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NEW NOISE AND STOCHASTIC CALCULUS

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ABSTRACT. In this paper, we will propose a new noise and establish a stochastic calculus driven by this noise in which we find its characteristics, the stochastic integration, Itô's formula and stochastic differential equation.

1. INTRODUCTION

The stochastic modeling on random phenomena is very efficient, evolving and widely used. We find in the field of finance, oceanography, modeling particle velocities in fluid turbulence,... The used models are almost driven by a motion Brownian, which may evolve through to our noise and the very complete stochastic calculus.

In 1996, Comte [4] used noise of a fractional Brownian motion by capturing the long memory. Then several authors find essential characteristics in this noise such as a spectrum at high frequency, nonstationarity implying an endless growth of the variance with time, fractal dimension, scaling, self-similarity (see [5]) and David Nualart [6] established the stochastic elements of the noise to facilitate its use. In this work of Lilly [5], we also find important properties of noise driven by

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the Matérn process which cannot be captured by fractional Brownian motion such as:

- an inclined spectrum which corresponds to a fractional Brownian motion at high frequency but which takes a constant value near zero frequency;
- a continuity between the two white noise regimes and a power law spectrum;
- the local behavior of a Matérn process which is diffusive (see [5]).

However, the stochastic calculus driven by the Matérn process don't yet exist, difficult to create and may be impossible in the sense of a Voltera kernel. For this, we will change the Matérn process for a Gaussian process with a Voltera kernel to follow the work of Alòs [1]. But the problem arises yet, is it a regular or singular Voltera kernel? What kind of noise is it? a fractional Brownian motion or a Matérn process? If it is other type, what are its characteristics?

2. Matérn process and its characteristics

Let $(\mathbb{R}, \mathbb{P})$ be a probability space.

2.1. Form of a Matérn Process. The Matérn process of Lilly [5] is of the form:

$$M_{H,\lambda}(t) = A\int_{-\infty}^{t} \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} e^{-\lambda(t-s)} dW_s$$

taking $\Omega = 0$, $\lambda, H > 0$, A a real, Γ is the Gamma function and W_s a Brownian motion.

2.2. Characteristics of a Matérn process and fractional Brownian motion. First, speaking of the roles of these parameters,

- A is the spectrum level setting
- H is a slope and scale parameter.
- λ acts as the inverse damping of the scale over time.

The spectrum of the Matérn process is of the form (see [5]):

$$\tilde{S}_M(w) = \frac{A^2}{(w^2+\lambda^2)^{H+\frac{1}{2}}}.$$

A spectrum which takes a constant value at any point. So the Matérn process is diffused at a uniform rate at long times.

Its autocovariance function is of the form:

$$R_M(t,\tau) = \sigma^2 \mathcal{M}_{H+\frac{1}{2}}(\lambda\tau)$$

which only depends on the time step τ. It has a short memory for long term behavior.

$$\sigma^2 = \frac{A^2 V_{H+\frac{1}{2}}}{\lambda^{2H}}$$

is its variance which is constant. So it is stationary.

$$\mathcal{M}_{H+\frac{1}{2}}(\lambda\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\tilde{S}_M(w)\lambda^{2H}e^{iw\tau}}{V_{H+\frac{1}{2}}A^2}dw$$

and

$$V_{H+\frac{1}{2}} = \frac{1}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(H)}{\Gamma(H+\frac{1}{2})}.$$

Recall that for the fractional Brownian motion with Hurst index $H \in (0.1)$, the spectrum is of the form:

$$\tilde{S}_{fBm}(w) = \frac{A^2}{|w|^{2H+1}}.$$

There has a singularity at zero frequency corresponding to an unbounded rate of dispersion.

The autocovariance function is

$$R_{fBm}(t,\tau) = \frac{A^2}{2}\left(|t+\tau|^{2H} + |t|^{2H} - |\tau|^{2H}\right).$$

The fractional Brownian motion has a short memory in $0 < H \leq \frac{1}{2}$ and a long memory elsewhere with variance depending of time:

$$\sigma^2(t) = |t|^{2H}.$$

That is, the noise driven by a fractional Brownian motion is nonstationary which has a tendency to diffuse with a rate that tends to increase without bound.

The isotropic diffusivity of a fractional Brownian motion is (see [5])

$$k_{fBm}(t) = \frac{A^2(H+\frac{3}{2})|t|^{2H+1}}{4H+2}$$

which tends to $+\infty$ as $t \nearrow +\infty$. So the behavior is superdiffusive.

For the Matérn process:

$$k_M = \frac{A^2}{4\lambda^{2H+1}}$$

which does not depend of time. So the behavior is diffusive.

The authors use the fractional Brownian motion for:

- a self-similarity property which is related to the slope of the spectrum. The slope is also related to two local properties: one associated with the low frequency slope where the autocovariance behavior has small time lags and the other associated with the high frequency slope where the autocovariance has large time lags
- the long term dependency or memory of process
- the fractal dimension which is a dimension measure of curve (see [3]).

2.3. **Matérn Process with a Voltera kernel.** Now, we are going to talk about the stochastic integral and for that we use the work of Alòs [1]. To use this work, we must have a Gaussian process of the form:

$$B_t = \int_0^t K(t,s)dW_s$$

where K is a regular or singular Voltera kernel (see [1]).

