

**FIRST-MODE BLOW-UP AND MITTAG–LEFFLER ASYMPTOTICS FOR
TIME-FRACTIONAL EVOLUTION EQUATIONS WITH NONLINEAR ROBIN
BOUNDARY FEEDBACK**Bi Youan Désiré Youan¹, Thibaut K. Kouakou, and Nabongo Diabate

ABSTRACT. We study a time-fractional reaction–diffusion equation with a super-linear interior source, a sublinear absorption term and a nonlinear Robin boundary feedback. The linear part is generated by a coercive Robin form, so that the weak formulation, the spectral decomposition and the boundary sign convention remain compatible. We prove local existence of nonnegative weak solutions by a regularized Galerkin scheme and a fractional compactness argument. A first-eigenfunction projection yields a large-data blow-up criterion for the projected mass, while global continuation and Mittag–Leffler decay are proved only under explicit dissipative assumptions on the energy and on the modal forcing. A reduced numerical study tests the complete behavior of the first-mode model and confirms, at the projected level, the threshold separation between the dissipative, balanced and amplifying regimes predicted by the theory.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary. We consider the time-fractional reaction–diffusion problem

¹corresponding author

2020 *Mathematics Subject Classification.* 35R11, 35K55, 35B44, 35B40, 35K20.

Key words and phrases. Caputo derivative, nonlinear Robin boundary feedback, spectral threshold, finite-time blow-up, Mittag–Leffler decay, first eigenfunction.

Submitted: 06.06.2026; *Accepted:* 21.06.2026; *Published:* 29.06.2026.

$$(1.1) \quad \begin{cases} \partial_t^\alpha u + Lu = au + bu^p - cu^q, & x \in \Omega, t > 0, \\ \partial_\nu^L u + \beta u = du^r, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega, \end{cases}$$

where $0 < \alpha < 1$, $p > 1$, $0 < q < 1$, $r > 1$, $a \in \mathbb{R}$, and $b, c, d \geq 0$. The derivative ∂_t^α is the Caputo derivative. The elliptic operator is

$$Lu = -\operatorname{div}(A(x)\nabla u) + V_0(x)u,$$

where $A(x)$ is symmetric, bounded and uniformly elliptic, while $V_0 \in L^\infty(\Omega)$ is nonnegative. The conormal derivative associated with L is denoted by ∂_ν^L .

The boundary condition in (1.1) is used with a fixed sign convention. After integration by parts, the nonlinear term du^r appears on the right-hand side of the weak formulation. It is therefore a nonlinear Robin feedback acting from the boundary, not a dissipative correction incorporated in the linear form. This sign convention matters throughout the paper. The same trace contribution strengthens the lower first-mode estimate, but it must be controlled separately when global bounds are required.

The three nonlinear terms affect the estimates in different ways. The interior power bu^p is a superlinear source and may force finite-time blow-up when the initial projected mass is large. The absorption $-cu^q$, with $0 < q < 1$, has the opposite sign and is effective near zero, but it is not locally Lipschitz at the origin. The nonlinear Robin term is a boundary source and must be estimated by trace inequalities. Finally, the Caputo derivative replaces the classical semigroup time scale by a Volterra time scale; stable decay is therefore measured by Mittag-Leffler functions rather than by exponentials.

The classical background includes Fujita's theory of finite-time blow-up [2], Kaplan's eigenfunction method [5], the abstract parabolic theory of Henry [3] and Pazy [12], and the theory of superlinear parabolic problems summarized by Quittner and Souplet [13]. Critical exponents and blow-up mechanisms for nonlinear evolution equations are discussed in Levine [8], while finite-time blow-up for fractional differential systems is treated in Mennouni and Youkana [9]. For fractional evolution equations, we use the Volterra theory of Pruss [10], the weak theory of Zacher [15], the maximum and comparison principles of Luchko [6]

and Sakamoto–Yamamoto [14], and standard references on fractional differential equations [1, 4, 11]. Recent works on time-fractional diffusion, blow-up, global solvability and Robin-type effects include [7, 16–25].

The first Robin eigenfunction is used only for the estimate it provides. If

$$m(t) = \int_{\Omega} u(x, t) \phi_1(x) dx,$$

then the projection along ϕ_1 isolates the least damped eigenmode. For large projected data it yields a scalar lower fractional inequality for $m(t)$, and this inequality is enough to force blow-up. It would be incorrect, however, to read this projected inequality as an upper estimate on u . For this reason, the continuation argument is developed at the full L^2 level, while the long-time analysis uses both the Robin energy and the decay of the higher spectral components.

The paper first establishes a local weak theory compatible with the boundary convention. The nonlinear Robin feedback is kept outside the coercive linear form and is treated through trace estimates. The sublinear absorption is regularized, Galerkin solutions are constructed for the regularized problem, and the limit is recovered by compactness and monotonicity. The second result is a first-mode instability criterion. The projected inequality contains the superlinear source, the sublinear absorption and the spectral damping in an explicit algebraic balance. It gives a large-data blow-up criterion and an explicit supercritical level that is sufficient for blow-up of the projected equation. The third result separates three conclusions that require different estimates: blow-up follows from a lower first-mode estimate, global continuation follows from a dissipative energy condition, and extinction follows from finite-memory assumptions on the higher modes together with a one-sided dissipative estimate on the first coefficient.

The paper is organized as follows. Section 2 fixes the Robin form, the weak formulation and the fractional estimates used later. Section 3 proves local existence, positivity, conditional uniqueness and continuation. Section 4 derives the first-mode inequality and the full energy estimate. Section 5 states the main results. Section 6 proves the continuation, blow-up and threshold statements, including a scalar Caputo blow-up lemma. Section 7 proves the Mittag–Leffler modal decay result. Section 8 tests the complete behavior of the reduced first-mode model. The paper ends with a conclusion and perspectives.

2. FUNCTIONAL SETTING AND FRACTIONAL ESTIMATES

We first state the assumptions under which the linear Robin part, the weak formulation and the spectral decomposition are compatible.

Assumption 2.1. *The matrix field $A \in L^\infty(\Omega; \mathbb{R}^{N \times N})$ is symmetric and uniformly elliptic: there exists $\nu_0 > 0$ such that*

$$A(x)\xi \cdot \xi \geq \nu_0 |\xi|^2 \quad \text{for a.e. } x \in \Omega, \quad \xi \in \mathbb{R}^N.$$

Moreover $V_0 \in L^\infty(\Omega)$, $V_0 \geq 0$, $\beta \in L^\infty(\partial\Omega)$, $\beta \geq 0$, and the Robin form

$$(2.1) \quad a(u, v) = \int_{\Omega} A(x) \nabla u \cdot \nabla v \, dx + \int_{\Omega} V_0(x) uv \, dx + \int_{\partial\Omega} \beta uv \, d\sigma$$

is coercive on $V = H^1(\Omega)$: there exists $\mu_0 > 0$ such that

$$(2.2) \quad a(v, v) \geq \mu_0 \|v\|_{H^1(\Omega)}^2 \quad (v \in H^1(\Omega)).$$

The self-adjoint operator associated with a in $L^2(\Omega)$ is denoted by A_R . It corresponds to the homogeneous linear Robin condition $\partial_\nu^f u + \beta u = 0$ on $\partial\Omega$. Only the linear part of the boundary condition is included in A_R ; the nonlinear feedback remains on the right-hand side of the weak formulation.

