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RATE OF CONVERGENCE AT p-LEBESGUE POINTS FOR A FAMILY OF INTEGRAL OPERATORS CONTAINING NON-INTEGRABLE FUNCTIONS

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ABSTRACT. In this study, the rate of convergence at p-Lebesgue points is investigated for a family of non-convolution type integral operators containing non-integrable functions that do not belong to $L^p(a,b)$.

1. INTRODUCTION

In approximation theory, determining the rate of convergence of families of integral operators is just as important as the convergence itself. The rate of convergence of various families of integral operators has been studied by many mathematicians. Some of these works are as follows. Gadjiev [1, 4] studied the rates of convergence of certain classes of singular integral operators and integral operators depending on two parameters. Mamedov [5] investigated the rate of convergence of the family of m-singular integral operators at generalized Lebesgue points and for one- and two-dimensional singular integrals. In 1973, Ibragimov and Gadjiev [6] studied the rate of convergence of singular integral operators of Cauchy–Stieltjes type. Esen [8] worked on the rate of approximation of integral operators with positive kernels in 2008, and Karslı–Gupta [9] studied the rate of convergence of nonlinear integral operators for functions of bounded variation. In

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2026, Esen Almalı [10] examined the rate of convergence of a family of integral operators containing non-integrable functions at Lebesgue points in $L^1(a,b)$. The reader may also consult [2, 7].

In the present paper, for $f \notin L^p(a,b)$ with $p \geq 1$, we investigate the rate of convergence at the p-Lebesgue point of the associated function $f^\varphi := f/\varphi$ for the family

$$T_\lambda(f;\, x) = \int_a^b f(t)\, K_\lambda(t,x)\, dt. \tag{1.1}$$

2. Main result

Theorem 2.1. *Let $\delta > 0$, $\varphi \in L^p(a,b)$, $f \notin L^p(a,b)$, and $f^\varphi := f/\varphi \in L^p(a,b)$. Let the non-negative kernel $K_\lambda(t,x)$ satisfy the following conditions:*

(1) *For fixed λ and x, regarded as a function of t, $K_\lambda(t,x)$ is non-decreasing on $[a,x]$ and non-increasing on $[x,b]$.*

(2) *For any given $\delta > 0$,*

$$\Lambda_\lambda := \int_{x-\delta}^{x+\delta} |t-x|^{\beta}\, K_\lambda(t,x)\, dt \;\longrightarrow\; 0 \quad \textit{as } \lambda \to \infty.$$

(3) *For any given $\delta > 0$, as $\lambda \to \infty$,*

$$K_\lambda(x-\delta, x) = o(\Lambda_\lambda).$$

(4) *The integrals*

$$\int_a^x \varphi(t)\, K_\lambda(t,x)\, dt = C_\lambda(x) \quad \textit{and} \quad \int_x^b \varphi(t)\, K_\lambda(t,x)\, dt = D_\lambda(x)$$

have finite values $C_\lambda(x)$ and $D_\lambda(x)$.

(5) *The functions φ and K_λ are differentiable with respect to t almost everywhere and, for each fixed λ and x,*

$$\varphi'(t) \cdot \frac{\partial}{\partial t} K_\lambda(t,x) \geq 0.$$

Suppose that at the point x, as $h \to 0$,

$$\int_0^h \left| \frac{f(x+t)}{\varphi(x+t)} - f^\varphi(x) \right|^p dt = o\big(h^{\beta+1}\big) \tag{2.1}$$

and

$$\int_0^h \left| \frac{f(x-t)}{\varphi(x-t)} - f^{\varphi}(x) \right|^p dt = o\big(h^{\beta+1}\big)\,. \tag{2.2}$$

Then, at this point x, as $\lambda \to \infty$,

$$\big| T_\lambda(f;\, x) - \big(f^{\varphi}(x)\, C_\lambda(x) + f^{\varphi}(x)\, D_\lambda(x)\big) \big| = o\Big(\Lambda_\lambda^{1/p}\Big)\,. \tag{2.3}$$

Proof. Using condition (4) of the theorem, we write

$$\begin{aligned}
&\big| T_\lambda(f;\, x) - \big(f^{\varphi}(x)\, C_\lambda(x) + f^{\varphi}(x)\, D_\lambda(x)\big) \big| \\
&\le \int_a^x \left| \frac{f(t)}{\varphi(t)} - f^{\varphi}(x) \right| \varphi(t)\, K_\lambda(t,x)\, dt + \int_x^b \left| \frac{f(t)}{\varphi(t)} - f^{\varphi}(x) \right| \varphi(t)\, K_\lambda(t,x)\, dt.
\end{aligned}$$