Here $B_0 = 0$ and for the Matérn process $M_{H,\lambda}(0) = A\int_{-\infty}^0 \frac{(-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} e^{\lambda s} dW_s \neq 0$.

So we are going to integrate the kernel of the Matérn process on $[0,t]$ instead of $]-\infty,t]$.

Hence our Voltera kernel is of the form:

$$K(t,s) = \frac{A(t-s)^{H-\frac{1}{2}} e^{-\lambda(t-s)}}{\Gamma(H+\frac{1}{2})}$$

and our noise is :

$$B_{H,\lambda}(t) = A\int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} e^{-\lambda(t-s)} dW_s,$$

Andrianantenainarinoro [2] used this process under the name of Matérn truncated process as an amplitude setting. He proved the existence in the sense of Riemann Stieltjes integral of $\int_0^t u_s dB_{H,\lambda}(s)$ if u_s is Hölder continuous trajectories of order larger than $1-H \leq \frac{1}{2}$. For $\lambda = 0$ and $H \in (0,1)$, the noise is a truncated fractional Brownian motion used by Comte [4] in 1996.

2.3.1. *Autocovariance function and spectrum.* We have

$$\begin{aligned} R(t,\tau) &= \mathbb{E}(B_{H,\lambda}(t+\tau)B_{H,\lambda}(t)) \\ &= \int_0^t K(t+\tau,u)K(t,u)du \\ &= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\int_0^t (t+\tau-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}e^{-\lambda(2t+\tau-2u)}du \\ &= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\int_0^t v^{H-\frac{1}{2}}\,(v+\tau)^{H-\frac{1}{2}}\,e^{-\lambda(\tau+2v)}dv \\ &\quad \text{by doing } v=t-u \\ &= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\frac{1}{2\pi}\int_{-\infty}^{+\infty}|G(t,w)|^2e^{iw\tau}dw \end{aligned}$$

where

$$G(t,w)=\int_0^{|t|}v^{H-\frac{1}{2}}e^{-\lambda v}e^{iwv}dv.$$

Using the work of Lilly [5], the spectrum is of the form

$$\begin{aligned} S(t,w) &= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\left[\left(\int_0^{|t|}y^{H-\frac{1}{2}}e^{-\lambda y}cos(wy)dy\right)^2\right. \\ &+ \left.\left(\int_0^{|t|}y^{H-\frac{1}{2}}e^{-\lambda y}sin(wy)dy\right)^2\right] \end{aligned}$$

2.3.2. *The noise is not a fractional Brownian motion nor a Matérn process and it is also nonstationary.* The autocovariance function is not that of the fractional Brownian motion nor of the Matérn process. Moreover,

$$\sigma_t^2=R(t,0)=\frac{A^2}{\Gamma(H+\frac{1}{2})^2(2\lambda)^{2H}}\gamma(2H,2\lambda t)$$

with

$$\gamma(r,s) = \int_0^s y^{r-1}e^{-y}dy$$

is the incomplete gamma function. So the noise has a tendency to diffuse with increasing rate. Which affirms that this noise is not a Matérn process. Moreover, σ_t^2 tends to the variance of a Matérn process as $t\nearrow\infty$. Thus, this noise keeps a finite rate for a long time. So it is also not a fractional Brownian motion. In short, the diffusion of the noise would be increasing and leveling off.

Speaking of stationarity, as the noise variance depends on the time t, then the noise is nonstationary.

This justifies that the Fourier Transform of a nonstationary autocovariance function $R(t, \tau)$ defines a spectrum depending of time.

According to the work of Lilly [5], the spectrum of a potentially nonstationary process is the time-averaged spectrum:

$$\tilde{S}(w) = \lim_{T \longrightarrow \infty} \bar{S}(t, w, T)$$

where $\bar{S}(t, w, T)$ is the mean of the Rihaczek distribution over global time in moving intervals of length T:

$$\bar{S}(t, w, T) = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} S(u, w) du.$$

We cannot to use this spectrum to find the autocovariance function but we calculate it with $S(t, w)$. However, across this spectrum it is better to know the behavior of noise.

2.3.3. *Noise behavior.* As $S(t, w)$ is pair then

$$\begin{aligned}\tilde{S}(w) &= \lim_{T \longrightarrow +\infty} \frac{2A^2}{T\Gamma(H + \frac{1}{2})^2} \int_0^{\frac{T}{2}} \left(\int_0^{|u|} y^{H-\frac{1}{2}} e^{-\lambda y} cos(wy) dy \right)^2 \\ &+ \left(\int_0^{|u|} y^{H-\frac{1}{2}} e^{-\lambda y} sin(wy) dy \right)^2 du.\end{aligned}$$

This limit is calculated by passing to the Proposition on the following real analysis:

Proposition 2.1. *If f is continuous on* $]0, +\infty[$ *and integrable at 0, we have* $\forall p \in \mathbb{N}^*$ *and* $\forall x, y \in]0, +\infty[$ *such that* $x \geq y$,

$$\frac{1}{x} \int_0^x \left(\int_0^y f(t) dt \right)^p dy = \left(\int_0^x f(t) dt \right)^p - \frac{p}{x} \int_0^x y f(y) \left(\int_0^y f(t) dt \right)^{p-1} dy$$

