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ON SOME TRACTABLE HARDY-HILBERT-TYPE INTEGRAL INEQUALITIES

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ABSTRACT. Numerous Hardy-Hilbert integral inequalities of varying complexity exist in the literature. In this paper, we develop Hardy-Hilbert-type integral inequalities featuring explicit constant factors and tractable weighted integral norms of the functions involved. Some of them consider the minimum and maximum of the variables. The proofs rely primarily on the Hölder integral inequality, the Fubini-Tonelli integral theorem, and several elementary analytical techniques. The relative simplicity of these proofs also lends the paper pedagogical value.

1. INTRODUCTION

The Hardy-Hilbert integral inequality is a classical and fundamental result in mathematical analysis. A standard statement of this result is given below. Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions. Then, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx \right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy \right)^{1/q},$$

provided that the integrals of the upper bound converge, i.e.,

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty.$$

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In addition to being tractable, the upper bound is known to be sharp. For further details and related discussions, see [5, 9]. This inequality has since been refined and extended in many directions. See, for example, [1,3,4,6–8,10]. Furthermore, the survey paper [2] provides a useful overview of several of these developments.

In this paper, we contribute to the theory by establishing several Hardy-Hilbert-type integral inequalities featuring explicit constant factors and tractable weighted integral norms of the functions involved. Some of them consider the minimum and maximum of the variables, i.e., $\min(x, y)$ and $\max(x, y)$, respectively. As an introductory illustration, our results include Hardy-Hilbert-type integral inequalities associated with the following two double integrals:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+(x+y)^2}dxdy$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)\min(x,y)}{1+x^2y^2}dxdy.$$

Our results complement a number of existing inequalities in the literature, while also providing formulations that are both effective and easy to apply in practice. The proofs rely primarily on the Hölder integral inequality, the Fubini-Tonelli integral theorem, and a collection of elementary analytical techniques. Due to their relative simplicity and clarity, the arguments presented here also give this paper a certain pedagogical value.

The rest of the paper concerns the results accompanied by their proofs in Section 2. A conclusion is given in Section 3.

2. Results with proofs

2.1. **First result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+x^2+y^2}dxdy.$$

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+x^2+y^2}dxdy$$
$$\leq \frac{\pi}{2}\left(\int_0^{+\infty}\frac{f^p(x)}{\sqrt{1+x^2}}dx\right)^{1/p}\left(\int_0^{+\infty}\frac{g^q(y)}{\sqrt{1+y^2}}dy\right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+x^2+y^2}dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)}{(1+x^2+y^2)^{1/p}}\times\frac{g(y)}{(1+x^2+y^2)^{1/q}}dxdy$$
$$\leq\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{1+x^2+y^2}dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{1+x^2+y^2}dxdy\right)^{1/q}.$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{1+x^2+y^2}dxdy=\int_0^{+\infty}f^p(x)\int_0^{+\infty}\frac{1}{1+x^2+y^2}dydx.$$

Using standard primitives and composition rules, we have

$$\int_0^{+\infty}\frac{1}{1+x^2+y^2}dy=\left[\frac{1}{\sqrt{1+x^2}}\arctan\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_{y=0}^{y\to+\infty}=\frac{\pi}{2\sqrt{1+x^2}}.$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{1+x^2+y^2}dxdy=\frac{\pi}{2}\int_0^{+\infty}\frac{f^p(x)}{\sqrt{1+x^2}}dx.$$

Similarly, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{1+x^2+y^2}dxdy=\frac{\pi}{2}\int_0^{+\infty}\frac{g^q(y)}{\sqrt{1+y^2}}dy.$$

Combining the equations above and using $1/p+1/q=1$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+x^2+y^2}dxdy$$
$$\leq\left(\frac{\pi}{2}\int_0^{+\infty}\frac{f^p(x)}{\sqrt{1+x^2}}dx\right)^{1/p}\left(\frac{\pi}{2}\int_0^{+\infty}\frac{g^q(y)}{\sqrt{1+y^2}}dy\right)^{1/q}$$

$$= \frac{\pi}{2} \left(\int_0^{+\infty} \frac{f^p(x)}{\sqrt{1+x^2}} dx \right)^{1/p} \left(\int_0^{+\infty} \frac{g^q(y)}{\sqrt{1+y^2}} dy \right)^{1/q}.$$