Let $1 < p < \infty$ and let q be the conjugate exponent of p. Applying Hölder's inequality gives

$$\begin{aligned}
&\big| T_\lambda(f;\, x) - \big(f^{\varphi}(x)\, C_\lambda(x) + f^{\varphi}(x)\, D_\lambda(x)\big) \big| \\
&\le \left(\int_a^x \left| \frac{f(t)}{\varphi(t)} - f^{\varphi}(x) \right|^p \varphi(t)\, K_\lambda(t,x)\, dt \right)^{1/p} \cdot \left(\int_a^x \varphi(t)\, K_\lambda(t,x)\, dt \right)^{1/q} \\
&+ \left(\int_x^b \left| \frac{f(t)}{\varphi(t)} - f^{\varphi}(x) \right|^p \varphi(t)\, K_\lambda(t,x)\, dt \right)^{1/p} \cdot \left(\int_x^b \varphi(t)\, K_\lambda(t,x)\, dt \right)^{1/q}.
\end{aligned}$$

Hence, for a fixed $\delta > 0$, splitting each integral over $[a, x-\delta]$, $[x-\delta, x]$, $[x, x+\delta]$, $[x+\delta, b]$, we obtain

$$\begin{aligned}
&\big| T_\lambda(f;\, x) - \big(f^{\varphi}(x)\, C_\lambda(x) + f^{\varphi}(x)\, D_\lambda(x)\big) \big|^p \\
&\le 2^p (F_{1,\lambda} + F_{2,\lambda} + F_{3,\lambda} + F_{4,\lambda}) \left(\int_a^b \varphi(t)\, K_\lambda(t,x)\, dt \right)^{p/q}.
\end{aligned}$$

Using the positivity of K_λ and [10], it is clear that

$$F_{1,\lambda} \le 2^p \cdot \varphi(x-\delta)\, K_\lambda(x-\delta, x) \Big(\|f^{\varphi}\|_{L^p}^p + |f^{\varphi}(x)|^p (b-a) \Big),$$

$$F_{4,\lambda} \le 2^p \cdot \varphi(x+\delta)\, K_\lambda(x+\delta, x) \Big(\|f^{\varphi}\|_{L^p}^p + |f^{\varphi}(x)|^p (b-a) \Big).$$

Moreover, from conditions (2.1), (2.2) and integration by parts,

$$F_{2,\lambda} \le (\beta+1)\,\varepsilon^p\,\varphi(x)\int_{x-\delta}^{x}(x-t)^\beta\,K_\lambda(t,x)\,dt,$$

$$F_{3,\lambda} \le (\beta+1)\,\varepsilon^p\,\varphi(x)\int_{x}^{x+\delta}(t-x)^\beta\,K_\lambda(t,x)\,dt.$$

Substituting these estimates and denoting the resulting absolute constants by C_1, C_2, C_3, we arrive at

$$\begin{aligned}&\left|T_\lambda(f;\,x)-\left(f^\varphi(x)\,C_\lambda(x)+f^\varphi(x)\,D_\lambda(x)\right)\right|^p\\ \le& C_1\,K_\lambda(x-\delta,x)+C_2\,K_\lambda(x+\delta,x)+\varepsilon\cdot C_3.\end{aligned}$$

Dividing both sides by Λ_λ and letting $\lambda\to\infty$, we use hypothesis (3) which states $K_\lambda(x\pm\delta,x)=o(\Lambda_\lambda)$, to get

$$0\le\limsup_{\lambda\to\infty}\frac{\left|T_\lambda(f;\,x)-\left(f^\varphi(x)\,C_\lambda(x)+f^\varphi(x)\,D_\lambda(x)\right)\right|^p}{\Lambda_\lambda}\le\varepsilon\cdot C_3.$$

Since $\varepsilon>0$ is arbitrary, the limit equals zero, i.e.

$$\left|T_\lambda(f;\,x)-\left(f^\varphi(x)\,C_\lambda(x)+f^\varphi(x)\,D_\lambda(x)\right)\right|^p=o(\Lambda_\lambda),$$

or equivalently,

$$\left|T_\lambda(f;\,x)-\left(f^\varphi(x)\,C_\lambda(x)+f^\varphi(x)\,D_\lambda(x)\right)\right|=o\left(\Lambda_\lambda^{1/p}\right),$$

which completes the proof. □

Conflict of interest

The author declares no competing interests.

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