Advances in Mathematics: Scientific Journal **15**(2) (2026), 243–254
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.15.2.5

SHARP BOUNDS FOR HANKEL DETERMINANT OF SECOND ORDER FOR INVERSE FUNCTIONS OF CERTAIN CLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we determine mostly sharp upper bounds for the Hankel determinant of second order for the inverse functions of functions from some classes of univalent functions.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disk $\mathbb{D} := |z| < 1$ and normalized such that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{1.1}$$

and let $\mathcal{S}$ be its subclass containing univalent functions.

The Hankel determinant $H_q(n)(f)$ of a given function f, for $q \geq 1$ and $n \geq 1$, is defined by

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

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2020 *Mathematics Subject Classification.* 30C45, 30C55.
Key words and phrases. Hankel determinant, second order, inverse functions, sharp bound, classes of univalent functions.
Submitted: 16.04.2026; *Accepted:* 22.05.2026; *Published:* 26.05.2026.

They find application in the theory of singularities [4], as well as in the study of power series with integer coefficients ([1, 5, 13]).

In the past several years much attention is given on finding upper bounds (preferably sharp) of the modulus of Hankel determinant. Since the study of the general case faces serious calculation challenges, the determinants of second and the third order, defined respectively by

$$H_2(2)(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

and

$$H_3(1)(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

are studied instead. Calculation difficulties also impose narrowing the focus on the subclasses of univalent functions. Some of the more significant results can be found in [3, 6, 7, 9–12, 16].

Recently, the second Hankel determinant for the inverse functions for the functions in different subclasses of $\mathcal{S}$ started attracting attention. Namely, the Koebe's 1/4 theorem guaranties that for every univalent function f in $\mathbb{D}$, exists inverse f^{-1} at least on the disk with radius 1/4. If

$$f^{-1}(w) = w + A_2w^2 + A_3w^3 + \cdots , \tag{1.2}$$

then, from (1.1) and (1.2), having in mind $f(f^{-1}(w)) = w$, we receive

$$\begin{cases} A_2 = -a_2, \\ A_3 = -a_3 + 2a_2^2, \\ A_4 = -a_4 + 5a_2a_3 - 5a_2^3. \end{cases} \tag{1.3}$$

So, the second Hankel determinant for the inverse function f^{-1} is

$$\begin{aligned} H_2(2)(f^{-1}) &= A_2A_4 - A_3^2 = a_2a_4 - a_3^2 - a_2^2(a_3 - a_2^2) \\ &= H_2(2)(f) - a_2^2(a_3 - a_2^2). \end{aligned} \tag{1.4}$$

In this paper we will give mostly sharp upper bound of the modulus of the second Hankel determinant for the inverse functions for the functions in different subclasses of $\mathcal{S}$ as listed bellow.

The classes of convex and starlike functions of order α, $0 \le \alpha < 1$, are defined in the following way, respectively:

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha,\, z \in \mathbb{D} \right\}$$

and

$$\mathcal{S}^\star(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\, z \in \mathbb{D} \right\}.$$

They are both subclasses of the class of univalent functions and also $\mathcal{C}(\alpha) \subset \mathcal{S}^\star(\alpha)$. The classes $\mathcal{C} \equiv \mathcal{C}(0)$ and $\mathcal{S}^\star \equiv \mathcal{S}^\star(0)$ are the well-known classes of *convex* and *starlike* functions, respectively. The upper bound of $|H_2(2)(f^{-1})|$ when f is in this two classes was studied recently also in [14]. For certain range of α we will correct the bound given in [14] for the class $\mathcal{C}(\alpha)$.

Further, let $\mathcal{G}(\alpha)$ $0 < \alpha \le 1$, be the class of functions $f \in \mathcal{A}$ such that

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] < 1 + \frac{1}{2}\alpha \qquad (z \in \mathbb{D}).$$

It is subclass of the class of starlike functions (see [8]).

For our consideration we need the next lemma proven in [15] (can be found also in [2]).

