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SOME NEW CYCLIC INEQUALITIES BASED ON AN INTEGRAL APPROACH AND THE TITU LEMMA

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ABSTRACT. This paper contributes to the theory of cyclic inequalities by introducing an underutilized proof strategy: combining integral representations with the Titu lemma. This approach yields new cyclic inequalities featuring well-known functions, such as the arctangent and logarithmic functions.

1. INTRODUCTION

Cyclic inequalities are a class of mathematical expressions where the inequality holds under a cyclic permutation of the variables. Given a set of n variables with $n \in \mathbb{N}\backslash\{0,1,2\}$, say $x_1, x_2, \ldots, x_n$, an inequality is cyclic if it remains invariant under the mapping $x_1 \to x_2, x_2 \to x_3, \ldots, x_n \to x_1$. These expressions are fundamental in optimization and analysis, often requiring sophisticated techniques.

A foundational example of this class is the Nesbitt inequality established in [7], which states that, for any $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

(the equality holds if $a = b = c$). This inequality serves as a primary benchmark for testing techniques in the study of cyclic and symmetric algebraic expressions,

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representing the simplest non-trivial case of many broader cyclic systems. For further developments on the theory of cyclic inequalities, we refer the reader to the works by [1–3, 5–8].

In this paper, we advance the study of cyclic inequalities by developing an underutilized methodological framework: establishing cyclic bounds through a synergy of integral representations and the Titu lemma (also known as the Titu form of the Cauchy-Schwarz inequality or the Bergström inequality).

The remainder of this paper is organized as follows: Section 2 provides the necessary preliminaries regarding the Titu lemma, distinguishing between its discrete and continuous formulations. In Section 3, we apply this framework to establish several new families of cyclic inequalities involving diverse function classes, notably including logarithmic and arctangent functions. Section 4 is devoted to an important remark. Finally, Section 5 offers concluding remarks.

2. Preliminaries

Before stating our main theorems, let us recall the two general results we will use.

(1) **The discrete Titu lemma:** For any $n \in \mathbb{N}\backslash\{0\}$, $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2, \ldots, b_n > 0$, we have

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i}.$$

(2) **The integral Titu lemma:** Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $b > a$. For any integrable functions $\phi, \psi : [a, b] \to (0, \infty)$, we have

$$\int_a^b \frac{\phi(x)^2}{\psi(x)} dx \geq \frac{\left(\int_a^b \phi(x) dx\right)^2}{\int_a^b \psi(x) dx}.$$

For a more comprehensive discussion of the Titu lemma, we refer the reader to [4] and the references therein.

3. Various cyclic inequalities

A new arctangent cyclic inequality is established in the theorem below.

Theorem 3.1. *For any $a, b, c > 0$, the following inequality holds:*

$$\frac{a^3}{b}\arctan\left(\frac{b}{a}\right)+\frac{b^3}{c}\arctan\left(\frac{c}{b}\right)+\frac{c^3}{a}\arctan\left(\frac{a}{c}\right)\geq\frac{\pi}{4}(a^2+b^2+c^2).$$

The equality holds if $a = b = c$.

Proof. We construct an arctangent integral development. For any $a, b > 0$, performing the substitution $u = (b/a)x$, we get

$$\int_0^1 \frac{a^4}{a^2+b^2x^2}dx=\int_0^{b/a}\frac{a^4}{a^2+a^2u^2}\left(\frac{a}{b}\right)du=\frac{a^3}{b}\int_0^{b/a}\frac{1}{1+u^2}du$$
$$=\frac{a^3}{b}\left[\arctan(u)\right]_0^{b/a}=\frac{a^3}{b}\arctan\left(\frac{b}{a}\right).$$

Using this representation, we can express the main sum as follows:

$$\frac{a^3}{b}\arctan\left(\frac{b}{a}\right)+\frac{b^3}{c}\arctan\left(\frac{c}{b}\right)+\frac{c^3}{a}\arctan\left(\frac{a}{c}\right)$$
$$=\int_0^1\left(\frac{a^4}{a^2+b^2x^2}+\frac{b^4}{b^2+c^2x^2}+\frac{c^4}{c^2+a^2x^2}\right)dx.$$

