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ON FIVE GENERAL THEOREMS FOR CYCLIC INEQUALITIES

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ABSTRACT. This paper establishes five general theorems for cyclic inequalities involving one or two functions, offering a unified framework that extends the Nesbitt inequality. By integrating convexity and monotonicity with the Chebyshev sum inequality, the Jensen inequality, the Nesbitt inequality itself, and the Titu lemma, this approach both recovers classical results and yields new inequalities under flexible hypotheses.

1. INTRODUCTION

In classical algebra, an inequality involving n variables with $n \in \mathbb{N}\backslash\{0,1,2\}$, say $a_1, a_2, \ldots, a_n$, is called cyclic if its form remains unchanged when the variables are shifted cyclically, that is, replacing $a_1 \to a_2$, $a_2 \to a_3$, $\ldots$, and $a_n \to a_1$. Unlike symmetric inequalities, which remain invariant under any permutation of variables, cyclic inequalities only require invariance under cyclic shifts. Cyclic sum notation compactly represents expressions generated by cyclic permutations of variables. For a function $f : \mathbb{R}^3 \to \mathbb{R}$, it is defined as follows:

$$\sum_{cyc} f(a,b,c) = f(a,b,c) + f(b,c,a) + f(c,a,b).$$

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Cyclic inequalities frequently appear in mathematical olympiads and research because they capture balance across variables without requiring full symmetry. Formulated by A. M. Nesbitt in 1903, the Nesbitt Inequality is one of the most famous three-variable cyclic inequalities. For any $a, b, c > 0$, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

The equality holds for $a = b = c$. In cyclic sum notation, the inequality is compactly written as follows:

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{3}{2}.$$

It can be demonstrated using diverse classical strategies, including the Cauchy-Schwarz inequality (or Titu lemma), the arithmetic mean-geometric mean inequality, the rearrangement inequality, and variable substitution. The result also forms the three-variable base case of the Shapiro inequality, a broader cyclic sum proposition known for failing in certain higher dimensions. Old and modern developments on cyclic inequalities can be found in [1–7].

In this paper, we propose five new general theorems for cyclic inequalities involving one or two functions. A common feature of these theorems is that they generalize the Nesbitt inequality in various directions, often by combining assumptions of convexity and monotonicity. This unified framework allows us to re-derive several existing inequalities and establish new results under flexible conditions. The key tools in our proofs are the Chebyshev sum inequality, the Jensen inequality, the Nesbitt inequality itself, and the Titu lemma.

The remainder of this paper is organized as follows: Section 2 presents our main theorems along with their proofs and some examples of application. Finally, Section 3 concludes the paper.

2. Five general theorems

2.1. **First theorem.** We begin by stating our first main theorem. We particularly emphasize the specific roles and conditions imposed on the functions f and g.

Theorem 2.1. *Let $f : (0, \infty) \to (0, \infty)$ be an increasing function and $g : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality* *holds:*

$$f(a)g\left(\frac{a}{b+c}\right)+f(b)g\left(\frac{b}{c+a}\right)+f(c)g\left(\frac{c}{a+b}\right)$$
$$\geq (f(a)+f(b)+f(c))\,g\left(\frac{1}{2}\right).$$

The equality holds for $a=b=c$.

Proof. The first step of the proof is to establish the conditions required to apply the Chebyshev sum inequality. To this end, without loss of generality, we may assume that

$$a \geq b \geq c.$$

This assumption is justified because the target inequality is completely symmetric with respect to a, b, and c.

We consider the two following sequences:

$$X=(x_1,x_2,x_3)=(f(a),f(b),f(c))$$

and

$$Y=(y_1,y_2,y_3)=\left(g\left(\frac{a}{b+c}\right),g\left(\frac{b}{c+a}\right),g\left(\frac{c}{a+b}\right)\right).$$

Since $a \geq b \geq c$ and f is an increasing function, we have $f(a) \geq f(b) \geq f(c)$, which implies that $x_1 \geq x_2 \geq x_3$.