Since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, A_R has compact resolvent. Hence there exist eigenvalues and eigenfunctions

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow +\infty, \quad A_R \phi_k = \lambda_k \phi_k,$$

forming an orthonormal basis of $L^2(\Omega)$. The first eigenfunction is chosen nonnegative and normalized by

$$\|\phi_1\|_{L^2(\Omega)} = 1, \quad M_1 := \int_{\Omega} \phi_1(x) \, dx > 0.$$

The variational characterization gives

$$(2.3) \quad a(v, v) \geq \lambda_1 \|v\|_{L^2(\Omega)}^2 \quad (v \in H^1(\Omega)).$$

The growth assumptions on the interior and boundary powers are fixed next.

Assumption 2.2. *The nonlinear exponents are subcritical. If $N \geq 3$, then*

$$(2.4) \quad 1 < p < \frac{N+2}{N-2}, \quad 1 < r < \frac{N}{N-2}.$$

In dimensions $N = 1, 2$, any fixed finite $p > 1$ and $r > 1$ are allowed.

Under Assumption 2.2, the embeddings

$$H^1(\Omega) \hookrightarrow L^{p+1}(\Omega), \quad \text{Tr } H^1(\Omega) \hookrightarrow L^{r+1}(\partial\Omega)$$

are compact in the ranges used below.

For a sufficiently regular scalar function y , the Caputo derivative is

$$\partial_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1.$$

We shall use the Hilbert-space convexity inequality

$$(2.5) \quad \langle \partial_t^\alpha w(t), w(t) \rangle_H \geq \frac{1}{2} \partial_t^\alpha \|w(t)\|_H^2,$$

and the standard fractional Gronwall estimate: if $z \geq 0$ and $\partial_t^\alpha z(t) \leq Cz(t) + h(t)$, then

$$z(t) \leq z(0)E_\alpha(Ct^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(C(t-s)^\alpha) h(s) ds.$$

With these conventions in place, the weak formulation used throughout the paper is the following.

Definition 2.1. *Let $T > 0$. A function u is a nonnegative weak solution of (1.1) on $(0, T)$ if*

$$u \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \quad \partial_t^\theta u \in L^\theta(0, T; H^1(\Omega)^*)$$

for some $\theta > 1$, $u \geq 0$ a.e., $u(0) = u_0$, and, for every $v \in H^1(\Omega)$ and a.e. $t \in (0, T)$,

$$(2.6) \quad \langle \partial_t^\alpha u(t), v \rangle + a(u(t), v) = \int_\Omega (au + bu^p - cu^q)v dx + d \int_{\partial\Omega} (\text{Tr } u)^r \text{Tr } v d\sigma.$$

Remark 2.1. *The sign in (2.6) is part of the model. The condition $\partial_\nu^L u + \beta u = du^r$ places the nonlinear boundary term on the right-hand side. With the alternative sign $\partial_\nu^L u + \beta u + du^r = 0$, the same term would be dissipative and would enter the estimates with the opposite sign.*

Remark 2.2. *The notation A_R refers only to the linear Robin operator generated by the closed form a . The nonlinear term du^r is never hidden in A_R . This separation prevents a common ambiguity: the boundary source may be favorable for proving a lower instability estimate, but it is unfavorable in the full energy inequality and must be controlled by a separate dissipative assumption.*

Remark 2.3. *The restrictions in (2.4) are used twice. First, they allow the interior power and the boundary feedback to be estimated at the H^1 energy level. Second, they give compactness of the trace map, which is needed to pass to the limit in $(\text{Tr } u)^r$. At critical trace growth, the present compactness argument would have to be replaced by a concentration analysis.*

We first record the link between the variational energy and the spectral expansion.

Lemma 2.1. *Let A_R be generated by the coercive form a . If*

$$v = \sum_{k=1}^{\infty} v_k \phi_k \in D(A_R^{1/2}),$$

then

$$(2.7) \quad a(v, v) = \|A_R^{1/2} v\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} \lambda_k |v_k|^2.$$

Moreover, the norm induced by a is equivalent to the $H^1(\Omega)$ norm.

Proof. The identity is the spectral theorem for positive self-adjoint operators associated with closed forms. Since the form domain of $A_R^{1/2}$ is $H^1(\Omega)$, one has

$$a(v, w) = (A_R^{1/2} v, A_R^{1/2} w)_{L^2(\Omega)}.$$

Expanding v in the orthonormal eigenbasis and applying Parseval's identity gives (2.7). The equivalence with the H^1 norm follows from coercivity (2.2) and boundedness of the coefficients in (2.1). \square

The next lemma gives the boundary compactness needed for the nonlinear Robin term.

Lemma 2.2. *Let r satisfy the trace condition in Assumption 2.2. Then, on bounded subsets of $H^1(\Omega)$, the map*

$$v \longmapsto (\text{Tr } v)^r$$

is bounded into $L^{(r+1)/r}(\partial\Omega)$. If $v_n \rightarrow v$ strongly in $L^{r+1}(\partial\Omega)$ and (v_n) is bounded in $H^1(\Omega)$, then

$$(\text{Tr } v_n)^r \rightarrow (\text{Tr } v)^r \quad \text{strongly in } L^{(r+1)/r}(\partial\Omega).$$

Proof. The boundedness follows from

$$\|(\operatorname{Tr} v)^r\|_{L^{(r+1)/r}(\partial\Omega)} = \|\operatorname{Tr} v\|_{L^{r+1}(\partial\Omega)}^r$$

and the trace embedding. The convergence follows from the continuity of the Nemytskii map on bounded subsets. Indeed,

$$|a^r - b^r| \leq C_r(|a|^{r-1} + |b|^{r-1})|a - b| \quad (a, b \geq 0),$$

and Hölder's inequality gives the strong convergence in $L^{(r+1)/r}(\partial\Omega)$. \square

Remark 2.4. *The letter C may change from line to line. It is independent of the Galerkin dimension and of the absorption regularization parameter. Constants depending on a bounded energy region are denoted by C_R or $C_{\mathcal{R}}$. This convention is used because the local construction, the global energy assumption and the modal finite-memory assumption require different bounds.*

The passage from Galerkin approximations to a weak solution uses the following fractional compactness principle.

Lemma 2.3. *Let (w_n) be bounded in $L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$. Assume also that $(\partial_t^\alpha w_n)$ is bounded in $L^\theta(0, T; H^1(\Omega)^*)$ for some $\theta > 1$. Then, after extraction of a subsequence,*

$$w_n \rightarrow w \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

Moreover, the traces converge strongly in $L^2(0, T; L^s(\partial\Omega))$ for every $s < 2(N - 1)/(N - 2)$ if $N \geq 3$, and for every finite s if $N = 1, 2$.

Proof. This is the fractional Aubin–Lions compactness principle for Volterra equations. The bound on $\partial_t^\alpha w_n$ gives a uniform fractional time-translation control in $H^1(\Omega)^*$ after applying the fractional integral representation. Since $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$ and $L^2(\Omega)$ is continuously embedded into $H^1(\Omega)^*$, the Aubin–Lions argument gives the strong convergence in $L^2(0, T; L^2(\Omega))$. The trace statement follows from the boundedness in $L^2(0, T; H^1(\Omega))$, the compact subcritical trace embedding and the same compactness argument on the boundary. \square

3. LOCAL WEAK SOLUTIONS AND CONTINUATION

The local construction has two difficulties: the non-Lipschitz absorption u^q at the origin and the nonlinear boundary source. We therefore regularize the absorption, build finite-dimensional approximations, prove estimates independent of the Galerkin dimension and the regularization parameter, and pass to the limit by compactness.

For uniqueness we isolate the one-sided condition that is actually used.