By the discrete Titu Lemma and standard integral developments, we obtain

$$\int_0^1\left(\frac{a^4}{a^2+b^2x^2}+\frac{b^4}{b^2+c^2x^2}+\frac{c^4}{c^2+a^2x^2}\right)dx$$
$$=\int_0^1\left(\frac{(a^2)^2}{a^2+b^2x^2}+\frac{(b^2)^2}{b^2+c^2x^2}+\frac{(c^2)^2}{c^2+a^2x^2}\right)dx$$
$$\geq\int_0^1\frac{(a^2+b^2+c^2)^2}{(a^2+b^2x^2)+(b^2+c^2x^2)+(c^2+a^2x^2)}dx$$
$$=\int_0^1\frac{(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)(1+x^2)}dx$$
$$=(a^2+b^2+c^2)\int_0^1\frac{1}{1+x^2}dx$$
$$=(a^2+b^2+c^2)\left[\arctan(x)\right]_0^1=\frac{\pi}{4}(a^2+b^2+c^2).$$

We immediately derive

$$\frac{a^3}{b}\arctan\left(\frac{b}{a}\right)+\frac{b^3}{c}\arctan\left(\frac{c}{b}\right)+\frac{c^3}{a}\arctan\left(\frac{a}{c}\right)\geq\frac{\pi}{4}(a^2+b^2+c^2).$$

The equality holds if $a = b = c$, in which case both sides simplify to $(3\pi/4)a^2$. This concludes the proof. $\square$

By substituting a, b, and c with $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$, respectively, in Theorem 3.1, we deduce the following cyclic inequality:

$$a\sqrt{\frac{a}{b}}\arctan\left(\sqrt{\frac{b}{a}}\right) + b\sqrt{\frac{b}{c}}\arctan\left(\sqrt{\frac{c}{b}}\right) + c\sqrt{\frac{c}{a}}\arctan\left(\sqrt{\frac{a}{c}}\right) \geq \frac{\pi}{4}(a+b+c).$$

Remark 3.1. *We can also derive the following inequalities:*

$$\frac{a}{b}\arctan\left(\frac{b}{a}\right) + \frac{b}{c}\arctan\left(\frac{c}{b}\right) + \frac{c}{a}\arctan\left(\frac{a}{c}\right) \geq \frac{\pi}{4} \cdot \frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

The proof follows the lines of the proof of Theorem 3.1.

A comprehensive cyclic inequality is established in the theorem below.

Theorem 3.2. *For any $a, b, c > 0$, the following inequality holds:*

$$a\sqrt{\frac{a}{b}} + b\sqrt{\frac{b}{c}} + c\sqrt{\frac{c}{a}} \geq a+b+c.$$

The equality holds if $a = b = c$.

Proof. We construct an integral development. For any $a, b > 0$, we have

$$\int_0^{\infty} \frac{a^2}{a+bx^2}dx = \frac{a^2}{b}\left[\sqrt{\frac{b}{a}}\arctan\left(x\sqrt{\frac{b}{a}}\right)\right]_0^{\infty}$$

$$= \frac{a^2}{\sqrt{ab}} \cdot \frac{\pi}{2} = \frac{\pi}{2}a\sqrt{\frac{a}{b}}.$$

Using this development, we can express the main sum as follows:

$$a\sqrt{\frac{a}{b}} + b\sqrt{\frac{b}{c}} + c\sqrt{\frac{c}{a}} = \frac{2}{\pi}\int_0^{\infty}\left(\frac{a^2}{a+bx^2} + \frac{b^2}{b+cx^2} + \frac{c^2}{c+ax^2}\right)dx.$$

By the discrete Titu Lemma and standard integral developments, we obtain

$$\begin{aligned}
&\int_0^\infty \left(\frac{a^2}{a+bx^2} + \frac{b^2}{b+cx^2} + \frac{c^2}{c+ax^2} \right) dx \\
&\geq \int_0^\infty \frac{(a+b+c)^2}{(a+bx^2)+(b+cx^2)+(c+ax^2)} dx \\
&= \int_0^\infty \frac{(a+b+c)^2}{(a+b+c)(1+x^2)} dx \\
&= (a+b+c) \int_0^\infty \frac{1}{1+x^2} dx \\
&= (a+b+c) \left[\arctan(x)\right]_0^\infty = \frac{\pi}{2}(a+b+c).
\end{aligned}$$

We immediately derive

$$a\sqrt{\frac{a}{b}} + b\sqrt{\frac{b}{c}} + c\sqrt{\frac{c}{a}} \geq \frac{2}{\pi} \cdot \frac{\pi}{2}(a+b+c) = a+b+c.$$