Since $a \geq b \geq c > 0$, we have $b+c \leq c+a \leq a+b$, so that

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$$

and

$$\frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b}.$$

Since g is increasing, we have

$$g\left(\frac{a}{b+c}\right) \geq g\left(\frac{b}{c+a}\right) \geq g\left(\frac{c}{a+b}\right),$$

which implies that $y_1 \geq y_2 \geq y_3$. Thus, both sequences X and Y are similarly ordered; we have $x_1 \geq x_2 \geq x_3$ and $y_1 \geq y_2 \geq y_3$.

Because of this, we can apply the Chebyshev sum inequality to X and Y, which yields

$$f(a)g\left(\frac{a}{b+c}\right)+f(b)g\left(\frac{b}{c+a}\right)+f(c)g\left(\frac{c}{a+b}\right)=\sum_{cyc}f(a)g\left(\frac{a}{b+c}\right)$$
$$=\sum_{i=1}^{3}x_iy_i\geq\frac{1}{3}\left(\sum_{i=1}^{3}x_i\right)\left(\sum_{i=1}^{3}y_i\right)=\frac{1}{3}\left(\sum_{cyc}f(a)\right)\left(\sum_{cyc}g\left(\frac{a}{b+c}\right)\right)$$
$$=\left(\sum_{cyc}f(a)\right)\left(\sum_{cyc}\frac{1}{3}g\left(\frac{a}{b+c}\right)\right).$$

Next, since g is a convex and increasing function, applying the Jensen inequality followed by the Nesbitt inequality yields

$$\sum_{cyc}\frac{1}{3}g\left(\frac{a}{b+c}\right)\geq g\left(\frac{\sum_{cyc}\frac{a}{b+c}}{3}\right)\geq g\left(\frac{\frac{3}{2}}{3}\right)=g\left(\frac{1}{2}\right).$$

Combining the inequalities above, noting that $\sum_{cyc}f(a)\geq 0$, we obtain

$$f(a)g\left(\frac{a}{b+c}\right)+f(b)g\left(\frac{b}{c+a}\right)+f(c)g\left(\frac{c}{a+b}\right)$$
$$\geq(f(a)+f(b)+f(c))\,g\left(\frac{1}{2}\right).$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $3f(a)g(1/2)$. This concludes the proof. □

Some examples of this theorem are presented below.

- Setting $f(x)=1$ and keeping $g:(0,\infty)\to\mathbb{R}$ as a convex and increasing function, Theorem 2.1 yields

$$g\left(\frac{a}{b+c}\right)+g\left(\frac{b}{c+a}\right)+g\left(\frac{c}{a+b}\right)\geq 3g\left(\frac{1}{2}\right).$$

- Setting $f(x)=1$ and $g(x)=x$, Theorem 2.1 gives

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2},$$

  which corresponds to the Nesbitt inequality.
- Setting $f(x)=x$ and $g(x)=x^2$, Theorem 2.1 yields

$$\frac{a^3}{(b+c)^2}+\frac{b^3}{(c+a)^2}+\frac{c^3}{(a+b)^2}\geq\frac{a+b+c}{4}.$$

- Setting $f(x) = x^2$ and $g(x) = x^2$, Theorem 2.1 gives

$$\frac{a^4}{(b+c)^2} + \frac{b^4}{(c+a)^2} + \frac{c^4}{(a+b)^2} \geq \frac{a^2+b^2+c^2}{4}.$$

- Setting $f(x) = x$ and $g(x) = e^x$, Theorem 2.1 gives

$$ae^{a/(b+c)} + be^{b/(c+a)} + ce^{c/(a+b)} \geq (a+b+c)\sqrt{e}.$$

- Setting $f(x) = e^x$ and $g(x) = x\ln(1+x)$, Theorem 2.1 yields

$$e^a \frac{a}{b+c} \ln\left(1 + \frac{a}{b+c}\right) + e^b \frac{b}{c+a} \ln\left(1 + \frac{b}{c+a}\right) + e^c \frac{c}{a+b} \ln\left(1 + \frac{c}{a+b}\right) \geq \frac{e^a+e^b+e^c}{2} \ln\left(\frac{3}{2}\right),$$

so that

$$\left(1 + \frac{a}{b+c}\right)^{e^a a/(b+c)} \left(1 + \frac{b}{c+a}\right)^{e^b b/(c+a)} \left(1 + \frac{c}{a+b}\right)^{e^c c/(a+b)} \geq \left(\frac{3}{2}\right)^{(1/2)(e^a+e^b+e^c)}.$$