This concludes the proof. □

The resulting constant factor is $\pi/2$, with the weight function defined by

$$w(x) = \frac{1}{\sqrt{1+x^2}}.$$

2.2. **Second result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{1+(x+y)^2} dx dy.$$

Theorem 2.2. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions. Then, we have*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{1+(x+y)^2} dx dy$$
$$\leq \left(\int_0^{+\infty} f^p(x) \arctan\left(\frac{1}{x}\right) dx \right)^{1/p} \left(\int_0^{+\infty} g^q(y) \arctan\left(\frac{1}{y}\right) dy \right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{1+(x+y)^2} dx dy$$
$$= \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)}{(1+(x+y)^2)^{1/p}} \times \frac{g(y)}{(1+(x+y)^2)^{1/q}} dx dy$$
$$\leq \left(\int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x)}{1+(x+y)^2} dx dy \right)^{1/p} \left(\int_0^{+\infty} \int_0^{+\infty} \frac{g^q(y)}{1+(x+y)^2} dx dy \right)^{1/q}.$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x)}{1+(x+y)^2} dx dy = \int_0^{+\infty} f^p(x) \int_0^{+\infty} \frac{1}{1+(x+y)^2} dy dx.$$

Using standard primitives and composition rules, we have

$$\int_0^{+\infty} \frac{1}{1+(x+y)^2} dy = [\arctan(x+y)]_{y=0}^{y\to+\infty} = \frac{\pi}{2} - \arctan(x) = \arctan\left(\frac{1}{x}\right).$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)}{1+(x+y)^2} dxdy = \int_0^{+\infty} f^p(x) \arctan\left(\frac{1}{x}\right) dx.$$

Similarly, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{g^q(y)}{1+(x+y)^2} dxdy = \int_0^{+\infty} g^q(y) \arctan\left(\frac{1}{y}\right) dy.$$

Combining the equations above, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{1+(x+y)^2} dxdy \\ &\leq \left(\int_0^{+\infty} f^p(x) \arctan\left(\frac{1}{x}\right) dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y) \arctan\left(\frac{1}{y}\right) dy\right)^{1/q}.\end{aligned}$$

This concludes the proof. □

The resulting constant factor is 1, with the weight function defined by

$$w(x) = \arctan\left(\frac{1}{x}\right).$$

2.3. **Third result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{1+x^2y^2} dxdy.$$

Theorem 2.3. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{1+x^2y^2} dxdy \leq \frac{\pi}{2} \left(\int_0^{+\infty} \frac{f^p(x)}{x} dx\right)^{1/p} \left(\int_0^{+\infty} \frac{g^q(y)}{y} dy\right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+x^2y^2}dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)}{(1+x^2y^2)^{1/p}}\times\frac{g(y)}{(1+x^2y^2)^{1/q}}dxdy\\
&\le\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{1+x^2y^2}dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{1+x^2y^2}dxdy\right)^{1/q}.
\end{aligned}$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{1+x^2y^2}dxdy=\int_0^{+\infty}f^p(x)\int_0^{+\infty}\frac{1}{1+x^2y^2}dydx.$$

Using standard primitives and composition rules, we have

$$\int_0^{+\infty}\frac{1}{1+x^2y^2}dy=\left[\frac{1}{x}\arctan(xy)\right]_{y=0}^{y\to+\infty}=\frac{\pi}{2x}.$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{1+x^2y^2}dxdy=\frac{\pi}{2}\int_0^{+\infty}\frac{f^p(x)}{x}dx.$$

Similarly, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{1+x^2y^2}dxdy=\frac{\pi}{2}\int_0^{+\infty}\frac{g^q(y)}{y}dy.$$

Combining the equations above and using $1/p+1/q=1$, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+x^2y^2}dxdy\\
&\le\left(\frac{\pi}{2}\int_0^{+\infty}\frac{f^p(x)}{x}dx\right)^{1/p}\left(\frac{\pi}{2}\int_0^{+\infty}\frac{g^q(y)}{y}dy\right)^{1/q}\\
&=\frac{\pi}{2}\left(\int_0^{+\infty}\frac{f^p(x)}{x}dx\right)^{1/p}\left(\int_0^{+\infty}\frac{g^q(y)}{y}dy\right)^{1/q}.
\end{aligned}$$