The equality is attained if $a = b = c$. This concludes the proof. □

Alternatively, setting $x = \sqrt{a}$, $y = \sqrt{b}$, and $z = \sqrt{c}$ allows us to apply the discrete Titu lemma in conjunction with the standard inequality $xy+yz+zx \leq x^2+y^2+z^2$, as follows:

$$\begin{aligned}
&a\sqrt{\frac{a}{b}} + b\sqrt{\frac{b}{c}} + c\sqrt{\frac{c}{a}} = \frac{x^3}{y} + \frac{y^3}{z} + \frac{z^3}{x} = \frac{(x^2)^2}{xy} + \frac{(y^2)^2}{yz} + \frac{(z^2)^2}{zx} \\
&\geq \frac{(x^2+y^2+z^2)^2}{xy+yz+zx} \geq \frac{(x^2+y^2+z^2)^2}{x^2+y^2+z^2} = x^2+y^2+z^2 = a+b+c.
\end{aligned}$$

A new cyclic inequality depending on one parameter is established in the theorem below.

Theorem 3.3. *For any $a, b, c > 0$ and $n > 1$, the following inequality holds:*

$$\frac{a^2}{b}\left(\frac{b}{c}\right)^{1/n} + \frac{b^2}{c}\left(\frac{c}{a}\right)^{1/n} + \frac{c^2}{a}\left(\frac{a}{b}\right)^{1/n} \geq a+b+c.$$

The equality holds if $a = b = c$.

Proof. For any $n > 1$, let us set

$$I_n = \int_0^\infty \frac{1}{1+x^n} dx$$

(we do not need its analytical form). We construct an integral development. For any $a, b, c > 0$, performing the substitution $x = u\,(b/c)^{1/n}$, we get

$$\int_0^\infty \frac{a^2}{b+cx^n}dx = \int_0^\infty \frac{a^2}{b+c\,(b/c)\,u^n}\left(\frac{b}{c}\right)^{1/n} du$$
$$= \frac{a^2}{b}\left(\frac{b}{c}\right)^{1/n}\int_0^\infty \frac{1}{1+u^n}du = \frac{a^2}{b}\left(\frac{b}{c}\right)^{1/n} I_n.$$

Using this development, we can express the main sum as follows:

$$\frac{a^2}{b}\left(\frac{b}{c}\right)^{1/n} + \frac{b^2}{c}\left(\frac{c}{a}\right)^{1/n} + \frac{c^2}{a}\left(\frac{a}{b}\right)^{1/n} = \frac{1}{I_n}\int_0^\infty \left(\frac{a^2}{b+cx^n} + \frac{b^2}{c+ax^n} + \frac{c^2}{a+bx^n}\right)dx.$$

By the discrete Titu Lemma and standard integral developments, we obtain

$$\begin{aligned}
&\int_0^\infty \left(\frac{a^2}{b+cx^n} + \frac{b^2}{c+ax^n} + \frac{c^2}{a+bx^n}\right)dx \\
&\geq \int_0^\infty \frac{(a+b+c)^2}{(b+cx^n)+(c+ax^n)+(a+bx^n)}dx \\
&= \int_0^\infty \frac{(a+b+c)^2}{(a+b+c)(1+x^n)}dx \\
&= (a+b+c)\int_0^\infty \frac{1}{1+x^n}dx \\
&= (a+b+c)I_n.
\end{aligned}$$

We immediately derive

$$\frac{a^2}{b}\left(\frac{b}{c}\right)^{1/n} + \frac{b^2}{c}\left(\frac{c}{a}\right)^{1/n} + \frac{c^2}{a}\left(\frac{a}{b}\right)^{1/n} \geq \frac{1}{I_n}\cdot I_n(a+b+c) = a+b+c.$$

The equality holds if $a = b = c$. This concludes the proof. $\square$

Setting $n = 2$ yields the

$$\frac{a^2}{\sqrt{bc}} + \frac{b^2}{\sqrt{ca}} + \frac{c^2}{\sqrt{ab}} \geq a+b+c.$$

Setting $n = 3$ yields

$$\frac{a^2}{b^{2/3}c^{1/3}} + \frac{b^2}{c^{2/3}a^{1/3}} + \frac{c^2}{a^{2/3}b^{1/3}} \geq a+b+c.$$

Remark 3.2. *We can also derive the following inequalities:*

$$\frac{a}{\sqrt{bc}}+\frac{b}{\sqrt{ca}}+\frac{c}{\sqrt{ab}}\geq\frac{(a+b+c)^2}{ab+bc+ca}\geq 3.$$

The proof of the first inequality follows the lines of the proof of Theorem 3.3 with the representation

$$\frac{a}{\sqrt{bc}}+\frac{b}{\sqrt{ca}}+\frac{c}{\sqrt{ab}}=\frac{1}{I_2}\int_0^{\infty}\left(\frac{a^2}{ab+acx^2}+\frac{b^2}{bc+bax^2}+\frac{c^2}{ca+cbx^2}\right)dx.$$

The second inequality concludes by using the known inequality $(a+b+c)^2\geq 3(ab+bc+ca)$.