- Setting $f(x) = x^3$ and $g(x) = \sqrt{x^2+1}$, Theorem 2.1 gives

$$a^3\sqrt{\frac{a^2}{(b+c)^2}+1} + b^3\sqrt{\frac{b^2}{(c+a)^2}+1} + c^3\sqrt{\frac{c^2}{(a+b)^2}+1} \geq \frac{a^3+b^3+c^3}{2}\sqrt{5}.$$

Many other examples of this type can be constructed in a similar manner.

Remark 2.1. *We now discuss a more general version of Theorem 2.1. Let* $\alpha, \beta > 0$*. The Titu lemma yields*

$$\sum_{cyc} \frac{a}{\alpha b + \beta c} = \sum_{cyc} \frac{a^2}{\alpha ab + \beta ac} \geq \frac{(a+b+c)^2}{\sum_{cyc}(\alpha ab + \beta ac)} = \frac{(a+b+c)^2}{(\alpha+\beta)(ab+bc+ca)}.$$

Using the known inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$*, we obtain*

$$\sum_{cyc} \frac{a}{\alpha b + \beta c} \geq \frac{3(ab+bc+ca)}{(\alpha+\beta)(ab+bc+ca)} = \frac{3}{\alpha+\beta}. \tag{2.1}$$

Using this instead of the Nesbitt inequality in the proof of Theorem 2.1, and assuming that $a \geq b \geq c$, we arrive at the following cyclic inequality:

$$\begin{aligned} & f(a)g\left(\frac{a}{\alpha b+\beta c}\right)+f(b)g\left(\frac{b}{\alpha c+\beta a}\right)+f(c)g\left(\frac{c}{\alpha a+\beta b}\right) \\ & \geq (f(a)+f(b)+f(c))\, g\left(\frac{1}{\alpha+\beta}\right). \end{aligned}$$

Note that assuming $a \geq b \geq c$ without loss of generality is invalid here, as the inequality is not fully symmetric with respect to a, b, and c. Under this assumption, however, the resulting inequality can be viewed as a two-parameter generalization of Theorem 2.1.

2.2. **Second theorem.** Our second main result is stated below, where the respective roles of the functions f and g are again highlighted.

Theorem 2.2. *Let $f:(0,\infty)\to(0,\infty)$ be an increasing function and $g:(0,\infty)\to\mathbb{R}$ be a convex and decreasing function. For any $a,b,c>0$, the following cyclic inequality holds:*

$$\begin{aligned} & f(a)g(b+c)+f(b)g(c+a)+f(c)g(a+b) \\ & \geq (f(a)+f(b)+f(c))\, g\left(\frac{2(a+b+c)}{3}\right). \end{aligned}$$

The equality holds for $a=b=c$.

Proof. The first step of the proof is to establish the conditions required to apply the Chebyshev sum inequality. To this end, since the target inequality is symmetric in a, b, and c, we may assume without loss of generality that

$$a \geq b \geq c.$$

We consider the two following sequences:

$$X=(x_1,x_2,x_3)=(f(a),f(b),f(c))$$

and

$$Y=(y_1,y_2,y_3)=(g(b+c),g(c+a),g(a+b)).$$

Since $a \geq b \geq c$ and f is an increasing function, we have $f(a) \geq f(b) \geq f(c)$, which implies that $x_1 \geq x_2 \geq x_3$.

Since $a \geq b \geq c$, we have $b+c \leq c+a \leq a+b$. Since g is a decreasing function, we have $g(b+c) \geq g(c+a) \geq g(a+b)$, which implies that $y_1 \geq y_2 \geq y_3$. Thus, both sequences X and Y are similarly ordered; we have $x_1 \geq x_2 \geq x_3$ and $y_1 \geq y_2 \geq y_3$.