Assumption 3.1. *For each $R_0 > 0$ there exists $L_{R_0} > 0$ such that, whenever $u, v \geq 0$ and $\|u\|_{H^1} + \|v\|_{H^1} \leq R_0$,*

$$\int_{\Omega} (a(u-v) + b(u^p - v^p))(u-v)^+ dx + d \int_{\partial\Omega} ((\text{Tr } u)^r - (\text{Tr } v)^r) \text{Tr}(u-v)^+ d\sigma \leq L_{R_0} \|(u-v)^+\|_{L^2(\Omega)}^2.$$

We can now state the local existence and continuation result.

Theorem 3.1. *Assume $0 < \alpha < 1$, $p > 1$, $0 < q < 1$, $r > 1$, $b > 0$, $c, d \geq 0$, and Assumptions 2.1–2.2. For every $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ a.e., there exist $T_{\max} \in (0, +\infty]$ and a nonnegative weak solution*

$$u \in L_{\text{loc}}^2(0, T_{\max}; H^1(\Omega)) \cap L_{\text{loc}}^\infty(0, T_{\max}; L^2(\Omega))$$

of (1.1). The solution is maximal in the sense that

$$(3.1) \quad T_{\max} = +\infty \quad \text{or} \quad \limsup_{t \uparrow T_{\max}} \|u(t)\|_{L^2(\Omega)} = +\infty.$$

If Assumption 3.1 holds on bounded energy sets, then the weak solution is unique in the above class.

Proof. We give the details because the argument is the main local foundation of the paper.

Step 1. Regularized Galerkin system. Let $V_n = \text{span}\{\phi_1, \dots, \phi_n\}$ and let P_n be the L^2 projection onto V_n . For $\varepsilon > 0$ define

$$\psi_\varepsilon(s) = ((s)^+ + \varepsilon)^q - \varepsilon^q, \quad s \in \mathbb{R}.$$

We seek

$$u_{n,\varepsilon}(t) = \sum_{k=1}^n g_k^{(n,\varepsilon)}(t) \phi_k$$

satisfying, for every $v \in V_n$,

$$(3.2) \quad \begin{aligned} \langle \partial_t^\alpha u_{n,\varepsilon}(t), v \rangle + a(u_{n,\varepsilon}(t), v) &= \int_\Omega (au_{n,\varepsilon} + b(u_{n,\varepsilon}^+)^p - c\psi_\varepsilon(u_{n,\varepsilon}))v \, dx \\ &+ d \int_{\partial\Omega} (\text{Tr } u_{n,\varepsilon}^+)^r \text{Tr } v \, d\sigma, \end{aligned}$$

with $u_{n,\varepsilon}(0) = P_n u_0$. Since all nonlinearities in (3.2) are continuous and locally bounded in finite dimension, the resulting Caputo system has a local solution.

Step 2. Positivity. Testing (3.2) with the finite-dimensional approximation of $u_{n,\varepsilon}^- := \max\{-u_{n,\varepsilon}, 0\}$ and passing to the limit in the approximation of the test function gives

$$\frac{1}{2} \partial_t^\alpha \|u_{n,\varepsilon}^-(t)\|_2^2 + a(u_{n,\varepsilon}^-(t), u_{n,\varepsilon}^-(t)) \leq a \|u_{n,\varepsilon}^-(t)\|_2^2.$$

The terms $b(u^+)^p$ and $d(\text{Tr } u^+)^r$ do not contribute against u^- , while $-c\psi_\varepsilon(u)$ is non-positive against u^- . Coercivity and the fractional Gronwall inequality give $u_{n,\varepsilon}^- = 0$ for nonnegative initial approximants. In a purely spectral Galerkin implementation, the negative-part test is justified by first regularizing with the positive resolvent of the Robin operator and then passing to the spectral limit; this avoids using any false positivity property of the L^2 spectral projection. Thus the approximating solutions, and consequently the limit solution, remain nonnegative.

Step 3. Uniform energy estimate. Taking $v = u_{n,\varepsilon}(t)$ in (3.2) and using (2.5), positivity and the favorable sign of the absorption gives

$$\frac{1}{2} \partial_t^\alpha \|u_{n,\varepsilon}\|_2^2 + a(u_{n,\varepsilon}, u_{n,\varepsilon}) \leq a \|u_{n,\varepsilon}\|_2^2 + b \|u_{n,\varepsilon}\|_{p+1}^{p+1} + d \|\text{Tr } u_{n,\varepsilon}\|_{L^{r+1}(\partial\Omega)}^{r+1}.$$

By the subcritical Sobolev and trace embeddings, for every $\eta > 0$ there exist $C_\eta > 0$ and exponents $\sigma_p, \sigma_r > 1$ such that

$$\begin{aligned} \|v\|_{p+1}^{p+1} &\leq \eta \|v\|_{H^1}^2 + C_\eta (1 + \|v\|_2^{2\sigma_p}), \\ \|\text{Tr } v\|_{L^{r+1}(\partial\Omega)}^{r+1} &\leq \eta \|v\|_{H^1}^2 + C_\eta (1 + \|v\|_2^{2\sigma_r}). \end{aligned}$$

Choosing η small and using coercivity, we obtain

$$(3.3) \quad \partial_t^\alpha Y_{n,\varepsilon}(t) + c_0 \|u_{n,\varepsilon}(t)\|_{H^1}^2 \leq C(1 + Y_{n,\varepsilon}(t)^\sigma), \quad Y_{n,\varepsilon} = \|u_{n,\varepsilon}\|_2^2,$$

for some $c_0 > 0$ and $\sigma > 1$ independent of n and ε . A fractional Bihari estimate applied to (3.3) yields a time $T > 0$, depending only on the data and the structural

constants, such that

$$(3.4) \quad \sup_{0 \leq t \leq T} \|u_{n,\varepsilon}(t)\|_2^2 + \int_0^T \|u_{n,\varepsilon}(t)\|_{H^1}^2 dt \leq C_T,$$

where C_T is independent of n and ε .

Step 4. Bound for the fractional derivative. From (3.2), (3.4), the Sobolev embedding, the trace embedding and the sublinear bound $0 \leq \psi_\varepsilon(s) \leq s^q$ for $s \geq 0$ up to an irrelevant constant, we obtain

$$(\partial_t^\alpha u_{n,\varepsilon}) \quad \text{bounded in } L^\theta(0, T; H^1(\Omega)^*)$$

for some $\theta > 1$ independent of n and ε .

Step 5. Compactness and passage to the limit. Lemma 2.3 gives, up to subsequences,

$$u_{n,\varepsilon} \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\Omega)), \quad u_{n,\varepsilon} \rightarrow u \quad \text{in } L^2(0, T; L^2(\Omega)).$$

The compact trace embedding gives strong convergence of the traces in the subcritical spaces. Consequently,

$$(u_{n,\varepsilon})^p \rightarrow u^p \quad \text{in } L_{\text{loc}}^{(p+1)/p}, \quad (\text{Tr } u_{n,\varepsilon})^r \rightarrow (\text{Tr } u)^r \quad \text{in } L_{\text{loc}}^{(r+1)/r},$$

after extracting a subsequence. The absorption is recovered by monotonicity and pointwise convergence: $\psi_\varepsilon(s) \rightarrow s^q$ for $s \geq 0$, and the uniform sublinear estimate gives uniform integrability on bounded energy sets. Passing first $n \rightarrow \infty$ and then $\varepsilon \downarrow 0$ in (3.2) yields (2.6). Positivity is preserved by the closedness of the positive cone under strong L^2 convergence.