If moreover, $f(y)$ *and* $yf(y)$ *are integrables trajectories at infinite, then*

$$\lim_{x \longrightarrow +\infty} \frac{1}{x} \int_0^x \left(\int_0^y f(t) dt \right)^p dy = \left(\int_0^{+\infty} f(t) dt \right)^p$$

Proof. Using integration by parts, we have

$$\begin{aligned}
\frac{1}{x}\int_0^x \left(\int_0^y f(t)dt\right)^p dy &= \frac{1}{2}\left(\int_0^x f(t)dt\right)^p + \frac{1}{2x}\int_0^x \left(\int_0^y f(t)dt\right)^p dy \\
&- \frac{p}{2x}\int_0^x yf(y)\left(\int_0^y f(t)dt\right)^{p-1} dy \\
&\quad \text{by doing } z = \frac{1}{y}\left(\int_0^y f(t)dt\right)^p \text{ and } v' = y. \\
&\quad \text{Then } z' = -\frac{1}{y^2}\left(\int_0^y f(t)dt\right)^p + \frac{pf(y)}{y} \\
&\quad \left(\int_0^y f(t)dt\right)^{p-1} \\
&\quad \text{and } v = \frac{y^2}{2}.
\end{aligned}$$

The second equation is the consequence of the limit at infinity. □

As a result,

$$\begin{aligned}
\tilde{S}(w) &= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\left[\left(\int_0^{+\infty} y^{H-\frac{1}{2}}e^{-\lambda y}cos(wy)dy\right)^2\right. \\
&\quad \left. + \left(\int_0^{+\infty} y^{H-\frac{1}{2}}e^{-\lambda y}sin(wy)dy\right)^2\right] \\
&= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\left[\left(\frac{cos\left((H+\frac{1}{2})\theta\right)\Gamma(H+\frac{1}{2})}{(\lambda^2+w^2)^{\frac{H}{2}+\frac{1}{4}}}\right)^2\right. \\
&\quad \left. + \left(\frac{sin\left((H+\frac{1}{2})\theta\right)\Gamma(H+\frac{1}{2})}{(\lambda^2+w^2)^{\frac{H}{2}+\frac{1}{4}}}\right)^2\right] \\
&\quad \text{where } \theta = \arg(\lambda + iw) \\
&= \frac{A^2}{(w^2+\lambda^2)^{H+\frac{1}{2}}} \\
&= \tilde{S}_M(w).
\end{aligned}$$

So the Matérn process and our noise provide the same spectrum. Thus the local behavior of our noise is diffusive. It provides continuity between the two white noise regimes and a power law spectrum that starts at the time scale $\frac{1}{\lambda}$. So more

λ is small, faster we recover the self–similar behavior. So they have the same behavior, it's like fractional Brownian motion and its truncated. But the difference is that these are all nonstationary and they have a rate of dispersion which increases without limit. However, the Matérn process is stationary and the noise diffuses uniformly while its truncated is nonstationary and diffuses with increasing and plateauing velocity. This explains why the Matérn process and its truncated are also very different.

In other words, if we want our noise for instance diffuse with varying velocity but controlled for a long time, then it has only one choice among these noises and it is the truncated Matérn process.

The Brownian motion is independent stationary increment but our noise is not stationary increment nor independent.

2.3.4. *The noise is nonstationary increment.* Let $t \geq 0$ and $h > 0$. We have

$$\begin{aligned} & \mathbb{E}(B_{H,\lambda}(t+h) - B_{H,\lambda}(t))^2 \\ = \quad & \frac{A^2}{\Gamma(H+\frac{1}{2})^2} \int_{\mathbb{R}} [(t+h-s)^{H-\frac{1}{2}} e^{-\lambda(t+h-s)} 1_{[0,t+h]} - (t-s)^{H-\frac{1}{2}} e^{-\lambda(t-s)} 1_{[0,t]}]^2 ds \\ = \quad & \frac{A^2}{\Gamma(H+\frac{1}{2})^2} \int_{\mathbb{R}} [(x+h)^{H-\frac{1}{2}} e^{-\lambda(x+h)} 1_{[-h,t]} - x^{H-\frac{1}{2}} e^{-\lambda x} 1_{[0,t]}]^2 dx \end{aligned}$$

(2.1) by doing $s = t - x$.

So this equality always depends on the time t. Thus, it is not stationary increment.

2.3.5. *The noise is non-independent increment.* The covariance between two increments $B_{H,\lambda}(t+h) - B_{H,\lambda}(t)$ and $B_{H,\lambda}(s+h) - B_{H,\lambda}(s)$, $0 < h \leq s+h \leq t$ is