Lemma 1.1. *Let*

$$\omega(z) = c_1 z + c_2 z^2 + \cdots \tag{1.5}$$

be a Schwartz function, i.e., a function analytic in $\mathbb{D}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Then $|c_1| \le 1$, $|c_2| \le 1 - |c_1|^2$, and $|c_3| \le 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}$.

2. Main results

Theorem 2.1. *Let $f \in \mathcal{C}(\alpha)$, $0 \le \alpha < 1$. Then we have the next estimations:*

$$|H_2(2)(f^{-1})| \le \begin{cases} \frac{1}{96}(12 - 28\alpha + 19\alpha^2), & 0 \le \alpha \le \frac{2}{5}, \\ \frac{(1-\alpha)^2}{9}, & \frac{2}{5} \le \alpha \le \frac{4}{5}, \\ \frac{(1-\alpha)^2}{6} \cdot \frac{\alpha(7\alpha-24)}{2\alpha^2-4\alpha+1}, & \frac{4}{5} \le \alpha < 1. \end{cases}$$

This result is sharp.

Proof. From the definition of the class $\mathcal{C}(\alpha)$ we have

$$1+\frac{zf''(z)}{f'(z)}=\alpha+(1-\alpha)\frac{1+\omega(z)}{1-\omega(z)}, \tag{2.1}$$

where ω is a Schwartz function, and from here

$$(zf'(z))'=\big(1+2(1-\alpha)[\omega(z)+\omega^2(z)+\ldots]\big)\,f'(z). \tag{2.2}$$

If we use the notations (1.1) and (1.5), and compare the coefficients on z, z^2, z^3 in the relation (2.2) then, after some simple calculations, we obtain

$$\begin{aligned} a_2&=(1-\alpha)c_1,\\ a_3&=\frac{1}{3}(1-\alpha)\left[c_2+(3-2\alpha)c_1^2\right],\\ a_4&=\frac{1}{6}(1-\alpha)\left[c_3+(5-3\alpha)c_1c_2+(2\alpha^2-7\alpha+6)c_1^3\right]. \end{aligned} \tag{2.3}$$

Now, by using (2.3), from (1.4), after some transformations we have

$$H_2(2)(f^{-1})=\frac{1}{6}(1-\alpha)^2\left[c_1c_3+\frac{5}{3}\left(\alpha-\frac{3}{5}\right)c_1^2c_2+\frac{4}{3}\alpha\left(\alpha-\frac{3}{4}\right)c_1^4-\frac{2}{3}c_2^2\right], \tag{2.4}$$

and further,

$$|H_2(2)(f^{-1})|\le\frac{1}{6}(1-\alpha)^2\left[|c_1||c_3|+\frac{5}{3}\left|\alpha-\frac{3}{5}\right||c_1|^2|c_2|+\frac{4}{3}|\alpha|\left|\alpha-\frac{3}{4}\right||c_1|^4+\frac{2}{3}|c_2|^2\right].$$

Applying the inequality for $|c_3|$ given in Lemma 1.1, we receive

$$\begin{aligned} |H_2(2)(f^{-1})|&\le\frac{1}{6}(1-\alpha)^2\left[|c_1|\left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right)\right.\\ &=\frac{1}{6}(1-\alpha)^2\left[|c_1|(1-|c_1|^2)+\frac{2-|c_1|}{3(1+|c_1|)}|c_2|^2\right.\\ &\qquad\left.+\frac{5}{3}|\alpha-\frac{3}{5}||c_1|^2|c_2|+\frac{4}{3}\alpha|\alpha-\frac{3}{4}||c_1|^4\right], \end{aligned} \tag{2.5}$$

and if we use $|c_2|\le 1-|c_1|^2$ (again from Lemma 1.1) and some calculations:

$$|H_2(2)(f^{-1})|\le\frac{1}{6}(1-\alpha)^2\left(\frac{2}{3}+b_1|c_1|^2+b_2|c_1|^4\right), \tag{2.6}$$

where

$$b_1 = \frac{5}{3}\left|\alpha - \frac{3}{5}\right| - \frac{1}{3},$$

(2.7)

$$b_2 = \frac{4}{3}\alpha\left|\alpha - \frac{3}{4}\right| - \frac{1}{3} - \frac{5}{3}\left|\alpha - \frac{3}{5}\right|.$$

As for (2.6) and (2.7), we have next cases.

