$$

with:

$$C_2=K\left(\varepsilon^2+\frac{1}{\varepsilon^2}\right),\quad C_3=K\frac{1}{\varepsilon^2},\quad C_4=K\varepsilon^2s+2.$$

We choose C_4 such that:

$$-\|\omega(s)\|^2+2\|\omega_1(x,s)\|_1^2\le C_2\int_0^s\|\omega\|^2dt+C_3\int_0^s\|\omega_1(x,t)\|_1^2dt.\tag{3.24}$$

Then we obtain:

$$\beta\left(\|\omega(s)\|^2+2\|\omega_1(x,s)\|_1^2\right)\le 0+\delta\int_0^s\left(\|\omega\|^2+\|\omega_1(x,t)\|_1^2\right)dt.\tag{3.25}$$

According to Grönwall's inequality, we have:

$$\omega(x,t)=0\Rightarrow u-v=0.\tag{3.26}$$

□

Furthermore, this opens the door to future research focused on optimizing the solution and exploring more advanced techniques.

CONFLICT OF INTEREST

The authors declare no competing interests.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARIEN NGOUABI
MASSENGO, BRAZZAVILLE,
CONGO.
Email address: `dieudonne.ampini@gmail.com`

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARIEN NGOUABI
21 RUE GAMBOMA (PLATEAU), BRAZZAVILLE,
CONGO.
Email address: `challoum.mouanda.dyaus@gmail.com`

DEPARTMENT OF MATEMATICS
UNIVERSITY OF MARIEN NGOUABI
KINTÉLÉ, BRAZZAVILLE,
CONGO.
Email address: `ndikoumanasimeon@gmail.com`