Because of this, we can apply the Chebyshev sum inequality to X and Y, which yields

$$f(a)g(b+c) + f(b)g(c+a) + f(c)g(a+b) = \sum_{cyc} f(a)g(b+c)$$
$$= \sum_{i=1}^{3} x_i y_i \geq \frac{1}{3}\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=1}^{3} y_i\right) = \frac{1}{3}\left(\sum_{cyc} f(a)\right)\left(\sum_{cyc} g(b+c)\right)$$
$$= \left(\sum_{cyc} f(a)\right)\left(\sum_{cyc} \frac{1}{3} g(b+c)\right).$$

Next, since g is convex, applying the Jensen inequality yields

$$\sum_{cyc} \frac{1}{3} g(b+c) \geq g\left(\frac{\sum_{cyc}(b+c)}{3}\right) \geq g\left(\frac{2(a+b+c)}{3}\right).$$

Combining the inequalities above, noting that $\sum_{cyc} f(a) \geq 0$, we obtain

$$f(a)g(b+c) + f(b)g(c+a) + f(c)g(a+b)$$
$$\geq (f(a) + f(b) + f(c))\, g\left(\frac{2(a+b+c)}{3}\right).$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $3f(a)g(2a)$. This ends the proof. $\square$

Some examples of this theorem are presented below.

- Setting $f(x) = x$ and keeping $g : (0, \infty) \to \mathbb{R}$ as a convex and decreasing function, Theorem 2.2 yields

$$ag(b+c) + bg(c+a) + cg(a+b) \geq (a+b+c)g\left(\frac{2(a+b+c)}{3}\right).$$

- Setting $f(x) = 1$ and $g(x) = 1/x$, Theorem 2.2 gives

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

  which corresponds to the Nesbitt inequality.

- Setting $f(x) = x$ and $g(x) = 1/x^2$, Theorem 2.2 yields

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}.$$

- Setting $f(x) = x^2$ and $g(x) = 1/x^2$, Theorem 2.2 gives

$$\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2} \geq \frac{9(a^2+b^2+c^2)}{4(a+b+c)^2}.$$

  Using this and the known inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we get

$$\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2} \geq \frac{3}{4}.$$

- Setting $f(x) = e^x$ and $g(x) = e^{-x}$, Theorem 2.2 yields

$$e^{a-(b+c)} + e^{b-(c+a)} + e^{c-(a+b)} \geq (e^a + e^b + e^c)e^{-(2/3)(a+b+c)}.$$

- Setting $f(x) = x^3$ and $g(x) = -\ln(x)$, Theorem 2.2 gives

$$\begin{aligned} & -a^3\ln(b+c) - b^3\ln(c+a) - c^3\ln(a+b) \\ & \geq -(a^3+b^3+c^3)\ln\left(\frac{2(a+b+c)}{3}\right), \end{aligned}$$

  so that

$$(b+c)^{a^3}(c+a)^{b^3}(a+b)^{c^3} \leq \left(\frac{2(a+b+c)}{3}\right)^{a^3+b^3+c^3}.$$

Many other examples of this type can be constructed in a similar manner.

Remark 2.2. *We now discuss a more general version of Theorem 2.2. Let $\alpha, \beta > 0$ such that $0 \leq \beta < 2\alpha$. Following the lines of the proof of Theorem 2.2, one can prove the following cyclic inequality:*

$$\begin{aligned} & f(a)g(\alpha(b+c) - \beta a) + f(b)g(\alpha(c+a) - \beta b) + f(c)g(\alpha(a+b) - \beta c) \\ & \geq (f(a)+f(b)+f(c))\, g\left(\frac{(2\alpha-\beta)(a+b+c)}{3}\right), \end{aligned}$$

(since the target inequality is symmetric with respect to $a, b,$ and c, we may assume without loss of generality that $a \geq b \geq c$). The resulting inequality can be viewed as a two-parameter generalization of Theorem 2.2.

2.3. **Third theorem.** The third main result is stated below, explicitly detailing the interplay between the functions f and g.