To establish the lower bound of 3*, a more direct approach applies the standard inequality* $x^2+y^2\geq 2xy$ *for all* $x,y\in\mathbb{R}$*, and the Nesbitt inequality, as follows:*

$$\frac{a}{\sqrt{bc}}+\frac{b}{\sqrt{ca}}+\frac{c}{\sqrt{ab}}\geq 2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\geq 2\cdot\frac{3}{2}=3.$$

A new logarithmic cyclic inequality is established in the theorem below.

Theorem 3.4. *For any* $a,b,c>0$*, the following inequality holds:*

$$\frac{a^2}{b}\ln\left(1+\frac{b}{a}\right)+\frac{b^2}{c}\ln\left(1+\frac{c}{b}\right)+\frac{c^2}{a}\ln\left(1+\frac{a}{c}\right)\geq\ln(2)(a+b+c),$$

so that

$$\left(1+\frac{b}{a}\right)^{a^2/b}\left(1+\frac{c}{b}\right)^{b^2/c}\left(1+\frac{a}{c}\right)^{c^2/a}\geq 2^{a+b+c}.$$

The equality holds if $a=b=c$.

Proof. We construct a logarithmic integral development. For any $a,b>0$, we have

$$\int_0^1\frac{a^2}{a+bx}dx=\frac{a^2}{b}\left[\ln(a+bx)\right]_0^1=\frac{a^2}{b}\ln\left(1+\frac{b}{a}\right).$$

Using this development, we can express the main sum as follows:

$$\frac{a^2}{b}\ln\left(1+\frac{b}{a}\right)+\frac{b^2}{c}\ln\left(1+\frac{c}{b}\right)+\frac{c^2}{a}\ln\left(1+\frac{a}{c}\right)$$
$$=\int_0^1\left(\frac{a^2}{a+bx}+\frac{b^2}{b+cx}+\frac{c^2}{c+ax}\right)dx.$$

By the discrete Titu Lemma and standard integral developments, we obtain

$$\begin{aligned}
&\int_0^1 \left(\frac{a^2}{a+bx}+\frac{b^2}{b+cx}+\frac{c^2}{c+ax}\right)dx\\
&\geq \int_0^1 \frac{(a+b+c)^2}{(a+bx)+(b+cx)+(c+ax)}dx\\
&= \int_0^1 \frac{(a+b+c)^2}{(a+b+c)(1+x)}dx\\
&= (a+b+c)\int_0^1 \frac{1}{1+x}dx\\
&= (a+b+c)\left[\ln(1+x)\right]_0^1 = \ln(2)(a+b+c).
\end{aligned}$$

We immediately derive

$$\frac{a^2}{b}\ln\left(1+\frac{b}{a}\right)+\frac{b^2}{c}\ln\left(1+\frac{c}{b}\right)+\frac{c^2}{a}\ln\left(1+\frac{a}{c}\right)\geq \ln(2)(a+b+c).$$

Standard logarithmic developments give

$$\ln\left(\left(1+\frac{b}{a}\right)^{a^2/b}\left(1+\frac{c}{b}\right)^{b^2/c}\left(1+\frac{a}{c}\right)^{c^2/a}\right)\geq \ln(2^{a+b+c}),$$

so that

$$\left(1+\frac{b}{a}\right)^{a^2/b}\left(1+\frac{c}{b}\right)^{b^2/c}\left(1+\frac{a}{c}\right)^{c^2/a}\geq 2^{a+b+c}.$$

The equality is attained if $a=b=c$. This concludes the proof. $\square$

Another new logarithmic cyclic inequality is established in the theorem below.