This concludes the proof. □

The resulting constant factor is $\pi/2$, with the weight function defined by

$$w(x)=\frac{1}{x}.$$

2.4. **Fourth result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)\min(x,y)}{1+x^2y^2}dxdy.$$

Theorem 2.4. *Let $p>1$, $q=p/(p-1)$, and $f,g:[0,+\infty)\to[0,+\infty)$ be measurable functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)\min(x,y)}{1+x^2y^2}dxdy$$
$$\le \left(\int_0^{+\infty} f^p(x)\left(\frac{1}{2x^2}\log(1+x^4)+\arctan\left(\frac{1}{x^2}\right)\right)dx\right)^{1/p}$$
$$\times \left(\int_0^{+\infty} g^q(y)\left(\frac{1}{2y^2}\log(1+y^4)+\arctan\left(\frac{1}{y^2}\right)\right)dy\right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)\min(x,y)}{1+x^2y^2}dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)\min^{1/p}(x,y)}{(1+x^2y^2)^{1/p}}\times\frac{g(y)\min^{1/q}(x,y)}{(1+x^2y^2)^{1/q}}dxdy$$
$$\le\left(\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)\min(x,y)}{1+x^2y^2}dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty} \frac{g^q(y)\min(x,y)}{1+x^2y^2}dxdy\right)^{1/q}.$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)\min(x,y)}{1+x^2y^2}dxdy=\int_0^{+\infty} f^p(x)\int_0^{+\infty}\frac{\min(x,y)}{1+x^2y^2}dydx.$$

Using standard primitives and composition rules, we have

$$\int_0^{+\infty}\frac{\min(x,y)}{1+x^2y^2}dy=\int_0^{x}\frac{\min(x,y)}{1+x^2y^2}dy+\int_x^{+\infty}\frac{\min(x,y)}{1+x^2y^2}dy$$
$$=\int_0^{x}\frac{y}{1+x^2y^2}dy+\int_x^{+\infty}\frac{x}{1+x^2y^2}dy$$
$$=\left[\frac{1}{2x^2}\log(1+x^2y^2)\right]_{y=0}^{y=x}+\left[\arctan(xy)\right]_{y=x}^{y\to+\infty}$$

$$= \frac{1}{2x^2}\log(1+x^4) + \frac{\pi}{2} - \arctan(x^2)$$
$$= \frac{1}{2x^2}\log(1+x^4) + \arctan\left(\frac{1}{x^2}\right).$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)\min(x,y)}{1+x^2y^2}dxdy$$
$$= \int_0^{+\infty} f^p(x)\left(\frac{1}{2x^2}\log(1+x^4) + \arctan\left(\frac{1}{x^2}\right)\right)dx.$$

Similarly, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{g^q(y)\min(x,y)}{1+x^2y^2}dxdy$$
$$= \int_0^{+\infty} g^q(y)\left(\frac{1}{2y^2}\log(1+y^4) + \arctan\left(\frac{1}{y^2}\right)\right)dy.$$

Combining the equations above, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)\min(x,y)}{1+x^2y^2}dxdy$$
$$\leq \left(\int_0^{+\infty} f^p(x)\left(\frac{1}{2x^2}\log(1+x^4) + \arctan\left(\frac{1}{x^2}\right)\right)dx\right)^{1/p}$$
$$\times \left(\int_0^{+\infty} g^q(y)\left(\frac{1}{2y^2}\log(1+y^4) + \arctan\left(\frac{1}{y^2}\right)\right)dy\right)^{1/q}.$$

This concludes the proof. □

The resulting constant factor is 1, with the weight function defined by

$$w(x) = \frac{1}{2x^2}\log(1+x^4) + \arctan\left(\frac{1}{x^2}\right).$$

2.5. **Fifth result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{1+xy\max(x,y)}dxdy.$$