Theorem 2.3. *Let* $f : (0,\infty) \to (0,\infty)$ *be a convex and increasing function and* $g : (0,\infty) \to (0,\infty)$ *be a convex and decreasing function. For any* $a, b, c > 0$*, the following cyclic inequality holds:*

$$f\left(\frac{a}{b+c}\right) g(b+c) + f\left(\frac{b}{c+a}\right) g(c+a) + f\left(\frac{c}{a+b}\right) g(a+b)$$
$$\geq 3f\left(\frac{1}{2}\right) g\left(\frac{2(a+b+c)}{3}\right).$$

The equality holds for $a = b = c$.

Proof. The first step of the proof is to establish the conditions required to apply the Chebyshev sum inequality. To this end, since the target inequality is symmetric in a, b, and c, we may assume without loss of generality that

$$a \geq b \geq c.$$

We consider the two following sequences:

$$X = (x_1, x_2, x_3) = \left(f\left(\frac{a}{b+c}\right), f\left(\frac{b}{c+a}\right), f\left(\frac{c}{a+b}\right)\right)$$

and

$$Y = (y_1, y_2, y_3) = (g(b+c), g(c+a), g(a+b)).$$

Since $a \geq b \geq c > 0$, we have $b+c \leq c+a \leq a+b$, so that

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$$

and

$$\frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b}.$$

Since f is increasing, we have

$$f\left(\frac{a}{b+c}\right) \geq f\left(\frac{b}{c+a}\right) \geq f\left(\frac{c}{a+b}\right),$$

which implies that $x_1 \geq x_2 \geq x_3$.

Since $a \geq b \geq c$, we have $b+c \leq c+a \leq a+b$. Since g is a decreasing function, we have $g(b+c) \geq g(c+a) \geq g(a+b)$, which implies that $y_1 \geq y_2 \geq y_3$. Thus, both sequences X and Y are similarly ordered; we have $x_1 \geq x_2 \geq x_3$ and $y_1 \geq y_2 \geq y_3$.

Because of this, we can apply the Chebyshev sum inequality to X and Y, which yields

$$\begin{aligned}
& f\left(\frac{a}{b+c}\right) g(b+c)+f\left(\frac{b}{c+a}\right) g(c+a)+f\left(\frac{c}{a+b}\right) g(a+b) \\
& =\sum_{cyc} f\left(\frac{a}{b+c}\right) g(b+c)=\sum_{i=1}^{3} x_i y_i \geq \frac{1}{3}\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=1}^{3} y_i\right) \\
& =\frac{1}{3}\left(\sum_{cyc} f\left(\frac{a}{b+c}\right)\right)\left(\sum_{cyc} g(b+c)\right) \\
& =3\left(\sum_{cyc} \frac{1}{3} f\left(\frac{a}{b+c}\right)\right)\left(\sum_{cyc} \frac{1}{3} g(b+c)\right).
\end{aligned}$$

Next, since f is a convex and increasing function, applying the Jensen inequality followed by the Nesbitt inequality yields

$$\sum_{cyc} \frac{1}{3} f\left(\frac{a}{b+c}\right) \geq f\left(\frac{\sum_{cyc} \frac{a}{b+c}}{3}\right) \geq f\left(\frac{\frac{3}{2}}{3}\right)=f\left(\frac{1}{2}\right).$$

Moreover, since g is convex, applying the Jensen inequality yields

$$\sum_{cyc} \frac{1}{3} g(b+c) \geq g\left(\frac{\sum_{cyc}(b+c)}{3}\right) \geq g\left(\frac{2(a+b+c)}{3}\right).$$

Combining the inequalities above, noting that the terms are positive, we obtain

$$\begin{aligned}
& f\left(\frac{a}{b+c}\right) g(b+c)+f\left(\frac{b}{c+a}\right) g(c+a)+f\left(\frac{c}{a+b}\right) g(a+b) \\
& \geq 3 f\left(\frac{1}{2}\right) g\left(\frac{2(a+b+c)}{3}\right).
\end{aligned}$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $3f(1/2)g(2a)$. This concludes the proof. □

Some examples of this theorem are presented below.

- Setting $f(x)=x$ and keeping $g:(0,\infty)\to(0,\infty)$ as a convex and decreasing function, Theorem 2.3 yields

$$\frac{a}{b+c} g(b+c)+\frac{b}{c+a} g(c+a)+\frac{c}{a+b} g(a+b) \geq \frac{3}{2} g\left(\frac{2(a+b+c)}{3}\right).$$

- Setting $f(x) = x$ and $g(x) = 1$, Theorem 2.3 gives

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

which corresponds to the Nesbitt inequality.