Theorem 3.5. *For any $a,b,c>0$ such that $a\neq b\neq c$, the following inequality holds:*

$$\frac{a^2}{a-b}\ln\left(\frac{a}{b}\right)+\frac{b^2}{b-c}\ln\left(\frac{b}{c}\right)+\frac{c^2}{c-a}\ln\left(\frac{c}{a}\right)\geq a+b+c.$$

Proof. We construct a logarithmic integral development. For any $a,b>0$ with $a\neq b$, performing the substitution $u=a(1-x)+bx$, we get

$$\begin{aligned}
&\int_0^1 \frac{a^2}{a(1-x)+bx}dx=\int_a^b \frac{a^2}{u}\cdot\frac{1}{b-a}du=\frac{a^2}{b-a}\left[\ln(u)\right]_a^b=\frac{a^2}{b-a}\ln\left(\frac{b}{a}\right)\\
&=\frac{a^2}{a-b}\ln\left(\frac{a}{b}\right).
\end{aligned}$$

Using this development, we can express the main sum as follows:

$$\frac{a^2}{a-b}\ln\left(\frac{a}{b}\right)+\frac{b^2}{b-c}\ln\left(\frac{b}{c}\right)+\frac{c^2}{c-a}\ln\left(\frac{c}{a}\right)$$
$$=\int_0^1\left(\frac{a^2}{a(1-x)+bx}+\frac{b^2}{b(1-x)+cx}+\frac{c^2}{c(1-x)+ax}\right)dx.$$

By the discrete Titu Lemma and standard integral developments, we obtain

$$\int_0^1\left(\frac{a^2}{a(1-x)+bx}+\frac{b^2}{b(1-x)+cx}+\frac{c^2}{c(1-x)+ax}\right)dx$$
$$\geq\int_0^1\frac{(a+b+c)^2}{(a(1-x)+bx)+(b(1-x)+cx)+(c(1-x)+ax)}dx$$
$$=\int_0^1\frac{(a+b+c)^2}{(1-x)(a+b+c)+x(a+b+c)}dx$$
$$=\int_0^1\frac{(a+b+c)^2}{a+b+c}dx=\int_0^1(a+b+c)dx$$
$$=(a+b+c)\left[x\right]_0^1=a+b+c.$$

We immediately derive

$$\frac{a^2}{a-b}\ln\left(\frac{a}{b}\right)+\frac{b^2}{b-c}\ln\left(\frac{b}{c}\right)+\frac{c^2}{c-a}\ln\left(\frac{c}{a}\right)\geq a+b+c.$$

This concludes the proof. □

Another new logarithmic cyclic inequality is established in the theorem below.

Theorem 3.6. *For any $a,b,c>0$, the following inequality holds:*

$$\left(\frac{a^2}{b}-\frac{a^3}{b^2}\ln\left(1+\frac{b}{a}\right)\right)+\left(\frac{b^2}{c}-\frac{b^3}{c^2}\ln\left(1+\frac{c}{b}\right)\right)+\left(\frac{c^2}{a}-\frac{c^3}{a^2}\ln\left(1+\frac{a}{c}\right)\right)$$
$$\geq(1-\ln(2))(a+b+c).$$

The equality is attained if $a=b=c$.

Proof. We construct a logarithmic integral development. For any $a,b>0$, performing the substitution $u=\sqrt{x}$ and integral developments, we get

$$
\begin{aligned}
\int_0^1 \frac{a^2}{a+b\sqrt{x}}dx &= \int_0^1 \frac{a^2(2u)}{a+bu}du = \frac{2a^2}{b}\int_0^1 \frac{bu}{a+bu}du \\
&= \frac{2a^2}{b}\int_0^1 \frac{a+bu-a}{a+bu}du = \frac{2a^2}{b}\int_0^1 \left(1-\frac{a}{a+bu}\right)du \\
&= \frac{2a^2}{b}\left[u-\frac{a}{b}\ln(a+bu)\right]_0^1 \\
&= \frac{2a^2}{b}\left(1-\frac{a}{b}\ln(a+b)+\frac{a}{b}\ln(a)\right) \\
&= \frac{2a^2}{b}\left(1-\frac{a}{b}\ln\left(\frac{a+b}{a}\right)\right) \\
&= 2\left(\frac{a^2}{b}-\frac{a^3}{b^2}\ln\left(1+\frac{b}{a}\right)\right).
\end{aligned}
$$