Theorem 2.5. *Let* $p > 1$*,* $q = p/(p-1)$*, and* $f, g : [0, +\infty) \to [0, +\infty)$ *be measurable functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+xy\max(x,y)}dxdy$$
$$\leq \left(\int_0^{+\infty} f^p(x)\left(\frac{1}{x^2}\log(1+x^3)+\frac{1}{\sqrt{x}}\arctan\left(\frac{1}{x\sqrt{x}}\right)\right)dx\right)^{1/p}$$
$$\times\left(\int_0^{+\infty} g^q(y)\left(\frac{1}{y^2}\log(1+y^3)+\frac{1}{\sqrt{y}}\arctan\left(\frac{1}{y\sqrt{y}}\right)\right)dy\right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{1+xy\max(x,y)}dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)}{(1+xy\max(x,y))^{1/p}}\times\frac{g(y)}{(1+xy\max(x,y))^{1/q}}dxdy$$
$$\leq\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{1+xy\max(x,y)}dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{1+xy\max(x,y)}dxdy\right)^{1/q}.$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{1+xy\max(x,y)}dxdy=\int_0^{+\infty}f^p(x)\int_0^{+\infty}\frac{1}{1+xy\max(x,y)}dydx.$$

Using standard primitives and composition rules, we have

$$\int_0^{+\infty}\frac{1}{1+xy\max(x,y)}dy$$
$$=\int_0^{x}\frac{1}{1+xy\max(x,y)}dy+\int_x^{+\infty}\frac{1}{1+xy\max(x,y)}dy$$
$$=\int_0^{x}\frac{1}{1+x^2y}dy+\int_x^{+\infty}\frac{1}{1+xy^2}dy$$
$$=\left[\frac{1}{x^2}\log(1+x^2y)\right]_{y=0}^{y=x}+\left[\frac{1}{\sqrt{x}}\arctan(\sqrt{x}y)\right]_{y=x}^{y\to+\infty}$$
$$=\frac{1}{x^2}\log(1+x^3)+\frac{1}{\sqrt{x}}\left(\frac{\pi}{2}-\arctan(x\sqrt{x})\right)$$

$$= \frac{1}{x^2}\log(1+x^3) + \frac{1}{\sqrt{x}}\arctan\left(\frac{1}{x\sqrt{x}}\right).$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)}{1+xy\max(x,y)}dxdy$$
$$= \int_0^{+\infty} f^p(x)\left(\frac{1}{x^2}\log(1+x^3) + \frac{1}{\sqrt{x}}\arctan\left(\frac{1}{x\sqrt{x}}\right)\right)dx.$$

Similarly, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{g^q(y)}{1+xy\max(x,y)}dxdy$$
$$= \int_0^{+\infty} g^q(y)\left(\frac{1}{y^2}\log(1+y^3) + \frac{1}{\sqrt{y}}\arctan\left(\frac{1}{y\sqrt{y}}\right)\right)dy.$$

Combining the equations above, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{1+xy\max(x,y)}dxdy$$
$$\leq \left(\int_0^{+\infty} f^p(x)\left(\frac{1}{x^2}\log(1+x^3) + \frac{1}{\sqrt{x}}\arctan\left(\frac{1}{x\sqrt{x}}\right)\right)dx\right)^{1/p}$$
$$\times \left(\int_0^{+\infty} g^q(y)\left(\frac{1}{y^2}\log(1+y^3) + \frac{1}{\sqrt{y}}\arctan\left(\frac{1}{y\sqrt{y}}\right)\right)dy\right)^{1/q}.$$

This concludes the proof. □

The resulting constant factor is 1, with the weight function defined by

$$w(x) = \frac{1}{x^2}\log(1+x^3) + \frac{1}{\sqrt{x}}\arctan\left(\frac{1}{x\sqrt{x}}\right).$$

2.6. **Sixth result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)\min^2(x,y)}{1+x^2y^2}dxdy.$$

Theorem 2.6. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0,+\infty) \to [0,+\infty)$ be measurable functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)\min^2(x,y)}{1+x^2y^2}dxdy$$
$$\leq\left(\int_0^{+\infty}f^p(x)\left(\frac{1}{x}\left(1-\frac{1}{x^2}\arctan(x^2)\right)+x\arctan\left(\frac{1}{x^2}\right)\right)dx\right)^{1/p}$$
$$\times\left(\int_0^{+\infty}g^q(y)\left(\frac{1}{y}\left(1-\frac{1}{y^2}\arctan(y^2)\right)+y\arctan\left(\frac{1}{y^2}\right)\right)dy\right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)\min^2(x,y)}{1+x^2y^2}dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)\min^{2/p}(x,y)}{(1+x^2y^2)^{1/p}}\times\frac{g(y)\min^{2/q}(x,y)}{(1+x^2y^2)^{1/q}}dxdy$$
$$\leq\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)\min^2(x,y)}{1+x^2y^2}dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)\min^2(x,y)}{1+x^2y^2}dxdy\right)^{1/q}.$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)\min^2(x,y)}{1+x^2y^2}dxdy=\int_0^{+\infty}f^p(x)\int_0^{+\infty}\frac{\min^2(x,y)}{1+x^2y^2}dydx$$