(a) $0 \le \alpha \le \frac{3}{5}$: In this case we have

$$|H_2(2)(f^{-1})| \le \frac{1}{18}(1-\alpha)^2\left[2 + (2-5\alpha)|c_1|^2 - 4(1-\alpha)^2|c_1|^4\right].$$

If, additionally, $\alpha \le \frac{2}{5}$, then

$$|H_2(2)(f^{-1})| \le \frac{1}{96}(12 - 28\alpha + 19\alpha^2).$$

Then $b_1 \ge 0$ and the maximum is attained for $|c_1|^2 = \frac{2-5\alpha}{8(1-\alpha)^2}$. The previous result is the best possible. Namely, if in (2.1) choose the next Schwartz function

(2.8)
$$\omega(z) = z\frac{z + c_1}{1 + c_1 z},$$

where

(2.9)
$$c_1 = \sqrt{\frac{2-5\alpha}{8(1-\alpha)^2}},$$

leading to

$$\omega(z) = c_1 z + (1 - c_1^2)z^2 - c_1(1 - c_1^2)z^3 + \cdots.$$

Now, for the function f_1 defined by (2.1) from (2.4), with c_1 given by (2.9), $c_2 = 1 - c_1^2$, $c_3 = -c_1(1 - c_1^2)$, we get

$$H_2(2)(f_1^{-1}) = -\frac{1}{18}(1-\alpha)^2\left[2 + (2-5\alpha)c_1^2 - 4(1-\alpha)^2 c_1^4\right] = -\frac{1}{96}(12 - 28\alpha + 19\alpha^2).$$

If $\frac{2}{5} \le \alpha \le \frac{3}{5}$, then from (2.7) we have $b_1 \le 0$ and $b_2 < 0$, which due to (2.6) implies

$$|H_2(2)(f_1^{-1})| \le \frac{(1-\alpha)^2}{9}.$$

The estimate is sharp for $c_1 = 0$ and $|c_2| = 1$, i.e. for the function defined by (2.1) with $\omega(z) = z^2$.

(b) Similar analysis will bring the result when $\frac{3}{5} \le \alpha < 1$. First let note that in this case $b_1 = \frac{5}{3}\left(\alpha - \frac{4}{5}\right)$.

Further, if $\frac{3}{5} \le \alpha \le \frac{3}{4}$, then $b_2 = -\frac{2}{3}(\alpha+1)(2\alpha-1) < 0$ and $-\frac{b_1}{2b_2} < 0$. So, $|H_2(2)(f_1^{-1})| \le \frac{(1-\alpha)^2}{9}$ is a sharp estimate with equality attained for $c_1 = 0$ and $|c_2| = 1$, i.e. for the function defined by (2.1) with $\omega(z) = z^2$.

Next, if $\frac{3}{4} \le \alpha \le \frac{4}{5}$, then $b_2 = \frac{2}{3}(2\alpha^2 - 4\alpha + 1) < 0$ and, again, $-\frac{b_1}{2b_2} < 0$, i.e., $|H_2(2)(f_1^{-1})| \le \frac{(1-\alpha)^2}{9}$ is a sharp with extremal function defined by (2.1) with $\omega(z) = z^2$, such that $c_1 = 0$ and $|c_2| = 1$.