Using this development, we can express the main sum as follows:

$$
\begin{aligned}
&\left(\frac{a^2}{b}-\frac{a^3}{b^2}\ln\left(1+\frac{b}{a}\right)\right)+\left(\frac{b^2}{c}-\frac{b^3}{c^2}\ln\left(1+\frac{c}{b}\right)\right)+\left(\frac{c^2}{a}-\frac{c^3}{a^2}\ln\left(1+\frac{a}{c}\right)\right) \\
&= \frac{1}{2}\int_0^1 \left(\frac{a^2}{a+b\sqrt{x}}+\frac{b^2}{b+c\sqrt{x}}+\frac{c^2}{c+a\sqrt{x}}\right)dx.
\end{aligned}
$$

By the discrete Titu Lemma and standard integral developments, we obtain

$$
\begin{aligned}
&\int_0^1 \left(\frac{a^2}{a+b\sqrt{x}}+\frac{b^2}{b+c\sqrt{x}}+\frac{c^2}{c+a\sqrt{x}}\right)dx \\
&\geq \int_0^1 \frac{(a+b+c)^2}{(a+b\sqrt{x})+(b+c\sqrt{x})+(c+a\sqrt{x})}dx \\
&= \int_0^1 \frac{(a+b+c)^2}{(a+b+c)+(a+b+c)\sqrt{x}}dx \\
&= \int_0^1 \frac{(a+b+c)^2}{(a+b+c)(1+\sqrt{x})}dx \\
&= (a+b+c)\int_0^1 \frac{1}{1+\sqrt{x}}dx.
\end{aligned}
$$

We evaluate this final integral using the same substitution $u=\sqrt{x}$, which yields

$$\int_0^1 \frac{1}{1+\sqrt{x}}dx = \int_0^1 \frac{2u}{1+u}du = 2\int_0^1 \left(1 - \frac{1}{1+u}\right) du$$
$$= 2\left[u - \ln(1+u)\right]_0^1 = 2(1-\ln(2)).$$

We immediately derive

$$\left(\frac{a^2}{b} - \frac{a^3}{b^2}\ln\left(1+\frac{b}{a}\right)\right) + \left(\frac{b^2}{c} - \frac{b^3}{c^2}\ln\left(1+\frac{c}{b}\right)\right) + \left(\frac{c^2}{a} - \frac{c^3}{a^2}\ln\left(1+\frac{a}{c}\right)\right)$$
$$\geq \frac{1}{2}\cdot 2(1-\ln(2))(a+b+c) = (1-\ln(2))(a+b+c).$$

The equality is attained if $a = b = c$. This concludes the proof. □

In the rest of the paper, we will investigate integral cyclic inequalities depending on three functions, f, g, and h.

Cyclic integral-ratio inequalities are presented in the theorem below.

Theorem 3.7. *Let* $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ *with* $b > a$, *and* $f, g, h : [a,b] \to (0,\infty)$ *be integrable functions. Then, the following cyclic inequalities hold:*

$$\int_a^b \frac{f(x)^2}{g(x)}dx + \int_a^b \frac{g(x)^2}{h(x)}dx + \int_a^b \frac{h(x)^2}{f(x)}dx$$
$$\geq \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx} + \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b h(x)dx} + \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b f(x)dx}$$
$$\geq \int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx.$$

The equality holds if $f(x) = g(x) = h(x)$ *almost everywhere on* $[a,b]$.

Proof. We apply the integral Titu lemma to each of the three terms on the left-hand side individually, so that

$$\int_a^b \frac{f(x)^2}{g(x)}dx \geq \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx}, \quad \int_a^b \frac{g(x)^2}{h(x)}dx \geq \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b h(x)dx}$$

and

$$\int_a^b \frac{h(x)^2}{f(x)}dx \geq \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b f(x)dx}.$$

Summing these three inequalities yields

$$\int_a^b \frac{f(x)^2}{g(x)}dx + \int_a^b \frac{g(x)^2}{h(x)}dx + \int_a^b \frac{h(x)^2}{f(x)}dx$$
$$\geq \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx} + \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b h(x)dx} + \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b f(x)dx}.$$

The first inequality is proved.

Now, we define

$$A = \int_a^b f(x)dx, \quad B = \int_a^b g(x)dx, \quad C = \int_a^b h(x)dx.$$

Because f, g, h are strictly positive on $[a, b]$, we are guaranteed that $A, B, C > 0$.