- Setting $f(x) = x$ and $g(x) = 1/x^2$, Theorem 2.3 yields

$$\frac{a}{(b+c)^3} + \frac{b}{(c+a)^3} + \frac{c}{(a+b)^3} \geq \frac{27}{8(a+b+c)^2}.$$

- Setting $f(x) = e^x$ and $g(x) = e^{-x}$, Theorem 2.3 yields

$$e^{a/(b+c)-(b+c)} + e^{b/(c+a)-(c+a)} + e^{c/(a+b)-(a+b)} \geq 3e^{1/2-(2/3)(a+b+c)}.$$

- Setting $f(x) = x\ln(1+x)$ and $g(x) = 1/x$, Theorem 2.3 gives

$$\frac{a}{(b+c)^2}\ln\left(1+\frac{a}{b+c}\right) + \frac{b}{(c+a)^2}\ln\left(1+\frac{b}{c+a}\right) + \frac{c}{(a+b)^2}\ln\left(1+\frac{c}{a+b}\right) \geq \ln\left(\frac{3}{2}\right)\frac{9}{4(a+b+c)},$$

so that

$$\left(1+\frac{a}{b+c}\right)^{a/(b+c)^2}\left(1+\frac{b}{c+a}\right)^{b/(c+a)^2}\left(1+\frac{c}{a+b}\right)^{c/(a+b)^2} \geq \left(\frac{3}{2}\right)^{(9/4)(1/(a+b+c))}.$$

Many other examples of this type can be constructed in a similar manner.

Remark 2.3. *We now discuss a more general version of Theorem 2.3. Let $\alpha, \beta > 0$. Assuming that $a \geq b \geq c$, and using the inequality* (2.1) *instead of the Nesbitt inequality in the proof of Theorem 2.3, we arrive at the following cyclic inequality:*

$$f\left(\frac{a}{\alpha b+\beta c}\right)g(b+c) + f\left(\frac{b}{\alpha c+\beta a}\right)g(c+a) + f\left(\frac{c}{\alpha a+\beta b}\right)g(a+b) \geq 3f\left(\frac{1}{\alpha+\beta}\right)g\left(\frac{2(a+b+c)}{3}\right).$$

Under the assumption $a \geq b \geq c$, the resulting inequality can be viewed as a two-parameter generalization of Theorem 2.3.

2.4. **Fourth theorem.** The fourth main result, presented below, does not invoke convexity in its statement; it requires only that f and g be increasing.

Theorem 2.4. *Let $f, g : (0,\infty) \to (0,\infty)$ be two increasing functions. For any $a, b, c > 0$, the following cyclic inequality holds:*

$$\frac{f(a)}{g(b)+g(c)} + \frac{f(b)}{g(c)+g(a)} + \frac{f(c)}{g(a)+g(b)} \geq \frac{3\,(f(a)+f(b)+f(c))}{2\,(g(a)+g(b)+g(c))}.$$

The equality holds for $a = b = c$, or if f and g are constants.

Proof. The first step of the proof is to establish the conditions required to apply the Chebyshev sum inequality. To this end, since the target inequality is symmetric in a, b, and c, we may assume without loss of generality that

$$a \geq b \geq c.$$

We consider the two following sequences:

$$X = (x_1, x_2, x_3) = (f(a), f(b), f(c))$$

and

$$Y = (y_1, y_2, y_3) = \left(\frac{1}{g(b)+g(c)}, \frac{1}{g(c)+g(a)}, \frac{1}{g(a)+g(b)}\right).$$

Since $a \geq b \geq c$ and f is an increasing function, we have $f(a) \geq f(b) \geq f(c)$, which implies that $x_1 \geq x_2 \geq x_3$.