Finally, in the case when $\frac{4}{5} \le \alpha < 1$, $b_2 = \frac{2}{3}(2\alpha^2 - 4\alpha + 1) < 0$, but $0 < -\frac{b_1}{2b_2} < 1$. Therefore,

$$|H_2(2)(f_1^{-1})| \le \frac{1}{6}(1-\alpha)^2\left(\frac{2}{3} + b_1|c_1|^2 + b_2|c_1|^4\right) = \frac{(1-\alpha)^2}{6} \cdot \frac{\alpha(7\alpha-24)}{2\alpha^2-4\alpha+1},$$

where $|c_1|^2 = -\frac{b_1}{2b_2} = \frac{4-5\alpha}{4(2\alpha^2-4\alpha+1)}$. The result is sharp due to the function defined by (2.1) with $\omega(z)$ as in (2.8) and $c_1 = \sqrt{\frac{4-5\alpha}{4(2\alpha^2-4\alpha+1)}}$. □

For $\alpha \in [0, 4/5]$, the upper bound from Theorem 2.1 is the same as the one from Theorem 3.2 from [14], while for $\alpha \in [4/5, 1)$ the bound is corrected. Namely, in [14] the authors claimed that for $\alpha \in [4/5, 1)$ the sharp bound is $\frac{\alpha(1-\alpha)^2(19\alpha-8)}{48(1+\alpha)(2\alpha-1)}$ which turns out to be less or equal to the corresponding bound from Theorem 2.1 supported with an extremal function.

Fixing $\alpha = 0$ in Theorem 2.1 brings the following corollary given earlier in [11].

Corollary 2.1. *If $f \in \mathcal{C}$, then $|H_2(2)(f^{-1})| \le \frac{1}{8}$ is a sharp estimate.*

Remark 2.1. *For a function f that is convex of order α, from the relation* (1.4)*, we receive*

$$\begin{aligned}|H_2(2)(f^{-1}) - H_2(2)(f)| &= |a_2|^2|a_3 - a_2^2| = \frac{1}{3}(1-\alpha)^3|c_1|^2\left(\alpha|c_1|^2 + |c_2|\right)\\ &\le \frac{1}{3}(1-\alpha)^3|c_1|^2\left[1 - (1-\alpha)|c_1|^2\right].\end{aligned}$$

Simple calculus shows that if $0 \le \alpha \le \frac{1}{2}$, then

$$|H_2(2)(f^{-1}) - H_2(2)(f)| \le \frac{1}{12}(1-\alpha)^2, \tag{2.10}$$

attained for $|c_1|^2 = \frac{1}{2(1-\alpha)}$*; and if* $\frac{1}{2} \le \alpha \le 1$*,*

$$|H_2(2)(f^{-1}) - H_2(2)(f)| \le \frac{1}{3}\alpha(1-\alpha)^3,$$

attained for $|c_1| = 1$*.*

Both these estimates are sharp.

Indeed, if $0 \le \alpha < \frac{1}{2}$*, then the function* f *defined by* (2.1) *and the Schwartz function* (2.8) *with* $c_1 = \sqrt{\frac{1}{2(1-\alpha)}}$*, is such that* $c_2 = 1 - c_1^2$*, and we receive equality in* (2.10)*.*

In the case when $\frac{1}{2} < \alpha \le 1$*, the extremal function showing the sharpness is* $f(z) = \frac{1}{2\alpha-1}[1-(1-z)^{2\alpha-1}]$*, such that* $1 + \frac{zf''(z)}{f'(z)} = \alpha + (1-\alpha)\frac{1+z}{1-z}$*, i.e., in the view of notation from* (2.1)*,* $\omega(z) = z$*,* $c_1 = 1$*,* $c_2 = c_3 = 0$*,* $a_2 = 1-\alpha$*,* $a_3 = \frac{1}{3}(1-\alpha)(3-2\alpha)$ *and*

$$|H_2(2)(f^{-1}) - H_2(2)(f)| = |a_2|^2|a_3 - a_2^2| = \frac{1}{3}\alpha(1-\alpha)^3.$$

For the case $\alpha = \frac{1}{2}$*, the extremal function is* $f(z) = -\log(1-z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots$*.*