Using standard primitives and composition rules, we have

$$\int_0^{+\infty}\frac{\min^2(x,y)}{1+x^2y^2}dy=\int_0^{x}\frac{\min^2(x,y)}{1+x^2y^2}dy+\int_x^{+\infty}\frac{\min^2(x,y)}{1+x^2y^2}dy$$
$$=\int_0^{x}\frac{y^2}{1+x^2y^2}dy+\int_x^{+\infty}\frac{x^2}{1+x^2y^2}dy$$
$$=\int_0^{x}\frac{1}{x^2}\left(1-\frac{1}{1+x^2y^2}\right)dy+\int_x^{+\infty}\frac{x^2}{1+x^2y^2}dy$$
$$=\frac{1}{x^2}\left[y-\frac{1}{x}\arctan(xy)\right]_{y=0}^{y=x}+\left[x\arctan(xy)\right]_{y=x}^{y\to+\infty}$$
$$=\frac{1}{x^2}\left(x-\frac{1}{x}\arctan(x^2)\right)+x\left(\frac{\pi}{2}-\arctan(x^2)\right)$$
$$=\frac{1}{x}\left(1-\frac{1}{x^2}\arctan(x^2)\right)+x\arctan\left(\frac{1}{x^2}\right).$$

Therefore, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)\min^2(x,y)}{1+x^2y^2}dxdy\\&=\int_0^{+\infty}f^p(x)\left(\frac{1}{x}\left(1-\frac{1}{x^2}\arctan(x^2)\right)+x\arctan\left(\frac{1}{x^2}\right)\right)dx.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)\min^2(x,y)}{1+x^2y^2}dxdy\\&=\int_0^{+\infty}g^q(y)\left(\frac{1}{y}\left(1-\frac{1}{y^2}\arctan(y^2)\right)+y\arctan\left(\frac{1}{y^2}\right)\right)dy.\end{aligned}$$

Combining the equations above, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)\min^2(x,y)}{1+x^2y^2}dxdy\\&\le\left(\int_0^{+\infty}f^p(x)\left(\frac{1}{x}\left(1-\frac{1}{x^2}\arctan(x^2)\right)+x\arctan\left(\frac{1}{x^2}\right)\right)dx\right)^{1/p}\\&\times\left(\int_0^{+\infty}g^q(y)\left(\frac{1}{y}\left(1-\frac{1}{y^2}\arctan(y^2)\right)+y\arctan\left(\frac{1}{y^2}\right)\right)dy\right)^{1/q}.\end{aligned}$$

This concludes the proof. □

The resulting constant factor is 1, with the weight function defined by

$$w(x)=\frac{1}{x}\left(1-\frac{1}{x^2}\arctan(x^2)\right)+x\arctan\left(\frac{1}{x^2}\right).$$

2.7. **Seventh result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)\min^{\alpha}(x,y)}{1+\max^2(x,y)}dxdy,$$

where α denotes an adjustable parameter.