The order of the two limits is fixed. For fixed ε , the regularized absorption is continuous and the Galerkin limit is identified by the strong convergence of $u_{n,\varepsilon}$ in $L^2(0, T; L^2(\Omega))$ and by the compact convergence of the traces. After this limit has been taken, the passage $\varepsilon \downarrow 0$ uses the monotone convergence of ψ_ε on $[0, +\infty)$ and the fact that the approximations are nonnegative. Thus the non-Lipschitz behavior of s^q at $s = 0$ is never treated as a locally Lipschitz perturbation. This is essential when the initial data vanish on subsets of positive measure.

The boundary term is handled in the same limit passage without changing the linear operator. Since $(\text{Tr } u_{n,\varepsilon})^r$ converges strongly in $L^{(r+1)/r}(\partial\Omega \times (0, T))$ along a subsequence, its action on $\text{Tr } v$ converges for every $v \in H^1(\Omega)$. Therefore the

weak formulation obtained in the limit keeps exactly the sign convention of (2.6). No part of the nonlinear boundary source is absorbed into the Robin form.

Step 6. Uniqueness under the one-sided condition. Let u and v be two nonnegative solutions on a common interval and set $w = (u - v)^+$. Testing the difference of the two equations with w , using coercivity, the monotonicity of $s \mapsto s^q$ and Assumption 3.1, gives

$$\partial_t^\alpha \|w(t)\|_2^2 \leq C_R \|w(t)\|_2^2$$

on every bounded energy range. Since $w(0) = 0$, the fractional Gronwall lemma gives $w = 0$. Interchanging u and v gives equality.

Step 7. Continuation. Suppose that $T_{\max} < +\infty$ and $\sup_{0 < t < T_{\max}} \|u(t)\|_2 < +\infty$. The local construction can be restarted at any $t_0 < T_{\max}$ with a lifespan depending only on this L^2 bound and the structural constants. Taking t_0 sufficiently close to T_{\max} extends the solution beyond T_{\max} , a contradiction. Hence (3.1) holds. \square

Remark 3.1. *The proof does not use the first eigenfunction to construct the solution. The first mode appears only later, as the quantity used to prove instability. Local existence and continuation are controlled by the full L^2 and H^1 estimates.*

4. FIRST SPECTRAL PROJECTION AND ENERGY ESTIMATES

We now separate two estimates with different roles. The projected estimate is one-sided and detects large-data instability. The energy estimate concerns the full solution and supports global continuation only under a dissipative energy assumption.

For a nonnegative weak solution define

$$m(t) = \int_{\Omega} u(x, t) \phi_1(x) dx.$$

The first step is to identify the exact scalar equation satisfied by this projection.

Lemma 4.1. *For a.e. $t > 0$,*

$$(4.1) \quad \partial_t^\alpha m(t) + \lambda_1 m(t) = \int_{\Omega} (au + bu^p - cu^q) \phi_1 dx + d \int_{\partial\Omega} (\text{Tr } u)^r \text{Tr } \phi_1 d\sigma.$$

Proof. The test function ϕ_1 belongs to $H^1(\Omega)$ and is admissible in (2.6). Since spatial projection onto a fixed L^2 function commutes with the Caputo derivative,

$$\langle \partial_t^\alpha u(t), \phi_1 \rangle = \partial_t^\alpha \int_{\Omega} u(x, t) \phi_1(x) dx = \partial_t^\alpha m(t).$$

Moreover the eigenvalue relation for the Robin form gives

$$a(u(t), \phi_1) = a(\phi_1, u(t)) = \lambda_1 m(t).$$

Substitution in the weak formulation proves (4.1). \square

We next convert the projected nonlinearities into powers of the scalar mass $m(t)$.

Lemma 4.2. *Let $u \geq 0$ and $m = \int_{\Omega} u \phi_1 dx$. Then*

$$(4.2) \quad \int_{\Omega} u^p \phi_1 dx \geq M_1^{1-p} m^p, \quad \int_{\Omega} u^q \phi_1 dx \leq M_1^{1-q} m^q.$$

Proof. Let $d\mu = M_1^{-1} \phi_1 dx$. Then μ is a probability measure and $\int u d\mu = m/M_1$. Jensen's inequality applied to the convex function s^p and the concave function s^q gives the two estimates in (4.2). \square

Combining the projected identity with the Jensen estimates gives the first lower inequality.

Proposition 4.1. *For a.e. $t > 0$,*

$$(4.3) \quad \partial_t^\alpha m(t) \geq F(m(t)),$$

where

$$(4.4) \quad F(s) = bM_1^{1-p} s^p - cM_1^{1-q} s^q - (\lambda_1 - a)s, \quad s \geq 0.$$

Proof. From (4.1), positivity of u , $d \geq 0$ and nonnegativity of the trace of ϕ_1 imply

$$\partial_t^\alpha m(t) \geq -(\lambda_1 - a)m(t) + b \int_{\Omega} u^p \phi_1 dx - c \int_{\Omega} u^q \phi_1 dx.$$

The Jensen estimates (4.2) give (4.3). \square

For large projected mass, the lower-order terms in F can be absorbed into the superlinear source.

Proposition 4.2. *Assume that $R > 0$ satisfies*

$$(4.5) \quad cM_1^{1-q} s^q \leq \frac{b}{4} M_1^{1-p} s^p \quad (s \geq R)$$

and

$$(4.6) \quad (\lambda_1 - a)s \leq \frac{b}{4}M_1^{1-p}s^p \quad (s \geq R).$$

Then, whenever $m(t) \geq R$,

$$\partial_t^\alpha m(t) \geq \frac{b}{2}M_1^{1-p}m(t)^p.$$

Proof. If $m(t) \geq R$, then (4.3), (4.5) and (4.6) give

$$\partial_t^\alpha m(t) \geq bM_1^{1-p}m^p - \frac{b}{4}M_1^{1-p}m^p - \frac{b}{4}M_1^{1-p}m^p = \frac{b}{2}M_1^{1-p}m^p.$$

Such an R exists because s^q and s are lower order than s^p at infinity. If $\lambda_1 - a \leq 0$, condition (4.6) is void.

An explicit sufficient choice is

$$R \geq \max \left\{ \left(\frac{4cM_1^{1-q}}{bM_1^{1-p}} \right)^{1/(p-q)}, \left(\frac{4(\lambda_1 - a)^+}{bM_1^{1-p}} \right)^{1/(p-1)} \right\},$$

with the convention that the first term is omitted when $c = 0$. □

Remark 4.1. *The number R is a sufficient supercritical level for the projected equation. It is not claimed to be the exact threshold for the PDE. Above R , the term s^p dominates the lower-order terms s^q and s . Below R , the scalar lower inequality is not enough to decide whether the full solution is bounded or unbounded.*

Remark 4.2. *The function $F(s) = bM_1^{1-p}s^p - cM_1^{1-q}s^q - (\lambda_1 - a)s$ contains three projected effects. Near the origin, the absorption term is dominant when $c > 0$, because $0 < q < 1$. At large amplitudes, the superlinear source dominates. This explains why the scalar balance may display a low-amplitude dissipative range and a high-amplitude amplifying range. The zeros of F describe the reduced balance only; they do not automatically classify all trajectories of (1.1).*

We now return to the full solution and record the energy inequality needed for continuation.