Theorem 2.2. *Let* $f \in \mathcal{S}^\star(\alpha)$*,* $0 \le \alpha < 1$*. Then*

$$|H_2(2)(f^{-1})| \le \begin{cases} \frac{1}{3}(1-\alpha)^2(20\alpha^2 - 28\alpha + 9), & 0 \le \alpha \le \frac{1}{10}, \\ \frac{1}{3}(1-\alpha)^2\frac{11-12\alpha^2}{1+8\alpha-10\alpha^2}, & \frac{1}{10} \le \alpha \le \frac{1}{2}, \\ (1-\alpha)^2 + \frac{1}{15}(2-3\alpha)^2, & \frac{1}{2} \le \alpha \le \frac{2}{3}, \\ (1-\alpha)^2, & \frac{2}{3} \le \alpha < 1. \end{cases}$$

If $0 \le \alpha \le \frac{1}{10}$ *or* $\frac{1}{2} \le \alpha < 1$*, then the corresponding estimates are sharp.*

Proof. We use the same method as in the previous theorem. From the definition of the class $\mathcal{S}^\star(\alpha)$, we have

$$\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)\frac{1+\omega(z)}{1-\omega(z)} \quad \left(= 2\alpha - 1 + 2(1-\alpha)\frac{1}{1-\omega(z)} \right), \tag{2.11}$$

where ω is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. From (2.11) we obtain

$$f'(z) = \left[1 + 2(1-\alpha)\left(\omega(z) + \omega^2(z) + \cdots\right)\right] \cdot \frac{f(z)}{z}. \tag{2.12}$$

If we use the notations from (1.1) and (1.5) for f and ω, and compare the coefficients of z, z^2, z^3 in the relation (2.12), after some calculations, we receive

$$\begin{aligned} a_2 &= 2(1-\alpha)c_1, \\ a_3 &= (1-\alpha)\left(c_2+(3-2\alpha)c_1^2\right), \\ a_4 &= \frac{2}{3}(1-\alpha)\left(c_3+(5-3\alpha)c_1c_2+(2\alpha^2-7\alpha+6)c_1^3\right). \end{aligned}$$

Then, by using the relation (1.4), we get

$$H_2(2)(f^{-1}) = \frac{4}{3}(1-\alpha)^2\left[c_1c_3+\left(3\alpha-\frac{5}{2}\right)c_1^2c_2+\left(5\alpha^2-7\alpha+\frac{9}{4}\right)c_1^4-\frac{3}{4}c_2^2\right],$$

and further,

$$\begin{aligned} |H_2(2)(f^{-1})| \le \frac{4}{3}(1-\alpha)^2 \Bigg(& |c_1||c_3| + \left|3\alpha-\frac{5}{2}\right||c_1|^2|c_2| \\ & + \left|5\alpha^2-7\alpha+\frac{9}{4}\right||c_1|^4+\frac{3}{4}|c_2|^2\Bigg). \end{aligned}$$

Applying the estimate for $|c_3|$ given in Lemma 1.1 and after that the estimate $|c_2| \le 1-|c_1|^2$ from the same lemma, similarly as in (2.5) and (2.6) we receive

$$|H_2(2)(f^{-1})| \le \frac{4}{3}(1-\alpha)^2\left(\frac{3}{4}+d_1|c_1|^2+d_2|c_1|^4\right) \equiv \varphi(|c_1|), \tag{2.13}$$

where

$$\begin{aligned} d_1 &= 3\left|\alpha-\frac{5}{6}\right|-\frac{1}{2}, \\ d_2 &= \left|5\alpha^2-7\alpha+\frac{9}{4}\right|-\frac{1}{4}-3\left|\alpha-\frac{5}{6}\right|, \end{aligned} \tag{2.14}$$

and $\varphi(t) = \frac{4}{3}(1-\alpha)^2\left(\frac{3}{4}+d_1t^2+d_2t^4\right)$, $t \in [0,1]$. Since

$$\left|5\alpha^2-7\alpha+\frac{9}{4}\right| = \begin{cases} 5\alpha^2-7\alpha+\frac{9}{4}, & \alpha \in [0,\frac{1}{2}] \cup [\frac{9}{10},1), \\ -\left(5\alpha^2-7\alpha+\frac{9}{4}\right), & \alpha \in [\frac{1}{2},\frac{9}{10}], \end{cases},$$

and similarly for $|\alpha-\frac{5}{6}|$, we must consider several cases as follows.