Theorem 2.7. *Let $p>1$, $q=p/(p-1)$, $\alpha>-1$, and $f,g:[0,+\infty)\to[0,+\infty)$ be measurable functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)\min^{\alpha}(x,y)}{1+\max^2(x,y)}dxdy$$
$$\leq\left(\int_0^{+\infty}f^p(x)\left(\frac{x^{\alpha+1}}{(\alpha+1)(1+x^2)}+x^{\alpha}\arctan\left(\frac{1}{x}\right)\right)dx\right)^{1/p}$$
$$\times\left(\int_0^{+\infty}g^q(y)\left(\frac{y^{\alpha+1}}{(\alpha+1)(1+y^2)}+y^{\alpha}\arctan\left(\frac{1}{y}\right)\right)dy\right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)\min^{\alpha}(x,y)}{1+\max^2(x,y)}dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)\min^{\alpha/p}(x,y)}{(1+\max^2(x,y))^{1/p}}\times\frac{g(y)\min^{\alpha/q}(x,y)}{(1+\max^2(x,y))^{1/q}}dxdy$$
$$\leq\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)\min^{\alpha}(x,y)}{1+\max^2(x,y)}dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)\min^{\alpha}(x,y)}{1+\max^2(x,y)}dxdy\right)^{1/q}.$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)\min^{\alpha}(x,y)}{1+\max^2(x,y)}dxdy=\int_0^{+\infty}f^p(x)\int_0^{+\infty}\frac{\min^{\alpha}(x,y)}{1+\max^2(x,y)}dydx.$$

Using standard primitives and composition rules, we have

$$\int_0^{+\infty}\frac{\min^{\alpha}(x,y)}{1+\max^2(x,y)}dy=\int_0^{x}\frac{\min^{\alpha}(x,y)}{1+\max^2(x,y)}dy+\int_x^{+\infty}\frac{\min^{\alpha}(x,y)}{1+\max^2(x,y)}dy$$
$$=\int_0^{x}\frac{y^{\alpha}}{1+x^2}dy+\int_x^{+\infty}\frac{x^{\alpha}}{1+y^2}dy$$
$$=\frac{1}{1+x^2}\left[\frac{y^{\alpha+1}}{\alpha+1}\right]_{y=0}^{y=x}+x^{\alpha}\left[\arctan(y)\right]_{y=x}^{y\to+\infty}$$
$$=\frac{x^{\alpha+1}}{(\alpha+1)(1+x^2)}+x^{\alpha}\left(\frac{\pi}{2}-\arctan(x)\right)$$
$$=\frac{x^{\alpha+1}}{(\alpha+1)(1+x^2)}+x^{\alpha}\arctan\left(\frac{1}{x}\right).$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)\min^\alpha(x,y)}{1+\max^2(x,y)}dxdy$$
$$=\int_0^{+\infty} f^p(x)\left(\frac{x^{\alpha+1}}{(\alpha+1)(1+x^2)}+x^\alpha\arctan\left(\frac{1}{x}\right)\right)dx.$$

Similarly, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)\min^\alpha(x,y)}{1+\max^2(x,y)}dxdy$$
$$=\int_0^{+\infty} g^q(y)\left(\frac{y^{\alpha+1}}{(\alpha+1)(1+y^2)}+y^\alpha\arctan\left(\frac{1}{y}\right)\right)dy.$$

Combining the equations above, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)\min^\alpha(x,y)}{1+\max^2(x,y)}dxdy$$
$$\leq\left(\int_0^{+\infty} f^p(x)\left(\frac{x^{\alpha+1}}{(\alpha+1)(1+x^2)}+x^\alpha\arctan\left(\frac{1}{x}\right)\right)dx\right)^{1/p}$$
$$\times\left(\int_0^{+\infty} g^q(y)\left(\frac{y^{\alpha+1}}{(\alpha+1)(1+y^2)}+y^\alpha\arctan\left(\frac{1}{y}\right)\right)dy\right)^{1/q}.$$

This concludes the proof. □

The resulting constant factor is 1, with the weight function defined by

$$w(x)=\frac{x^{\alpha+1}}{(\alpha+1)(1+x^2)}+x^\alpha\arctan\left(\frac{1}{x}\right).$$

2.8. **Eighth result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x^2y^2+\max(x,y)}dxdy.$$

Theorem 2.8. *Let $p>1$, $q=p/(p-1)$, and $f,g:[0,+\infty)\to[0,+\infty)$ be measurable functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x^2y^2+\max(x,y)}dxdy$$
$$\leq\left(\int_0^{+\infty} f^p(x)\left(\frac{1}{x\sqrt{x}}\arctan(x\sqrt{x})+\ln\left(1+\frac{1}{x^3}\right)\right)dx\right)^{1/p}$$

$$\times\left(\int_0^{+\infty} g^q(y)\left(\frac{1}{y\sqrt{y}}\arctan(y\sqrt{y})+\ln\left(1+\frac{1}{y^3}\right)\right)dy\right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x^2y^2+\max(x,y)}dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)}{(x^2y^2+\max(x,y))^{1/p}}\times\frac{g(y)}{(x^2y^2+\max(x,y))^{1/q}}dxdy\\
&\leq\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{x^2y^2+\max(x,y)}dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{x^2y^2+\max(x,y)}dxdy\right)^{1/q}.
\end{aligned}$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{x^2y^2+\max(x,y)}dxdy=\int_0^{+\infty}f^p(x)\int_0^{+\infty}\frac{1}{x^2y^2+\max(x,y)}dydx.$$