Substituting these variables into the right-hand side, we get

$$\frac{A^2}{B} + \frac{B^2}{C} + \frac{C^2}{A}.$$

By the discrete Titu Lemma, we obtain

$$\frac{A^2}{B} + \frac{B^2}{C} + \frac{C^2}{A} \geq \frac{(A+B+C)^2}{B+C+A}.$$

Since the numerator is squared, we can simplify this expression directly, as follows:

$$\frac{(A+B+C)^2}{A+B+C} = A+B+C = \int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx.$$

Therefore, we have

$$\frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx} + \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b h(x)dx} + \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b f(x)dx}$$
$$\geq \int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx.$$

The second inequality is proved.

Combining the inequalities above, we immediately derive

$$\int_a^b \frac{f(x)^2}{g(x)}dx + \int_a^b \frac{g(x)^2}{h(x)}dx + \int_a^b \frac{h(x)^2}{f(x)}dx$$

$$\geq \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx} + \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b h(x)dx} + \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b f(x)dx}$$
$$\geq \int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx.$$

The equality holds if $f(x) = g(x) = h(x)$ almost everywhere on $[a, b]$. This concludes the proof. □

In a sense, these cyclic inequalities bridge the discrete and integral forms of the Titu lemma.

Remark 3.3. *By applying the discrete Titu lemma to the functions f, g, and h, we directly obtain the lower bound of Theorem 3.7, as follows:*

$$\int_a^b \frac{f(x)^2}{g(x)}dx + \int_a^b \frac{g(x)^2}{h(x)}dx + \int_a^b \frac{h(x)^2}{f(x)}dx = \int_a^b \left(\frac{f(x)^2}{g(x)} + \frac{g(x)^2}{h(x)} + \frac{h(x)^2}{f(x)}\right) dx$$
$$\geq \int_a^b \left(\frac{(f(x)+g(x)+h(x))^2}{g(x)+h(x)+f(x)}\right) dx = \int_a^b (f(x)+g(x)+h(x))\, dx$$
$$= \int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx.$$

The main significance of Theorem 3.7 lies in the order of the resulting chain of three inequalities.

The theorem below presents further cyclic integral-ratio inequalities depending on one parameter.

Theorem 3.8. *Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $b > a$, $f, g, h : [a, b] \to (0, \infty)$ be integrable functions, and $c \geq 0$. Then, the following cyclic inequalities hold:*

$$\int_a^b \frac{f(x)^2}{f(x)+cg(x)}dx + \int_a^b \frac{g(x)^2}{g(x)+ch(x)}dx + \int_a^b \frac{h(x)^2}{h(x)+cf(x)}dx$$
$$\geq \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b f(x)dx + c\int_a^b g(x)dx} + \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b g(x)dx + c\int_a^b h(x)dx} + \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b h(x)dx + c\int_a^b f(x)dx}$$
$$\geq \frac{1}{c+1}\left(\int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx\right).$$

The equality holds if $f(x) = g(x) = h(x)$ *almost everywhere on* $[a, b]$*, or if* $c = 0$*.*

Proof. We apply the integral Titu lemma to each of the three terms on the left-hand side individually, so that

$$\int_a^b \frac{f(x)^2}{f(x)+cg(x)}dx \geq \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b \left(f(x)+cg(x)\right)dx} = \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b f(x)dx + c\int_a^b g(x)dx},$$

$$\int_a^b \frac{g(x)^2}{g(x)+ch(x)}dx \geq \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b \left(g(x)+ch(x)\right)dx} = \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b g(x)dx + c\int_a^b h(x)dx}$$

and

$$\int_a^b \frac{h(x)^2}{h(x)+cf(x)}dx \geq \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b \left(h(x)+cf(x)\right)dx} = \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b h(x)dx + c\int_a^b f(x)dx}.$$

Summing these three inequalities gives

$$\int_a^b \frac{f(x)^2}{f(x)+cg(x)}dx + \int_a^b \frac{g(x)^2}{g(x)+ch(x)}dx + \int_a^b \frac{h(x)^2}{h(x)+cf(x)}dx$$
$$\geq \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b f(x)dx + c\int_a^b g(x)dx} + \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b g(x)dx + c\int_a^b h(x)dx} + \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b h(x)dx + c\int_a^b f(x)dx}.$$

The first inequality is proved.

Now, we define

$$A = \int_a^b f(x)dx, \quad B = \int_a^b g(x)dx, \quad C = \int_a^b h(x)dx.$$

Because f, g, h are strictly positive on $[a, b]$, we are guaranteed that $A, B, C > 0$.