$$\begin{aligned} & \frac{A^2}{\Gamma(H+\frac{1}{2})^2} \int_0^{s+h} v^{H-\frac{1}{2}} (v+t-s)^{H-\frac{1}{2}} e^{-\lambda(t-s+2v)} dv \\ - \quad & \frac{A^2}{\Gamma(H+\frac{1}{2})^2} \int_0^{s} v^{H-\frac{1}{2}} (v+t-s+h)^{H-\frac{1}{2}} e^{-\lambda(t-s+h+2v)} dv \\ - \quad & \frac{A^2}{\Gamma(H+\frac{1}{2})^2} \int_0^{s+h} v^{H-\frac{1}{2}} (v+t-s-h)^{H-\frac{1}{2}} e^{-\lambda(t-s-h+2v)} dv \\ + \quad & \frac{A^2}{\Gamma(H+\frac{1}{2})^2} \int_0^{s} v^{H-\frac{1}{2}} (v+t-s)^{H-\frac{1}{2}} e^{-\lambda(t-s+2v)} dv \end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\int_0^{s+h} v^{H-\frac{1}{2}}\left(v+nh\right)^{H-\frac{1}{2}} e^{-\lambda(nh+2v)}dv \\
&- \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\int_0^{s} v^{H-\frac{1}{2}}\left(v+(n+1)h\right)^{H-\frac{1}{2}} e^{-\lambda((n+1)h+2v)}dv \\
&- \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\int_0^{s+h} v^{H-\frac{1}{2}}\left(v+(n-1)h\right)^{H-\frac{1}{2}} e^{-\lambda((n-1)h+2v)}dv \\
&+ \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\int_0^{s} v^{H-\frac{1}{2}}\left(v+nh\right)^{H-\frac{1}{2}} e^{-\lambda(nh+2v)}dv \\
&\quad \text{by doing } t-s=nh \\
&= \frac{A^2 e^{-\lambda nh} n^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})^2}\int_0^{s+h} v^{H-\frac{1}{2}}e^{-2\lambda v}\left[\left(h+\frac{v}{n}\right)^{H-\frac{1}{2}}\right. \\
&- \left.\left(\left(1-\frac{1}{n}\right)h+\frac{v}{n}\right)^{H-\frac{1}{2}} e^{\lambda h}\right] dv \\
&+ \frac{A^2 e^{-\lambda nh} n^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})^2}\int_0^{s} v^{H-\frac{1}{2}}e^{-2\lambda v}\left[\left(h+\frac{v}{n}\right)^{H-\frac{1}{2}}\right. \\
&- \left.\left(\left(1+\frac{1}{n}\right)h+\frac{v}{n}\right)^{H-\frac{1}{2}} e^{-\lambda h}\right] dv \\
&\approx \frac{A^2 e^{-\lambda nh} n^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})^2}\left[\int_0^{s+h} v^{H-\frac{1}{2}}e^{-2\lambda v}h(1-e^{\lambda h})dv + \int_0^{s} v^{H-\frac{1}{2}}e^{-2\lambda v}h(1-e^{-\lambda h})dv\right] \\
&\quad \text{for } n \text{ large enough and we denote it by } \rho(n).
\end{aligned}
$$

Since $|\rho(n)|n^2$ tends to 0 when $n \nearrow \infty$, then $\sum|\rho(n)| < \infty$. So the dependency of noise has an intermittency behavior.

2.3.6. *The noise has a short memory.* We have

$$
\begin{aligned}
r_n &= R(n+1,1) - R(n,1) \\
&= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\int_0^1 v^{H-\frac{1}{2}}\left((v+n)^{H-\frac{1}{2}}e^{-\lambda(n+2v)} - (v+n-1)^{H-\frac{1}{2}}e^{-\lambda(n-1+2v)}\right)dv.
\end{aligned}
$$

So

$$\begin{aligned}
r_n n^2 &= \frac{A^2 e^{-\lambda n} n^2}{\Gamma(H+\frac{1}{2})^2} \int_0^1 v^{H-\frac{1}{2}} \left((v+n)^{H-\frac{1}{2}} e^{-2\lambda v} - (v+n-1)^{H-\frac{1}{2}} e^{-\lambda(-1+2v)} \right) dv \\
&= \frac{A^2 e^{-\lambda n} n^{H+\frac{3}{2}}}{\Gamma(H+\frac{1}{2})^2} \int_0^1 v^{H-\frac{1}{2}} \left(\left(1+\frac{v}{n}\right)^{H-\frac{1}{2}} e^{-2\lambda v} \right. \\
&\quad \left. - \left(1+\frac{v}{n}-\frac{1}{n}\right)^{H-\frac{1}{2}} e^{-\lambda(-1+2v)} \right) dv
\end{aligned}$$

which tends to 0 when $n \nearrow \infty$.

2.3.7. *Continuity.* We have

$$(2.1) \quad = \frac{A^2 h^{2H}}{\Gamma(H+\frac{1}{2})^2} \int_{\mathbb{R}} [(1-u)^{H-\frac{1}{2}} e^{-\lambda h(1-u)} 1_{[-\frac{t}{h},1]} - (-u)^{H-\frac{1}{2}} e^{\lambda h u} 1_{[-\frac{t}{h},0]}]^2 du$$

by doing $u = -\dfrac{x}{h}$.