Proposition 4.3. *Every nonnegative weak solution satisfies, for a.e. $t > 0$,*

$$(4.7) \quad \frac{1}{2} \partial_t^\alpha \|u(t)\|_2^2 + a(u(t), u(t)) \leq a\|u(t)\|_2^2 + b\|u(t)\|_{p+1}^{p+1} \\ - c\|u(t)\|_{q+1}^{q+1} + d\|\text{Tr } u(t)\|_{L^{r+1}(\partial\Omega)}^{r+1}.$$

Consequently,

$$(4.8) \quad \begin{aligned} & \frac{1}{2} \partial_t^\alpha \|u(t)\|_2^2 + (\lambda_1 - a) \|u(t)\|_2^2 + c \|u(t)\|_{q+1}^{q+1} \\ & \leq b \|u(t)\|_{p+1}^{p+1} + d \|\operatorname{Tr} u(t)\|_{L^{r+1}(\partial\Omega)}^{r+1}. \end{aligned}$$

Proof. Taking $v = u(t)$ in (2.6) and using (2.5) gives (4.7). The estimate (4.8) follows by using (2.3) and moving the linear reaction and absorption terms to the left-hand side. \square

To turn the energy inequality into a global bound, we impose the following dissipative closure.

Assumption 4.1. *There exist a positively invariant energy region $\mathcal{R} \subset H^1(\Omega)$ and constants $\theta \in (0, 1)$, $C_{\mathcal{R}} > 0$, $\rho > 0$ such that, for every $v \in \mathcal{R}$,*

$$(4.9) \quad b \|v\|_{p+1}^{p+1} + d \|\operatorname{Tr} v\|_{L^{r+1}(\partial\Omega)}^{r+1} \leq \theta a(v, v) + C_{\mathcal{R}},$$

and

$$(4.10) \quad (1 - \theta) a(v, v) - a \|v\|_2^2 \geq \rho \|v\|_2^2.$$

Under this closure, the full energy estimate becomes a scalar dissipative inequality.

Corollary 4.1. *If the trajectory satisfies Assumption 4.1, namely $u(t) \in \mathcal{R}$ for a.e. t , then*

$$(4.11) \quad \partial_t^\alpha \|u(t)\|_2^2 + 2\rho \|u(t)\|_2^2 \leq 2C_{\mathcal{R}}.$$

Proof. Insert (4.9) into (4.7) and drop the favorable absorption term. Then absorb $\theta a(u, u)$ into the left-hand side and use (4.10). Multiplying by two gives (4.11). \square

Remark 4.3. *Proposition 4.1 is a scalar lower estimate. It keeps the least damped direction and is used to prove instability. Proposition 4.3 is a full energy estimate and is the only estimate that can support continuation. The first estimate may discard the boundary source because the sign is favorable; the second cannot discard it when proving global boundedness.*

Remark 4.4. *Assumption 4.1 can be checked, for example, in a positively invariant H^1 ball where the Sobolev and trace terms are dominated by the coercive form, or*

under smallness conditions that control both the interior source and the boundary feedback. What is needed is a bound on the full energy, not smallness of the projected mass alone.

5. MAIN RESULTS

We state the results in a form that separates what is unconditional, what follows from a lower projected inequality, and what requires additional assumptions on the full energy or on the modal forcing.

The first result is the global consequence of the dissipative energy closure.

Theorem 5.1. *Assume the hypotheses of Theorem 3.1. Let u be a maximal weak solution. If Assumption 4.1 holds along the trajectory, then $T_{\max} = +\infty$ and*

$$\sup_{t \geq 0} \|u(t)\|_{L^2(\Omega)} < +\infty.$$

The complementary large-data result is obtained from the lower first-mode inequality.

Theorem 5.2. *Assume the hypotheses of Theorem 3.1. Let $R > 0$ satisfy (4.5)–(4.6). If*

$$m(0) = \int_{\Omega} u_0(x) \phi_1(x) dx \geq R,$$

then the maximal weak solution blows up in finite time in the sense that there exists $T^ < +\infty$ such that*

$$\limsup_{t \uparrow T^*} \|u(t)\|_{L^2(\Omega)} = +\infty.$$

The next statement records what the algebraic zero of the reduced balance proves, and what it does not prove for the full PDE.

Theorem 5.3. *Let F be defined by (4.4). Assume that F has a first positive zero m_c , that*

$$F(s) \leq 0 \quad (0 \leq s \leq m_c),$$

and that there exist $R > m_c$ and $\kappa > 0$ such that

$$(5.1) \quad F(s) \geq \kappa s^p \quad (s \geq R).$$

Then m_c is an invariant upper balance level for the reduced scalar problem

$$\partial_t^\alpha y = F(y), \quad y(0) = y_0 \in [0, m_c].$$

In contrast, every full solution of (1.1) satisfying $m(0) \geq R$ blows up in the sense of Theorem 5.2. The number m_c is therefore a threshold for the reduced first-mode balance, not a global threshold for the full PDE.

The last main result describes decay when the higher modes have finite memory and the first coefficient is dissipative.

Theorem 5.4. Assume that u is a global nonnegative weak solution and that

$$(5.2) \quad \sup_{t \geq 0} \|u(t)\|_{D(A_R^{1/2})} < +\infty.$$

For $k \geq 1$, set

$$(5.3) \quad F_k(t) = \int_{\Omega} (au + bu^p - cu^q)\phi_k \, dx + d \int_{\partial\Omega} (\text{Tr } u)^r \text{Tr } \phi_k \, d\sigma.$$

Assume that

$$(5.4) \quad F_k \in L^1(0, +\infty) \quad (k \geq 2).$$

Then

$$(5.5) \quad u_k(t) \rightarrow 0 \quad (k \geq 2), \quad \|u(t) - m(t)\phi_1\|_{L^2(\Omega)} \rightarrow 0.$$

If, in addition, there exist $\delta > 0$ and a nonnegative function η such that

$$(5.6) \quad \partial_t^\alpha m(t) + \delta m(t) \leq \eta(t)$$

for a.e. $t > 0$, and

$$(5.7) \quad \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha) \eta(s) \, ds \rightarrow 0 \quad (t \rightarrow +\infty),$$

then $m(t) \rightarrow 0$ and $u(t) \rightarrow 0$ in $L^2(\Omega)$. If $\eta \equiv 0$ for all large times and

$$(5.8) \quad \|u(t) - m(t)\phi_1\|_{L^2(\Omega)} \leq \frac{C}{1+t^\alpha}$$

for all large t , then

$$(5.9) \quad \|u(t)\|_{L^2(\Omega)} \leq \frac{C}{1+t^\alpha}$$

for all large t .

Remark 5.1. *The four results have distinct mathematical roles. The local theory is structural. The blow-up result is a large-data consequence of a lower first-mode inequality. The global theorem is based on the full energy estimate and on Assumption 4.1. The asymptotic theorem is a modal conditional result and does not follow from boundedness alone.*

Remark 5.2. *A full PDE threshold theorem below the reduced level m_c would require an upper comparison estimate or an invariant energy region expressed in terms of the whole solution. The lower inequality (4.3) cannot provide such a result. This is why Theorem 5.3 is stated only at the level justified by the proof.*

Remark 5.3. *The structural hypotheses on the operator and exponents define the problem. The large-data condition in Theorem 5.2 is a checkable spectral condition. The energy assumption in Theorem 5.1 and the modal assumptions in Theorem 5.4 are additional hypotheses. Keeping these levels separate prevents the first-mode estimate from being used beyond what it proves.*

6. PROOFS OF THE FINITE-TIME AND THRESHOLD RESULTS

We begin with the scalar comparison and blow-up facts used in the first-mode argument. The first lemma is a comparison statement for Caputo inequalities on a bounded interval of amplitudes.