Using standard primitives and composition rules, we have

$$\begin{aligned}
&\int_0^{+\infty}\frac{1}{x^2y^2+\max(x,y)}dy\\
&=\int_0^{x}\frac{1}{x^2y^2+\max(x,y)}dy+\int_x^{+\infty}\frac{1}{x^2y^2+\max(x,y)}dy\\
&=\int_0^{x}\frac{1}{x^2y^2+x}dy+\int_x^{+\infty}\frac{1}{x^2y^2+y}dy\\
&=\frac{1}{x}\int_0^{x}\frac{1}{xy^2+1}dy+\int_x^{+\infty}\frac{1}{y(x^2y+1)}dy\\
&=\frac{1}{x}\left[\frac{1}{\sqrt{x}}\arctan(\sqrt{x}y)\right]_{y=0}^{y=x}+\int_x^{+\infty}\left(\frac{1}{y}-\frac{x^2}{x^2y+1}\right)dy\\
&=\frac{1}{x\sqrt{x}}\arctan(x\sqrt{x})+\left[\ln(y)-\ln(x^2y+1)\right]_{y=x}^{y\to+\infty}\\
&=\frac{1}{x\sqrt{x}}\arctan(x\sqrt{x})+\left[\ln\left(\frac{y}{x^2y+1}\right)\right]_{y=x}^{y\to+\infty}\\
&=\frac{1}{x\sqrt{x}}\arctan(x\sqrt{x})+\ln\left(\frac{1}{x^2}\right)+\ln\left(\frac{x^3+1}{x}\right)
\end{aligned}$$

$$= \frac{1}{x\sqrt{x}} \arctan(x\sqrt{x}) + \ln\left(1 + \frac{1}{x^3}\right).$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)}{x^2y^2 + \max(x,y)} dxdy$$
$$= \int_0^{+\infty} f^p(x)\left(\frac{1}{x\sqrt{x}} \arctan(x\sqrt{x}) + \ln\left(1 + \frac{1}{x^3}\right)\right) dx.$$

Similarly, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{g^q(y)}{x^2y^2 + \max(x,y)} dxdy$$
$$= \int_0^{+\infty} g^q(y)\left(\frac{1}{y\sqrt{y}} \arctan(y\sqrt{y}) + \ln\left(1 + \frac{1}{y^3}\right)\right) dy.$$

Combining the equations above, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x^2y^2 + \max(x,y)} dxdy$$
$$\leq \left(\int_0^{+\infty} f^p(x)\left(\frac{1}{x\sqrt{x}} \arctan(x\sqrt{x}) + \ln\left(1 + \frac{1}{x^3}\right)\right) dx\right)^{1/p}$$
$$\times \left(\int_0^{+\infty} g^q(y)\left(\frac{1}{y\sqrt{y}} \arctan(y\sqrt{y}) + \ln\left(1 + \frac{1}{y^3}\right)\right) dy\right)^{1/q}.$$

This concludes the proof. □

The resulting constant factor is 1, with the weight function defined by

$$w(x) = \frac{1}{x\sqrt{x}} \arctan(x\sqrt{x}) + \ln\left(1 + \frac{1}{x^3}\right).$$

2.9. **Ninth result.** The result below presents a Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{\max^2(x,y) + \min(x,y)} dxdy.$$

Theorem 2.9. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{\max^2(x,y)+\min(x,y)}dxdy$$
$$\leq\left(\int_0^{+\infty}f^p(x)\left(\ln\left(1+\frac{1}{x}\right)+\frac{1}{\sqrt{x}}\arctan\left(\frac{1}{\sqrt{x}}\right)\right)dx\right)^{1/p}$$
$$\times\left(\int_0^{+\infty}g^q(y)\left(\ln\left(1+\frac{1}{y}\right)+\frac{1}{\sqrt{y}}\arctan\left(\frac{1}{\sqrt{y}}\right)\right)dy\right)^{1/q},$$

provided that the integrals of the upper bound converge.