Suppose

$$\Theta(h) = \int_{\mathbb{R}} [(1-u)^{H-\frac{1}{2}} e^{-\lambda h(1-u)} 1_{[-\frac{t}{h},1]} - (-u)^{H-\frac{1}{2}} e^{\lambda h u} 1_{[-\frac{t}{h},0]}]^2 du.$$

We have (see [6], page 3)

$$\Theta(h) \underset{h \longrightarrow 0^+}{\longrightarrow} C_H^2 = \int_{\mathbb{R}} [(1-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}]^2 du < \infty.$$

Let $\delta > 0$, $h < \delta$. We have

$$\Theta(h) < C_H^2 + \frac{1}{2}.$$

For $h \geq \delta$,

$$\begin{aligned}
\Theta(h) &\leq \int_{-\infty}^1 (1-u)^{2H-1} e^{-2\lambda h(1-u)} du \\
&= \frac{\Gamma(2H)}{(2\lambda)^{2H} h^{2H}} \\
&\leq \frac{\Gamma(2H)}{(2\lambda)^{2H} \delta^{2H}}.
\end{aligned}$$

So

$$(2.2) \qquad \mathbb{E}(B_{H,\lambda}(t) - B_{H,\lambda}(s))^2 \leq |t-s|^{2H} M_{H,\lambda}$$

where
$$M_{H,\lambda} = \frac{A^2}{\Gamma(H+\frac{1}{2})^2} \max\left(C_H^2 + \frac{1}{2}, \frac{\Gamma(2H)}{(2\lambda\delta)^{2H}}\right).$$
By Kolmogrove's continuity criterion, our noise had a continuous trajectories. Then, through the Garsia–Rodemich–Rumsey inequality, we have $\forall \epsilon > 0$ and $T > 0$, $\exists G_{\epsilon,T}$ a non-negative random variable such that $\mathbb{E}(|G_{\epsilon,T}|^p) < \infty \ \forall p \geq 1$ and
$$|B_{H,\lambda}(t) - B_{H,\lambda}(s)| \leq G_{\epsilon,T}|t-s|^{H-\epsilon}.$$
Thus the trajectories of our noise are $(H-\epsilon)$–Hölder continuous.

2.3.8. *The kernel of our noise is a regular Voltera.* Let $\Delta = \{0 = t_0, \ldots, T = t_{n+1}\}$ be a subdivision of $[0,T]$. We have
$$\begin{aligned}
\int_0^T |K|((s,T],s)^2 ds &= \int_0^T \int_s^T |K|(dt,s)^2 ds \\
&= \lim_{|\Delta|\downarrow 0} \int_0^T \sum_{i=1}^n |K((t_{i+1},s) - K(t_i,s)|^2 ds \\
&= \lim_{|\Delta|\downarrow 0} \sum_{i=1}^n \int_0^T |K((t_{i+1},s) - K(t_i,s)|^2 ds \\
&= \lim_{|\Delta|\downarrow 0} \sum_{i=1}^n \int_{\mathbb{R}} |(t_{i+1}-s)^{H-\frac{1}{2}} e^{-\lambda(t_{i+1}-s)} 1_{[0,t_{i+1}]} - \\
&\quad (t_i - s)^{H-\frac{1}{2}} e^{-\lambda(t_i-s)} 1_{[0,t_i]}|^2 ds \frac{A^2}{\Gamma(H+\frac{1}{2})^2} \\
&\leq \lim_{|\Delta|\downarrow 0} \sum_{i=1}^n |t_{i+1}-t_i|^{2H} M_{H,\lambda} \text{ thanks to (2.1) and (2.2)} \\
&\leq T^{2H} M_{H,\lambda} < \infty.
\end{aligned}$$

3. Stochastic integral and Itô's formula

We have two methods to develop a stochastic integral either trajectories by trajectories or by Malliavin calculus and we will use the latter.

Let $B = \{B_{H,\lambda}(t), t \geq 0\}$ be our Gaussian noise with mean 0 and variance covariance $\mathbb{E}(B_{H,\lambda}(t)B_{H,\lambda}(s)) = R(t,s)$.

Let $t \in [0,T]$ and $u \in \mathbb{D}^{1,2}(\mathcal{H}_{K_r})$ (set of random variables with value in $\mathcal{H}_{K_r}$) where $\mathcal{H}_{K_r}$ is the completion of ξ mini of the semi-norm

$$||\varphi||^2_{K_r} = \int_0^T \varphi(s)^2 K(s^+, s)ds + \int_0^T \left(\int_s^T |\varphi(t)||K|(dt, s) \right)^2 ds$$

and ξ is the set of step functions on $[0,T]$.

Please see the details in the work of Alòs [1].

$\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ mini of the norm

$$||F||^p_{B,k,p} = ||F||^p_{L^p(\Omega)} + \sum_{j=1}^{k} ||D^{B,j}F||^p_{L^p(\Omega,\mathcal{H}^{\otimes}j)}$$

where $D^{B,k} : L^p(\Omega) \longrightarrow L^p(\Omega, \mathcal{H}^{\otimes k})$ is the iterated derivative operator.

$\mathcal{S}$ is the set of smooth and cylindrical random variables of the form:

$$F = f(B(\varphi_1), \ldots, B(\varphi_n)) \tag{3.1}$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and its derivatives are bounded), $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$, $\mathcal{H}$ is a Hilbert space defined as the closure of the linear span of the indicator functions $1_{[0,t]}$, $t \in [0,T]$ with respect to the scalar product $< 1_{[0,t]}, 1_{[0,s]} >= R(t,s)$.