Lemma 6.1. *Let $I \subset [0, +\infty)$ and let $G : I \rightarrow \mathbb{R}$ satisfy the one-sided Lipschitz condition*

$$G(\xi) - G(\eta) \leq L_I(\xi - \eta) \quad (\xi, \eta \in I, \xi \geq \eta).$$

If $y, z \in C([0, T])$ take values in I and satisfy

$$\partial_t^\alpha y(t) \geq G(y(t)), \quad \partial_t^\alpha z(t) = G(z(t)), \quad y(0) \geq z(0),$$

then $y(t) \geq z(t)$ on $[0, T]$.

Proof. Writing the Caputo equations in Volterra form gives

$$z(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(z(s)) ds,$$

and the corresponding inequality for y . Set $w(t) = (z(t) - y(t))^+$. On the set where $w(s) > 0$, one has $z(s) > y(s)$ and hence $G(z(s)) - G(y(s)) \leq L_I w(s)$. Therefore

$$w(t) \leq \frac{L_I}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s) ds.$$

The fractional Gronwall lemma gives $w \equiv 0$. □

The comparison lemma is combined with the following finite-time blow-up criterion.

Lemma 6.2. *Let $K > 0$, $p > 1$ and $y_0 > 0$. A nonnegative continuous function satisfying*

$$(6.1) \quad y(t) \geq y_0 + \frac{K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)^p ds$$

cannot remain finite for all $t > 0$. Equivalently, every nonnegative Caputo supersolution of $\partial_t^\alpha y \geq Ky^p$ with positive initial value blows up in finite time.

Proof. We recall the fractional Osgood argument in the form needed here. The Volterra inequality (6.1) is stronger than the equality problem

$$z(t) = y_0 + \frac{K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)^p ds.$$

By the scalar comparison lemma, $y \geq z$ as long as both are finite. It is therefore enough to prove finite-time blow-up for z .

Let $M_n = 2^n y_0$ and denote by t_n the first time at which z reaches M_n . Positivity of the kernel and the integral formulation imply that, once the level M_n is reached, the accumulated contribution of the interval during which the solution stays above $M_n/2$ yields

$$t_{n+1} - t_n \leq C_\alpha K^{-1/\alpha} M_n^{(1-p)/\alpha},$$

where C_α depends only on α . This is the standard dyadic form of the fractional Bihari–Osgood estimate; it follows by inserting the lower bound $z \geq M_n/2$ into the Volterra integral over the last interval before the crossing of M_{n+1} . Since $p > 1$,

$$\sum_{n=0}^{\infty} M_n^{(1-p)/\alpha} = y_0^{(1-p)/\alpha} \sum_{n=0}^{\infty} 2^{-n(p-1)/\alpha} < +\infty.$$

Thus the sequence (t_n) has a finite limit. Because $M_n \rightarrow +\infty$, z cannot be continued as a finite continuous solution beyond this limit. Hence y , which dominates z , also blows up in finite time. \square

Remark 6.1. *The scalar lemma is used only to prove finiteness of the first time at which the projected mass becomes unbounded. No explicit formula for that time is required. The proof of Theorem 5.2 uses the following precise implication: if the projected mass stays above a positive level and satisfies a superlinear Caputo lower inequality, then it cannot remain finite for all positive times. This is why the dyadic estimate in Lemma 6.2 is sufficient for the PDE argument.*

Proof of Theorem 5.1. Let $Y(t) = \|u(t)\|_2^2$. Corollary 4.1 gives

$$\partial_t^\alpha Y(t) + 2\rho Y(t) \leq 2C_{\mathcal{R}}.$$

Comparison with the linear Caputo problem

$$\partial_t^\alpha Z(t) + 2\rho Z(t) = 2C_{\mathcal{R}}, \quad Z(0) = Y(0),$$

yields

$$Z(t) = Y(0)E_\alpha(-2\rho t^\alpha) + 2C_{\mathcal{R}}t^\alpha E_{\alpha, \alpha+1}(-2\rho t^\alpha).$$

The first term decays and the second remains bounded. Hence Y is bounded on the maximal interval. The continuation alternative in Theorem 3.1 gives $T_{\max} = +\infty$. \square

Proof of Theorem 5.2. Set $K = (b/2)M_1^{1-p}$. Proposition 4.2 gives

$$\partial_t^\alpha m(t) \geq Km(t)^p$$

whenever $m(t) \geq R$. The interval $[R, +\infty)$ is invariant for this lower comparison: comparing m with the scalar solution starting from R shows that $m(t)$ cannot exit below R while the solution exists. Lemma 6.2 then implies that $m(t)$ becomes unbounded at some finite time T^* . Since $\|\phi_1\|_2 = 1$,

$$m(t) = \int_{\Omega} u(t)\phi_1 dx \leq \|u(t)\|_2.$$

Therefore the L^2 norm of the full solution diverges as $t \uparrow T^*$. \square

Proof of Theorem 5.3. For $0 \leq \eta \leq \xi \leq m_c$,

$$F(\xi) - F(\eta) \leq bM_1^{1-p}(\xi^p - \eta^p) + |\lambda_1 - a|(\xi - \eta) \leq L_{m_c}(\xi - \eta).$$

The negative term $-cM_1^{1-q}s^q$ only improves the one-sided Lipschitz estimate. Since $F(m_c) = 0$, the constant function $z(t) = m_c$ is a supersolution of the reduced scalar equation. Lemma 6.1 gives $y(t) \leq m_c$ whenever $0 \leq y(0) \leq m_c$. Thus $[0, m_c]$ is invariant for the reduced scalar equation.

If $m(0) \geq R$, then (4.3) and (5.1) give $\partial_t^\alpha m \geq \kappa m^p$ while $m \geq R$. Lemma 6.2 and the preceding argument prove finite-time blow-up of the full solution in the sense of Theorem 5.2. The implication below m_c is restricted to the reduced scalar equation; no upper control of the full PDE follows from the lower projected inequality. \square

7. MITTAG–LEFFLER ASYMPTOTICS AND MODAL DECAY

We now prove Theorem 5.4. Write

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\phi_k, \quad u_k(t) = \int_{\Omega} u(x, t)\phi_k(x) dx.$$

Projection of the weak equation gives

$$\partial_t^\alpha u_k(t) + \lambda_k u_k(t) = F_k(t),$$

where F_k is defined by (5.3). Hence

$$(7.1) \quad u_k(t) = u_k(0)E_\alpha(-\lambda_k t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k(t-s)^\alpha) F_k(s) ds.$$

The proof of the modal decay result relies on the integrability of the Mittag–Leffler resolvent kernel.

Lemma 7.1. *Let $\lambda > 0$ and*

$$K_\lambda(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha).$$

*Then $K_\lambda \in L^1(0, +\infty)$ and $K_\lambda * h(t) \rightarrow 0$ as $t \rightarrow +\infty$ for every $h \in L^1(0, +\infty)$.*

Proof. The Laplace transform identity

$$\mathcal{L}\{t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\}(\xi) = \frac{1}{\xi^\alpha + \lambda}, \quad \xi > 0,$$

gives total mass $1/\lambda$ as $\xi \downarrow 0$. Thus $K_\lambda \in L^1(0, +\infty)$. The convolution of two L^1 functions on the half-line, extended by zero to \mathbb{R} , belongs to $C_0(\mathbb{R})$; hence it vanishes at infinity. \square

Proof of Theorem 5.4. For $k \geq 2$, assumption (5.4) and Lemma 7.1 imply from (7.1) that $u_k(t) \rightarrow 0$. We then pass from coefficientwise convergence to convergence of the whole higher-mode remainder. By Parseval,

$$\|u(t) - m(t)\phi_1\|_2^2 = \sum_{k \geq 2} |u_k(t)|^2.$$

For each fixed N , the sum over $2 \leq k \leq N$ tends to zero. The tail is uniformly controlled by (5.2):

$$\sum_{k > N} |u_k(t)|^2 \leq \frac{1}{\lambda_{N+1}} \sum_{k > N} \lambda_k |u_k(t)|^2 \leq \frac{C}{\lambda_{N+1}}.$$

Letting first $t \rightarrow \infty$ and then $N \rightarrow \infty$ proves (5.5).