(i) In this case $\alpha \in [0,\frac{1}{2}]$, from (2.14) we have

$$d_1 = 2-3\alpha \ge 0 \quad \text{and} \quad d_2 = \frac{1}{2}(10\alpha^2-8\alpha-1) \le 0.$$

So, from (2.13) and elementary consideration we conclude that

$$|H_2(2)(f^{-1})| \leq \frac{4}{3}(1-\alpha)^2\left(5\alpha^2 - 7\alpha + \frac{9}{4}\right)$$

for $\alpha \in [0, \frac{1}{10}]$ (attained for $|c_1| = 1$); and

$$|H_2(2)(f^{-1})| \leq \frac{1}{3}(1-\alpha)^2\frac{11-12\alpha^2}{1+8\alpha-10\alpha^2},$$

for $\alpha \in [\frac{1}{10}, \frac{1}{2}]$ (attained for $|c_1|^2 = -\frac{d_1}{2d_2} = \frac{2-3\alpha}{1+8\alpha-10\alpha^2}$). The estimate for $\alpha \in [0, 1/10]$ is sharp, i.e., the best possible, with equality sign if in (2.11) we can choose the function f with $\omega(z) = z$.

(ii) If $\alpha \in [\frac{1}{2}, \frac{2}{3}]$, then $d_1 \geq 0$, $d_2 < 0$ and the maximal value of $\varphi(|c_1|)$ on the interval $[0,1]$ is $(1-\alpha)^2 + \frac{1}{15}(2-3\alpha)^2$, attained for $c_1 = \frac{2-3\alpha}{10(1-\alpha)^2}$. The extremal function showing sharpness of the estimate is defined by (2.11) for ω given by (2.8).

(iii) If the remaining case, $\alpha \in [\frac{2}{3}, 1]$, $d_1 \leq 0$, $d_2 \leq 0$ and the maximal value of $\varphi(|c_1|)$ on the interval $[0,1]$ is $(1-\alpha)^2$, attained for $c_1 = 0$. The extremal function is the one obtained for $\omega(z) = z^2$ in (2.11). □

In the case when $\frac{1}{10} < \alpha < \frac{1}{2}$, the sharp upper bound is given in Theorem 3.1 from [14], such that, when $\alpha < \alpha_0$, the sharp bound is the same as in the case $\alpha \leq \frac{1}{10}$, and when $\alpha > \alpha_0$, the sharp bound is the same as in the case $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$. Here, $\alpha_0 = 0.232\ldots$ is the solution of the equation $100x^4 - 340x^3 + 401x^2 - 188x + 26 = 0$ on the interval $[0,1)$.

Theorem 2.2, with $\alpha = 0$, leads to the next result, given earlier in [11].

Corollary 2.2. *If $f \in \mathcal{S}^\star$, then $|H_2(2)(f^{-1})| \leq 3$ is a sharp estimate.*

Remark 2.2. *Let $f \in \mathcal{S}^\star(\alpha)$. From the relation (1.4) we have*

$$|H_2(2)(f^{-1}) - H_2(2)(f)| \leq 4(1-\alpha)^3|c_1|^2(|c_2| + |2\alpha-1||c_1|^2)$$

and by using $|c_2| \leq 1 - |c_1|^2$:

$$|H_2(2)(f^{-1}) - H_2(2)(f)| \leq 4(1-\alpha)^3|c_1|^2\left[1 + (|2\alpha-1|-1)|c_1|^2\right].$$

Further,

$$|H_2(2)(f^{-1}) - H_2(2)(f)| \le \begin{cases} 4(1-\alpha)^3(1-2\alpha), & 0 \le \alpha \le \frac{1}{4} \\ \frac{(1-\alpha)^3}{2\alpha}, & \frac{1}{4} \le \alpha \le \frac{1}{2} \\ \frac{(1-\alpha)^2}{2}, & \frac{1}{2} \le \alpha \le \frac{3}{4} \\ 4(1-\alpha)^3(2\alpha-1), & \frac{3}{4} \le \alpha < 1. \end{cases}$$