Given a random variable F of the form (3.1), we define its derivatives as random variables with value in $\mathcal{H}$ given by:

$$D^B F = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(B(\varphi_1), \ldots, B(\varphi_n)).\varphi_j.$$

Let u be an indefinite integral driven by the divergence δB defined by

$$\begin{aligned}
\int_0^t u_s \delta B_{H,\lambda}(s) &= \int_0^t (K^* u)_s \delta W_s \\
&= \int_0^t \left(u_s K(s^+, s) + \int_s^t u_r K(dr, s) \right) \delta W_s \\
&= \lim_{|\epsilon|\downarrow 0} \int_0^t \left(u_s K(s+\epsilon, s) + \int_s^t u_r \frac{\partial K(r+\epsilon, s)}{\partial r} dr \right) \delta W_s
\end{aligned}$$

where K^* is the adjoint of the operator K (see [1]). The integral in the sense of divergence is an Itô's integral, also called Skohorod integral (see [7]).

Proposition 3.1. *Let F be a function of class $\mathcal{C}^2(\mathbb{R})$ such that:*

$$\max\{|F(x)|, ||F'(x)|, |F''(x)|\} \leq ce^{\nu x^2}$$

for all positive real constants c, $\nu < \frac{\Gamma(H+\frac{1}{2})^2(2\lambda)^{2H}}{4A^2\gamma(2H,2\lambda T)}$ and γ is the incomplete gamma function. Then the process $F'(B_{H,\lambda}(t)) \in \mathbb{D}^{1,2}(\mathcal{H}_{K_r})$ and for each $t \in [0,T]$, the following formula holds:

$$\begin{aligned} F(B_{H,\lambda}(t)) &= F(0) + \int_0^t F'(B_{H,\lambda}(s))\delta B_{H,\lambda}(s) \\ &+ \frac{A^2}{2\Gamma(H+\frac{1}{2})^2}\int_0^t F''(B_{H,\lambda}(s))s^{2H-1}e^{-2\lambda s}ds \end{aligned}$$

Proof. We have already seen that the kernel $K(t,s)$ satisfies the condition (K4) of Alòs [1] (a regular kernel). $B_{H,\lambda}(t)$ is a Gaussian process with mean 0 and covariance $R(t,s)$ of the form

$$R(t,s) = \int_0^{t\wedge s} K(t,r)K(s,r)dr$$

and we have

$$\begin{aligned} R_s &= \sigma^2(s) \\ &= \frac{A^2}{\Gamma(H+\frac{1}{2})^2(2\lambda)^{2H}}\int_0^{2\lambda s} v^{2H-1}e^{-v}dv \\ &= \frac{A^2}{\Gamma(H+\frac{1}{2})^2}\int_0^s y^{2H-1}e^{-2\lambda y}dy. \end{aligned}$$

So

$$dR_s = \frac{A^2}{\Gamma(H+\frac{1}{2})^2}s^{2H-1}e^{-2\lambda s}ds.$$

Then the result follows through the Theorem 2 of Alòs [1]. □

If $H > \frac{1}{2}$ (therefore $K(s^+,s) = 0$) and $u = \{u_t, t \in [0,T]\}$ is bounded in the norm of the space $\mathbb{D}^{1,p}$ $\forall p \geq 2$, then

$$\begin{aligned} (K_t^* u)_s &= \int_s^t u_r \frac{\partial K(r,s)}{\partial r}dr \\ &= \frac{A}{\Gamma(H+\frac{1}{2})}\int_s^t u_r\left[\left(H-\frac{1}{2}\right)(r-s)^{H-\frac{3}{2}} - \lambda(r-s)^{H-\frac{1}{2}}\right]e^{-\lambda(r-s)}dr. \end{aligned}$$

Let $u \in \mathbb{D}^{1,2}(\mathcal{H}_{K_r})$ and $H > \frac{1}{2}$. We define the Stratonovich integral driven by $B_{H,\lambda}$ by

$$\int_0^t u_s dB_{H,\lambda}(s) = \int_0^t u_s \delta B_{H,\lambda}(s) + \int_0^t \left(\int_s^t D_s u_r K(dr, s) \right) ds$$

where $dB_{H,\lambda}(s)$ defines the Stratonovich differential driven by $B_{H,\lambda}(s)$ and the operator D is defined by if $u_t = f(B_t)$, then $D_t u_s = f'(B_s)K(t,s)$. Therefore, we get Itô's formula version for the Stratonovich integral:

$$F(B_{H,\lambda}(t)) = F(0) + \int_0^t F'(B_{H,\lambda}(s)) dB_{H,\lambda}(s).$$

On the other hand, the kernel $K(t,s)$ satisfies the conditions (i), (ii') and (iii') in the work of Alòs [1] with $0 < \alpha = H - \frac{1}{2} \leq \frac{1}{2}$. By Proposition 3 in the works of Alòs [1], if a process $u = u_t, t \in [0,T]$ is bounded in the norm of the space $\mathbb{D}^{1,p}$ $\forall p \geq 2$, then $u \in \mathbb{D}^{1,p}(\mathcal{H}_{K_r})$ and the integral undefined $X_t = \int_0^t u_s \delta B_{H,\lambda}(s)$ satisfies $\mathbb{E}|X_t - X_s|^p \leq c|t-s|^{\frac{p}{2}(H+\frac{1}{2})}$. Moreover, if u is bounded in $\mathbb{D}^{2,4}$ then we obtain the following generalized Itô's formula: $\forall F \in C_b^2(\mathbb{R})$

$$\begin{aligned} F(X_t) &= F(0) + \int_0^t F'(X_s) u_s \delta B_{H,\lambda}(s) \\ &+ \int_0^t F''(X_s) u_s \left(\int_0^s \frac{\partial K}{\partial s}(s,r) \left(\int_0^s D_r (K_s^* u)_\theta \delta W_\theta \right) dr \right) ds \\ &+ \int_0^t F''(X_s) \frac{\partial}{\partial s} \left(\int_0^s (K_s^* u)_r^2 dr \right) ds. \end{aligned}$$

For Itô's formula in the Stratonovich integral term.