From (5.6), the scalar comparison formula gives

$$m(t) \leq m(0)E_\alpha(-\delta t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha) \eta(s) ds.$$

The first term tends to zero and the second tends to zero by (5.7); hence $m(t) \rightarrow 0$. Together with (5.5), this gives $u(t) \rightarrow 0$ in $L^2(\Omega)$.

If η eventually vanishes, then $m(t) \geq 0$ and

$$0 \leq m(t) \leq C E_\alpha(-\delta t^\alpha) \leq \frac{C}{1+t^\alpha}$$

for all large t . Combining this with (5.8) gives (5.9). \square

Remark 7.1. *The integrability condition on F_k is a finite-memory condition for each higher mode. It is stronger than boundedness, but it is appropriate for Caputo equations because the linear fractional resolvent decays only at the Mittag–Leffler scale.*

8. NUMERICAL CONFIRMATION OF THE COMPLETE REDUCED FIRST-MODE BEHAVIOR

This section tests the complete behavior of the reduced first-mode model, not the full fractional Robin boundary-value problem. The computation concerns the scalar equation obtained from the projected balance. It checks the three regimes

predicted by the analysis: decay below the reduced balance level, persistence at the balance level, and finite numerical amplification above it.

After normalization of the positive constants in the lower first-mode inequality, the reduced balance is represented by

$$(8.1) \quad \partial_t^\alpha y(t) = y(t)^2 - \gamma y(t), \quad t > 0, \quad y(0) = y_0 > 0.$$

Here

$$F(y) = y(y - \gamma),$$

so that $y = \gamma$ is the balance level of the reduced model. If $0 < y_0 < \gamma$, the linear damping is stronger than the quadratic source at the initial level. If $y_0 = \gamma$, the constant function $y(t) = \gamma$ solves the reduced equation. If $y_0 > \gamma$, the quadratic source dominates the damping term and the reduced solution enters the amplifying range.

To make the numerical tests unambiguous, we use the following convention.

Definition 8.1. Fix $\gamma > 0$ and a stopping level $Y_* > \gamma$ for (8.1). A computed trajectory is called *subcritical* when $0 < y_0 < \gamma$, *balanced* when $y_0 = \gamma$, and *supercritical* when $y_0 > \gamma$. In the supercritical case, the numerical hitting time is

$$T_{\text{num}} = \min\{t_n : y^n \geq Y_*\}.$$

In the computations below we take $Y_* = 10^2$.

We use a uniform grid $t_n = n\tau$ and approximate the Caputo derivative by the L1 formula. The discrete derivative is

$$D_\tau^\alpha y^n = \frac{1}{\Gamma(2 - \alpha)\tau^\alpha} \sum_{j=0}^{n-1} a_j (y^{n-j} - y^{n-j-1}), \quad a_j = (j+1)^{1-\alpha} - j^{1-\alpha}.$$

The numerical update is

$$(8.2) \quad D_\tau^\alpha y^n = (y^{n-1})^2 - \gamma y^n, \quad n \geq 1.$$

The quadratic source is evaluated at the previous time level and the damping term at the new time level. Thus each time step is a scalar linear solve. The constant sequence $y^n = \gamma$ satisfies (8.2) exactly, which is useful for checking the balanced case.

The parameters are

$$\alpha = 0.70, \quad \gamma = 1, \quad y_0 \in \{0.30, 1.00, 1.50\}.$$

The three initial values respectively represent the subcritical, balanced and supercritical cases. In the supercritical computation, the run is stopped when the computed value first reaches the prescribed level 10^2 .

Table 1 confirms the three expected behaviors of the reduced model. The subcritical computation stays below the balance level and decreases on the observed interval. The balanced computation remains constant, up to round-off error. The supercritical computation reaches the stopping level before $t = 0.6$. This reproduces the qualitative separation predicted by the scalar first-mode balance.

TABLE 1. Complete reduced first-mode test for $\tau = 5 \times 10^{-3}$.

y_0	position relative to γ	observed behavior	final value or crossing time
0.30	below γ	decay	$y(10) = 2.4353 \times 10^{-2}$
1.00	equal to γ	equilibrium	$y(10) = 1.0000$
1.50	above γ	amplification	$T_{\text{num}} = 0.580$

We next check that this separation is not an artifact of a single time step. The same three initial values are computed with three decreasing values of τ .

TABLE 2. Robustness of the three reduced regimes with respect to the time step.

y_0	τ	observed behavior	final value or crossing time
0.30	10^{-2}	decay	$y(10) = 2.4363 \times 10^{-2}$
0.30	5×10^{-3}	decay	$y(10) = 2.4353 \times 10^{-2}$
0.30	2.5×10^{-3}	decay	$y(10) = 2.4348 \times 10^{-2}$
1.00	10^{-2}	equilibrium	$y(10) = 1.0000$
1.00	5×10^{-3}	equilibrium	$y(10) = 1.0000$
1.00	2.5×10^{-3}	equilibrium	$y(10) = 1.0000$
1.50	10^{-2}	amplification	$T_{\text{num}} = 0.630$
1.50	5×10^{-3}	amplification	$T_{\text{num}} = 0.580$
1.50	2.5×10^{-3}	amplification	$T_{\text{num}} = 0.550$

Table 2 is not used to claim an order of convergence. Its purpose is to check that the reduced threshold behavior is stable under time-step refinement. In all subcritical runs, the computed solution stays below γ and decays. In all balanced

runs, the value $y = \gamma$ is preserved. In all supercritical runs, the computed solution reaches the fixed stopping level in finite numerical time. The changes in T_{num} only reflect the fact that the reported value is the first grid time at which the level 10^2 is crossed.

The interpretation is limited to the scalar reduced problem. The tests confirm the complete behavior of the reduced first-mode model. They do not validate the full fractional Robin problem. A numerical treatment of (1.1) would require a spatial discretization of the Robin form, a consistent approximation of the non-linear boundary trace and a discrete energy argument. The present computation concerns only the projected equation used in Sections 4 and 6.

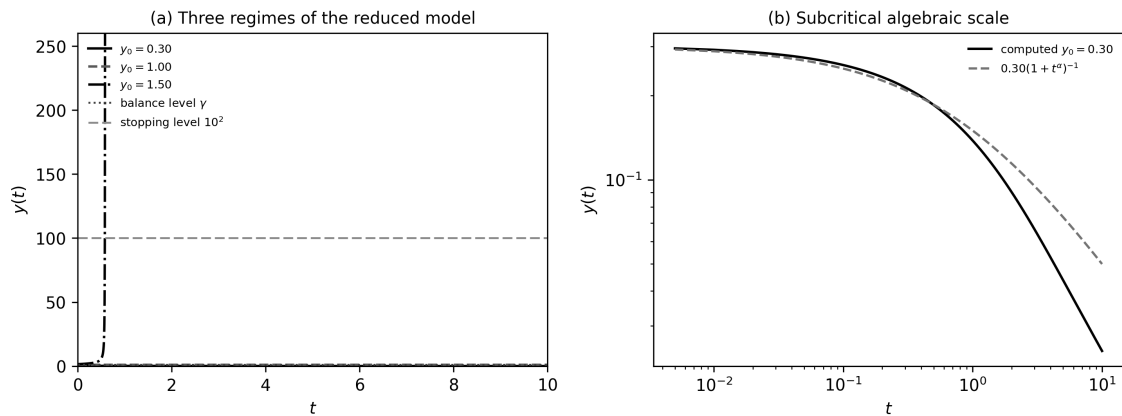


FIGURE 1. Numerical confirmation of the complete behavior of the reduced first-mode model for (8.1). Panel (a) shows the subcritical, balanced and supercritical cases. Panel (b) compares the subcritical computation with an algebraic Mittag–Leffler scale.