All these estimates are sharp. Namely, as an extremal function, in the first and the fourth case we can choose the function for which $\omega(z) = z$ in (2.11)*, and in the second and third cases we can choose the functions with ω of the type* (2.8)*, and $c_1 = \frac{1}{\sqrt{4\alpha}}$ and $c_1 = \frac{1}{\sqrt{4(1-\alpha)}}$, respectively.*

Theorem 2.3. *If $f \in \mathcal{G}(\alpha)$, $0 < \alpha \le 1$, then*

$$|H_2(2)(f^{-1})| \le \begin{cases} \frac{\alpha^2(68+20\alpha-7\alpha^2)}{1152(2-\alpha^2)}, & 0 < \alpha \le \frac{3}{4}, \\ \frac{\alpha^2(2\alpha^2+5\alpha+2)}{144}, & \frac{3}{4} \le \alpha \le 1. \end{cases}$$

This second estimate is sharp.

Proof. From the definition of the class $\mathcal{G}(\alpha)$ we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{1}{2}\cdot\alpha - \frac{\alpha}{2}\cdot\frac{1+\omega(z)}{1-\omega(z)},$$

where ω is a Schwartz function. From here

$$(zf'(z))' = \left[1 - \alpha(\omega(z) + \omega^2(z) + \cdots)\right]\cdot f'(z).$$

If we use the notations (1.1) and (1.5), and compare the coefficients on z, z^2, z^3 in the relation (2.2), after some simple calculations, we obtain

$$\begin{aligned} a_2 &= -\frac{\alpha}{2}c_1, \\ a_3 &= -\frac{\alpha}{6}\left[c_2 + (1-\alpha)c_1^2\right], \\ a_4 &= -\frac{\alpha}{24}\left[2c_3 + (4-3\alpha)c_1c_2 + (\alpha^2 - 3\alpha + 2)c_1^3\right]. \end{aligned}$$

As in two previous theorems, by using the relation (1.4) and after some simple transformations, we obtain

$$H_2(2)(f^{-1}) = \frac{\alpha^2}{144}\left[6c_1c_3 + (5\alpha+4)c_1^2c_2 + (2\alpha^2+5\alpha+2)c_1^4 - 4c_2^2\right],$$

and

$$|H_2(2)(f^{-1})| \leq \frac{\alpha^2}{144}\left[6|c_1||c_3| + (5\alpha+4)|c_1|^2|c_2| + (2\alpha^2+5\alpha+2)|c_1|^4 + 4|c_2|^2\right].$$

Further, using the same method as in Theorem 2.1 and Theorem 2.2, we get

$$|H_2(2)(f^{-1})| \leq \frac{\alpha^2}{144}\left[4 + (5\alpha+2)|c_1|^2 - 2(2-\alpha^2)|c_1|^4\right],$$

and the statement of the theorem easily follows from the last function on $|c_1|^2$.

Those estimate is sharp for $\frac{3}{4} \leq \alpha \leq 1$ with an extremal function given by (1.5) with $\omega(z) = z$. □

For $\alpha = 1$ in Theorem 2.2 we have that the next corollary given earlier in [11].

Corollary 2.3. *If $f \in \mathcal{G}$, then $|H_2(2)(f^{-1})| \leq \frac{1}{16}$, is valid and sharp.*

Remark 2.3. *For functions f from $\mathcal{G}$, from the relation* (1.4)*, we have*

$$|H_2(2)(f^{-1}) - H_2(2)(f)| \leq \frac{\alpha^3}{48}|c_1|^2\left(2|c_2| + (2+\alpha)|c_1|^2\right),$$

and by using $|c_2| \leq 1 - |c_1|^2$,

$$|H_2(2)(f^{-1}) - H_2(2)(f)| \leq \frac{\alpha^3}{48}|c_1|^2(2+\alpha|c_1|^2) \leq \frac{\alpha^3}{48}(2+\alpha).$$

This estimate is sharp with equality sign for the extremal function for which $\omega(z) = z$ in (2.9).

Conflict of interest

The authors declare no competing interests.

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