Let $Y_t = \int_0^t u_s dB_{H,\lambda}(s)$.

We have

$$F(Y_t) = F(0) + \int_0^t F'(Y_s) u_s dB_{H,\lambda}(s).$$

4. Riemann sum approximation

By Proposition 5 of Alòs [1], if u is a continuous adapted process in the norm of $\mathbb{D}^{1,2}$ satisfying

$$\sup_{s \leq T} \int_0^s \mathbb{E}|D_r u_s|^{2q} dr < \infty; \forall q > H - \frac{1}{2} > 0$$

then

$$\lim_{|\Delta|\downarrow 0} \sum_{i=0}^{n} u_{s_i}(B_{H,\lambda}(s_{i+1}) - B_{H,\lambda}(s_i)) = \int_0^T u_t dB_{H,\lambda}(t).$$

In the sense that u is not adapted, we can consider an approximation by the integral in the sense of Itô (integral of Skohorod).

5. Stochastic differential equation

Let $B = \{B_{H,\lambda}(t), t \geq 0\}$ be our noise of dimension d and index $H \in (\frac{1}{2}, 1)$ and $\lambda > 0$.

Let's define a stochastic differential equation in $\mathbb{R}^d$ driven by this noise

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds,\ t \in [0, T] \tag{5.1}$$

where $X_0 : \Omega \longrightarrow \mathbb{R}^d$ and $\sigma^{i,j}, b^i : \Omega \times [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ are measurable functions.

5.1. Existence of the stochastic integral.

We knew that the stochastic integral $\int_0^t \sigma(s, X_s) dB_s$ exists in the sense of Riemann Stieltjes if σ is Hölder continuous trajectories of order larger than $1 - H$, but here we will look at another condition.

Choose α such that $0 < 1 - H < \alpha < \frac{1}{2}$. Since B is $(H - \epsilon)$–Hölder continuous trajectories, then $B \in C^{1-\alpha-\epsilon}(0,T)$ (ϵ >0). Moreover, the support of $C^{1-\alpha+\epsilon}(0,T)$ namely $[0,T]$ is compact then the Banach space $(C^{1-\alpha+\epsilon}(0,T), ||.||_{1-\alpha+\epsilon})$ is separable.

$C^{1-\alpha+\epsilon}(0,T)$ is a Hölder space of continuous functions f on $[0,T]$ mini of the norm,

$$||f||_{1-\alpha+\epsilon} = \sup_{0 \leq y < x \leq T} \left(\frac{|f(x) - f(y)|}{(x-y)^{1-\alpha+\epsilon}} \right) < \infty.$$

$W_T^{1-\alpha,\infty}(0,T)$ is a space of measurable functions g such that

$$||g||_{1-\alpha,\infty,T} = \sup_{0 \leq y < x \leq T} \left(\frac{|g(x) - g(y)|}{(x-y)^{1-\alpha}} + \int_y^x \frac{|g(u) - g(y)|}{(u-y)^{2-\alpha}} du \right) < \infty.$$

We have $\forall 0 < \epsilon < \alpha$

$$C^{1-\alpha+\epsilon}(0,T) \subset W_T^{1-\alpha,\infty}(0,T) \subset C^{1-\alpha}(0,T).$$

For $g \in W_T^{1-\alpha,\infty}(0,T)$, define

$$\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0\leq s<t\leq T} |D_{t^-}^{1-\alpha} g(s)| \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} ||g||_{1-\alpha,\infty,T}$$

where

$$D_{t^-}^{1-\alpha} g(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{g(x)}{(t-x)^\alpha} + \alpha \int_x^b \frac{|g(x)-g(y)|}{(y-x)^{\alpha+1}} dy \right).$$

By Fernique's Theorem, $\forall \delta \in (0,2)$, $\exists x_t > 0$ such that

$$\mathbb{E}\left(e^{(x_t ||B||^\delta_{1-\alpha+\epsilon})} \right) < \infty.$$

Let $y = \sup\limits_{0\leq t\leq T} x_t$ and $x = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}$. We have

$$\mathbb{E}\left(e^{(x||B||^\delta_{1-\alpha,\infty,T})} \right) < \mathbb{E}\left(e^{y\left\| \left(\frac{x}{y}\right)^{\frac{1}{\delta}} B \right\|^\delta_{1-\alpha+\epsilon}} \right) < \infty.$$