Figure 1 summarizes these computations. The subcritical solution decays slowly, with the fractional algebraic scale expected from the Mittag–Leffler behavior. The balanced solution remains at the reduced threshold. The supercritical solution increases rapidly and reaches the prescribed stopping level. Thus the numerical section confirms the complete behavior predicted by the reduced first-mode model. A spatially discrete study of the full Robin boundary-value problem is left for future work.

CONCLUSION AND PERSPECTIVES

We have studied a time-fractional reaction–diffusion problem with nonlinear Robin boundary feedback. The coercive Robin form keeps the weak formulation, the spectral decomposition and the boundary sign convention in a single variational setting. With the sign adopted here, the nonlinear Robin term remains an explicit boundary source in the variational identity.

The analysis clarifies both the strength and the limitation of the first Robin mode. Its projection detects the least damped direction and yields a scalar lower inequality, which gives finite-time blow-up for sufficiently large projected data. This estimate is one-sided. It proves instability, but it does not control the whole solution. Global continuation and asymptotic extinction therefore require separate dissipative assumptions involving the full energy, the higher-mode memory and the dissipative behavior of the principal coefficient.

The Caputo derivative changes the long-time scale. Exponential decay is replaced by Mittag–Leffler decay. The numerical tests confirm the complete behavior of the reduced first-mode model: decay below the balance level, persistence at the balance level and finite numerical amplification above it. Possible extensions include deriving the modal condition $F_k \in L^1(0, +\infty)$ from intrinsic energy estimates, treating spatially nonlocal or fractional diffusion operators, and studying systems in which several low modes interact.

A direct numerical continuation of the present work is a one-dimensional discretization of the full Robin problem, with an L1 approximation in time, finite elements or finite differences in space, and a boundary update that keeps the nonlinear Robin feedback in the weak formulation. Such a computation would test the nonlinear trace source itself, instead of only the scalar first-mode equation considered here.

Another possible continuation is to replace the present sufficient large-data condition by a more precise comparison between the first coefficient and the full energy. This would require an upper estimate that is not available from the lower projected inequality alone. For this reason, the paper keeps the threshold statement at the reduced level and states the global and decay results under explicit assumptions.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

REFERENCES

- [1] K. DIETHELM: *The Analysis of Fractional Differential Equations*, Springer, 2010.
- [2] H. FUJITA: *On the blowing up of solutions of the Cauchy problem*, J. Fac. Sci. Univ. Tokyo, **13** (1966), 109–124.
- [3] D. HENRY: *Geometric Theory of Semilinear Parabolic Equations*, Springer, 1981.
- [4] A.A. KILBAS, H.M. SRIVASTAVA, J.J. TRUJILLO: *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
- [5] S. KAPLAN: *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math., **16** (1963), 305–330.
- [6] Y. LUCHKO: *Maximum principle for the generalized time-fractional diffusion equation*, J. Math. Anal. Appl., **351** (2009), 218–223.
- [7] Z. LI, Y. LIU: *A compactness method for time-fractional PDEs*, SIAM J. Math. Anal., **50** (2018), 3963–3995.
- [8] H.A. LEVINE: *The role of critical exponents in blow-up problems*, SIAM Rev., **32** (1990), 262–288.
- [9] A. MENNOUNI, A. YOUKANA: *Finite time blow-up of solutions for a nonlinear system of fractional differential equations*, Electron. J. Differential Equations, **2017**(152) (2017), 1–15.
- [10] J. PRUSS: *Evolutionary Integral Equations and Applications*, Birkhauser, 1993.
- [11] I. PODLUBNY: *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [12] A. PAZY: *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.
- [13] P. QUITTNER, P. SOUPLET: *Superlinear Parabolic Problems*, Birkhauser, 2007.
- [14] K. SAKAMOTO, M. YAMAMOTO: *Initial value/boundary value problems for fractional diffusion-wave equations*, Appl. Math. Comput., **218** (2011), 256–265.
- [15] R. ZACHER: *Weak solutions of abstract evolutionary integro-differential equations*, Funkcial. Ekvac., **52** (2009), 1–18.
- [16] M. ALLEN, L. CAFFARELLI, A. VASSEUR: *A parabolic problem with a fractional time derivative*, Arch. Ration. Mech. Anal., **221** (2016), 603–630.
- [17] B. DE ANDRADE, G. SIRACUSA, A. VIANA: *A nonlinear fractional diffusion equation: well-posedness, comparison results, and blow-up*, J. Math. Anal. Appl., **505** (2022), Article 125524.
- [18] Q. ZHANG, Y. LI: *Global well-posedness and blow-up solutions of the Cauchy problem for a time-fractional superdiffusion equation*, J. Evol. Equ., **19** (2019), 271–303.
- [19] Q. ZHANG: *On the blow-up of solutions for a fractional diffusion equation with nonlinear memory and reaction terms in a bounded domain*, Mediterr. J. Math., **20** (2023), Article 190.

- [20] Z. GOU, J. SHI: *Blow-up phenomena and global existence for nonlinear parabolic problems under nonlinear boundary conditions*, AIMS Math., **8** (2023), 11822–11836.
- [21] L. SHI, W. CHENG, J. MAO, T. XU: *Blow-up and global existence of solutions for the time fractional reaction–diffusion equation*, Mathematics, **9** (2021), Article 3248.
- [22] G. FLORIDIA, Y. LIU, M. YAMAMOTO: *Blowup in $L^1(\Omega)$ -norm and global existence for time-fractional diffusion equations with polynomial semilinear terms*, Adv. Nonlinear Anal., **12** (2023), Article 20230121.
- [23] A.-M. MOSNEAGU: *On some local and nonlocal reaction-diffusion models with Robin boundary conditions*, Discrete Contin. Dyn. Syst. Ser. S, **16** (2023), 104–125.
- [24] X. HUANG, Y. LIU, M. YAMAMOTO: *Blow-up for time-fractional diffusion equations with superlinear convex semilinear terms*, arXiv:2310.14295, 2023.
- [25] D. FENG, M. YAMAMOTO: *Global and local existence of solutions for nonlinear systems of time-fractional diffusion equations*, arXiv:2405.16462, 2024.

UNIVERSITE ALASSANE OUATTARA
UFR DES SCIENCES ET TECHNOLOGIE (UFR-ST)
BP V 18 BOUAKE 01,
COTE D'IVOIRE.
Email address: biyouandesirey@gmail.com

UNIVERSITE NANGUI ABROGOUA
UFR DES SCIENCES FONDAMENTALES ET APPLIQUEES
DEPARTEMENT DE MATHEMATIQUES ET INFORMATIQUE
02 BP 801 ABIDJAN 02,
COTE D'IVOIRE.
Email address: kkthibaut@yahoo.fr

UNIVERSITE ALASSANE OUATTARA
UFR DES SCIENCES ET TECHNOLOGIES
01 BP V 18 BOUAKE 01,
COTE D'IVOIRE.
Email address: nabongo_diabate@yahoo.fr