Proof. Applying the Hölder integral inequality, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{\max^2(x,y)+\min(x,y)}dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)}{(\max^2(x,y)+\min(x,y))^{1/p}}\times\frac{g(y)}{(\max^2(x,y)+\min(x,y))^{1/q}}dxdy$$
$$\leq\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{\max^2(x,y)+\min(x,y)}dxdy\right)^{1/p}\times$$
$$\left(\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{\max^2(x,y)+\min(x,y)}dxdy\right)^{1/q}.$$

The Fubini-Tonelli integral theorem gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{\max^2(x,y)+\min(x,y)}dxdy$$
$$=\int_0^{+\infty}f^p(x)\int_0^{+\infty}\frac{1}{\max^2(x,y)+\min(x,y)}dydx.$$

Using standard primitives and composition rules, we have

$$\int_0^{+\infty}\frac{1}{\max^2(x,y)+\min(x,y)}dy$$
$$=\int_0^{x}\frac{1}{\max^2(x,y)+\min(x,y)}dy+\int_x^{+\infty}\frac{1}{\max^2(x,y)+\min(x,y)}dy$$
$$=\int_0^{x}\frac{1}{x^2+y}dy+\int_x^{+\infty}\frac{1}{y^2+x}dy$$
$$=\left[\ln(x^2+y)\right]_{y=0}^{y=x}+\left[\frac{1}{\sqrt{x}}\arctan\left(\frac{y}{\sqrt{x}}\right)\right]_{y=x}^{y\to+\infty}$$

$$= \ln(x^2 + x) - \ln(x^2) + \frac{1}{\sqrt{x}} \left(\frac{\pi}{2} - \arctan(\sqrt{x}) \right)$$
$$= \ln\left(1 + \frac{1}{x}\right) + \frac{1}{\sqrt{x}} \arctan\left(\frac{1}{\sqrt{x}}\right).$$

Therefore, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x)}{\max^2(x, y) + \min(x, y)} dxdy$$
$$= \int_0^{+\infty} f^p(x) \left(\ln\left(1 + \frac{1}{x}\right) + \frac{1}{\sqrt{x}} \arctan\left(\frac{1}{\sqrt{x}}\right) \right) dx.$$

Similarly, we obtain

$$\int_0^{+\infty} \int_0^{+\infty} \frac{g^q(y)}{\max^2(x, y) + \min(x, y)} dxdy$$
$$= \int_0^{+\infty} g^q(y) \left(\ln\left(1 + \frac{1}{y}\right) + \frac{1}{\sqrt{y}} \arctan\left(\frac{1}{\sqrt{y}}\right) \right) dy.$$

Combining the equations above, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{\max^2(x, y) + \min(x, y)} dxdy$$
$$\leq \left(\int_0^{+\infty} f^p(x) \left(\ln\left(1 + \frac{1}{x}\right) + \frac{1}{\sqrt{x}} \arctan\left(\frac{1}{\sqrt{x}}\right) \right) dx \right)^{1/p}$$
$$\times \left(\int_0^{+\infty} g^q(y) \left(\ln\left(1 + \frac{1}{y}\right) + \frac{1}{\sqrt{y}} \arctan\left(\frac{1}{\sqrt{y}}\right) \right) dy \right)^{1/q}.$$

This concludes the proof. □

The resulting constant factor is 1, with the weight function defined by

$$w(x) = \ln\left(1 + \frac{1}{x}\right) + \frac{1}{\sqrt{x}} \arctan\left(\frac{1}{\sqrt{x}}\right).$$

3. Conclusion

In this paper, we established several new Hardy-Hilbert-type integral inequalities involving explicit constant factors and tractable weighted integral norms. The results complement a variety of existing inequalities in the literature while maintaining formulations that are both effective and accessible. Our proofs rely mainly on the Hölder integral inequality, the Fubini-Tonelli integral theorem, and elementary analytical arguments, yielding concise and transparent derivations. We hope that the methods and results presented here will stimulate further investigations into related classes of integral inequalities and their applications.

Conflict of interest

The author declares no competing interests.

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