Substituting these variables into the right-hand side, we get

$$\frac{A^2}{A+cB} + \frac{B^2}{B+cC} + \frac{C^2}{C+cA}.$$

The discrete Titu Lemma gives

$$\frac{A^2}{A+cB} + \frac{B^2}{B+cC} + \frac{C^2}{C+cA} \geq \frac{(A+B+C)^2}{(A+cB)+(B+cC)+(C+cA)}$$

$$
\begin{aligned}
&= \frac{(A+B+C)^2}{(c+1)(A+B+C)} = \frac{1}{c+1}(A+B+C) \\
&= \frac{1}{c+1}\left(\int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx\right).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b f(x)dx + c\int_a^b g(x)dx} + \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b g(x)dx + c\int_a^b h(x)dx} + \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b h(x)dx + c\int_a^b f(x)dx} \\
&\geq \frac{1}{c+1}\left(\int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx\right).
\end{aligned}
$$

The second inequality is proved.

Combining the inequalities above, we immediately derive

$$
\begin{aligned}
&\int_a^b \frac{f(x)^2}{f(x)+cg(x)}dx + \int_a^b \frac{g(x)^2}{g(x)+ch(x)}dx + \int_a^b \frac{h(x)^2}{h(x)+cf(x)}dx \\
&\geq \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b f(x)dx + c\int_a^b g(x)dx} + \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b g(x)dx + c\int_a^b h(x)dx} + \frac{\left(\int_a^b h(x)dx\right)^2}{\int_a^b h(x)dx + c\int_a^b f(x)dx} \\
&\geq \frac{1}{c+1}\left(\int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx\right).
\end{aligned}
$$

The equality holds if $f(x) = g(x) = h(x)$ almost everywhere on $[a, b]$. This concludes the proof. □

4. A REMARK

A unifying feature of some inequalities presented in the previous section is that they can be expressed in the form

$$
af\left(\frac{b}{a}\right) + bf\left(\frac{c}{b}\right) + cf\left(\frac{a}{c}\right) \geq (a+b+c)f(1),
$$

for a suitable function $f : (0, \infty) \to \mathbb{R}$. This pattern inspires the theorem below, where convexity plays a central role.

Theorem 4.1. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function. Then, for any $a,b,c>0$, the following inequality holds:*

$$af\left(\frac{b}{a}\right)+bf\left(\frac{c}{b}\right)+cf\left(\frac{a}{c}\right)\geq(a+b+c)f(1).$$

The equality holds if $a=b=c$.

Proof. Modifying the weights and applying the Jensen inequality to the convex function f, we obtain

$$\begin{aligned}
&af\left(\frac{b}{a}\right)+bf\left(\frac{c}{b}\right)+cf\left(\frac{a}{c}\right)\\
&=(a+b+c)\left(\frac{a}{a+b+c}f\left(\frac{b}{a}\right)+\frac{b}{a+b+c}f\left(\frac{c}{b}\right)+\frac{c}{a+b+c}f\left(\frac{a}{c}\right)\right)\\
&\geq(a+b+c)f\left(\frac{a(b/a)}{a+b+c}+\frac{b(c/b)}{a+b+c}+\frac{c(a/c)}{a+b+c}\right)\\
&=(a+b+c)f\left(\frac{a+b+c}{a+b+c}\right)=(a+b+c)f(1).
\end{aligned}$$

The equality holds if $a=b=c$. This concludes the proof. □

If f is concave instead of convex, then the inequality is reversed.

For example, consider $f(x)=1/\sqrt{x}$, which is convex because the second derivative satisfies $f''(x)=(3/4)x^{-5/2}\geq 0$. Then Theorem 4.1 gives

$$a\sqrt{\frac{a}{b}}+b\sqrt{\frac{b}{c}}+c\sqrt{\frac{c}{a}}\geq a+b+c.$$

This also corresponds to Theorem 3.2.

However, the possible relationship between the integral-Titu lemma framework and the convexity assumption on f remains an open methodological question. Furthermore, verifying the convexity of f can be technically challenging in complex settings. Consequently, the two approaches should be viewed as complementary.

5. Conclusion

In this paper, we have advanced the study of cyclic inequalities by establishing a unified framework that combines integral representations with both discrete and continuous formulations of the Titu lemma. Through this approach, we derived several new families of cyclic inequalities across diverse function classes, notably including logarithmic and arctangent functions. These results highlight the versatility and power of this underutilized methodology, offering a robust way for tackling complex analytical bounds and opening promising directions for future research in functional inequalities.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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