So

$$\mathbb{E}\left(e^{\Lambda_\alpha(B)^\delta} \right) < \infty.$$

Therefore, if $u = \{u_t, t \in [0,T]\}$ is a stochastic process whose trajectories belong to $W_0^{\alpha,1}(0,T)$ almost surely, the generalized stieltjes integral by trajectory $\int_0^T u_s dB_s$ exists and we have the estimate

$$\left| \int_0^T u_s dB_s \right| \leq G||u||_{\alpha,1}$$

where $W_0^{\alpha,1}(0,T)$ is the set of measurable functions f on $[0,T]$ such that

$$||f||_{\alpha,1} = \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s)-f(y)|}{(s-y)^{\alpha+1}} dy ds < \infty.$$

Moreover, if the trajectories of the process $u \in W_0^{\alpha,\infty}(0,T)$, set of measurable functions f mini of the norm:

$$||f||_{\alpha,\infty} = \sup_{0\leq t\leq T} \left(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty,$$

then the indefinite integral $U_t = \int_0^t u_s dB_s$ is Hölder continuous trajectories of order $1-\alpha$ and its trajectories also belong to $W_0^{\alpha,\infty}(0,T)$.

5.2. **Existence and uniqueness of the solution of the stochastic differential equation** (5.1)**.** If the coefficients b and σ satisfy the conditions H1 and H2 below where the constants M_N and L_N depend on $w \in \Omega$, and $\beta > 1-H$, $\delta > \frac{1}{H}-1$, then through the Theorem 9 of Alòs [1], for a $\alpha \in]1-H, min(\frac{1}{2}, \beta, \frac{\delta}{\delta+1})[$, the stochastic differential equation (5.1) on $\mathbb{R}^d$ admits a unique solution in $(W_0^{\alpha,\infty}(0,T))^d$. Moreover, the solution is Hölder continuous trajectories of order $1-\alpha$.

H1: $\sigma(t,x)$ is a differentiable function at x and there exist constants $0 < \beta, \delta \leq 1$ and for all $N \geq 0$, there exists $M_N > 0$ such that the following properties hold:

$$\begin{aligned} |\sigma(t,x) - \sigma(t,y)| &\leq M_0|x-y|, \forall x, y \in \mathbb{R}^d, \forall t \in [0,T] \\ |\partial_{x_k}\sigma(t,x) - \partial_{x_k}\sigma(t,y)| &\leq M_N|x-y|^\delta, \forall |x|, |y| \leq N, \\ &\qquad \forall t \in [0,T] \\ |\sigma(t,x) - \sigma(s,x)| + |\partial_{x_k}\sigma(t,x) - \partial_{x_k}\sigma(s,x)| &\leq M_0|t-s|^\beta, \\ &\qquad \forall x, y \in \mathbb{R}^d, \forall t, s \in [0,T] \end{aligned}$$

H2: $b(t,x)$ verifies: for all $N \geq 0$, there is a constant $L_N > 0$,

$$\begin{aligned} |b(t,x) - b(t,y)| &\leq L_N|x-y|, \forall |x|, |y| \leq N, \forall t \in [0,T] \\ |b(t,x)| &\leq L_0(|x|+1), \forall x \in \mathbb{R}^d, \forall t, s \in [0,T]. \end{aligned}$$

5.3. **Estimate of the solution.** We will simply follow the work of Alòs [1].

Suppose X_0 is bounded and the constants do not depend on w. If moreover

$$|\sigma(t,x)| \leq K_0(1+|x|^\gamma)$$

where $0 \leq \gamma \leq 1$, then

$$||X||_{\alpha,\infty} \leq c_1 e^{c_2 \Lambda_\alpha(B)^\kappa}$$

where

$$\kappa = \begin{cases} \frac{\gamma}{1-2\alpha} \text{ if } \frac{1-2\alpha}{1-\alpha} \leq \gamma \leq 1 \\ \frac{1}{1-\alpha} \text{ elsewhere} \end{cases}$$

So for all $p \geq 1$,

$$\mathbb{E}(||X||^p_{\alpha,\infty}) \leq c_1^p e^{pc_2\Lambda_\alpha(B)^\kappa} < \infty$$

for all $\kappa < 2$, or the moment is finite if $\frac{\gamma}{4} + \frac{1}{2} \leq H$ and $1-H < \alpha < -\frac{\gamma}{4} + \frac{1}{2}$.

- If $\gamma = 1$, then $\alpha < \frac{1}{4}$ and $\frac{3}{4} < H$.
- If $\gamma < 2 - \frac{1}{H}$, we can take any α such that $1-H < \alpha < \frac{1}{2}$.

6. Conclusion

We proposed a new noise and established a stochastic calculus driven by this noise. One can now use this noise in all the modelings of the random phenomena. Section 2 presents the interests on its use and its specificity compared to other noises. The stochastic calculus start in section 3 where we studied the stochastic integral and Itô's formula in all domains of the noise parameter (λ and H). But we restrict ourselves to $H \in (\frac{1}{2}, 1)$ on the stochastic differential equation because of the space $(W_0^{\alpha,\infty}(0,T))^d$ which contains our solution and if we want to expand on $H \geq 1$, we will have to search for a new space which contains the solution. Andrianantenainarinoro [2] used this noise in the field of financial engineering on the stock markets.

Conflict of interest

The authors declare no conflict of interest.

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