Since $a \geq b \geq c$ and g is an increasing function, we have $g(a) \geq g(b) \geq g(c)$, which implies that $g(b)+g(c) \leq g(c)+g(a) \leq g(a)+g(b)$, so that

$$\frac{1}{g(b)+g(c)} \geq \frac{1}{g(c)+g(a)} \geq \frac{1}{g(a)+g(b)},$$

which implies that $y_1 \geq y_2 \geq y_3$. Thus, both sequences X and Y are similarly ordered; we have $x_1 \geq x_2 \geq x_3$ and $y_1 \geq y_2 \geq y_3$.

Because of this, we can apply the Chebyshev sum inequality to X and Y, which yields

$$\begin{aligned}
&\frac{f(a)}{g(b)+g(c)} + \frac{f(b)}{g(c)+g(a)} + \frac{f(c)}{g(a)+g(b)} \\
&= \sum_{cyc} \frac{f(a)}{g(b)+g(c)} = \sum_{i=1}^{3} x_i y_i \geq \frac{1}{3}\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=1}^{3} y_i\right) \\
&= \frac{1}{3}\left(\sum_{cyc} f(a)\right)\left(\sum_{cyc} \frac{1}{g(b)+g(c)}\right).
\end{aligned}$$

Next, the Titu lemma yields

$$\sum_{cyc} \frac{1}{g(b)+g(c)} \geq \frac{(1+1+1)^2}{\sum_{cyc}(g(b)+g(c))} = \frac{9}{2(g(a)+g(b)+g(c))}.$$

Combining the inequalities above, noting that the terms are positive, we obtain

$$\frac{f(a)}{g(b)+g(c)} + \frac{f(b)}{g(c)+g(a)} + \frac{f(c)}{g(a)+g(b)} \geq \frac{3(f(a)+f(b)+f(c))}{2(g(a)+g(b)+g(c))}.$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $(3/2)(f(a)/g(a))$, or if f and g are constants, with both sides equal to $3/2$. This completes the proof. $\square$

Notably, as in the Nesbitt inequality, the constant $3/2$ arises naturally in the lower bound despite the involvement of two distinct functions.

Some examples of this theorem are presented below.

- Setting $f(x)=x$ and keeping $g:(0,\infty)\to(0,\infty)$ as an increasing function, Theorem 2.4 yields

$$\frac{a}{g(b)+g(c)} + \frac{b}{g(c)+g(a)} + \frac{c}{g(a)+g(b)} \geq \frac{3(a+b+c)}{2(g(a)+g(b)+g(c))}.$$

- Setting $f(x)=x$ and $g(x)=x$, Theorem 2.4 gives

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

which corresponds to the Nesbitt inequality.

- Setting $f(x)=x$ and $g(x)=x^2$, Theorem 2.4 yields

$$\frac{a}{b^2+c^2} + \frac{b}{c^2+a^2} + \frac{c}{a^2+b^2} \geq \frac{3(a+b+c)}{2(a^2+b^2+c^2)}.$$

- Setting $f(x)=e^x$ and $g(x)=\sqrt{x}$, Theorem 2.4 gives

$$\frac{e^a}{\sqrt{b}+\sqrt{c}} + \frac{e^b}{\sqrt{c}+\sqrt{a}} + \frac{e^c}{\sqrt{a}+\sqrt{b}} \geq \frac{3\left(e^a+e^b+e^c\right)}{2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}.$$

- Setting $f(x)=\arctan(x)$ and $g(x)=e^x$, Theorem 2.4 yields

$$\frac{\arctan(a)}{e^b+e^c} + \frac{\arctan(b)}{e^c+e^a} + \frac{\arctan(c)}{e^a+e^b} \geq \frac{3(\arctan(a)+\arctan(b)+\arctan(c))}{2(e^a+e^b+e^c)}.$$

Many other examples of this type can be constructed in a similar manner.

2.5. **Fifth theorem.** Our fifth and final main result departs from the preceding ones in several aspects, most notably in its underlying setting and method of proof. It is stated below, this time highlighting the central role played by the single function f and the parameters α, β, γ, η, and θ.

Theorem 2.5. *Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c, \alpha, \beta, \gamma > 0$ and $\eta, \theta \geq 0$, the following cyclic inequality holds (with respect to a, b, c):*

$$\alpha f\left(\frac{\eta a^2 + \theta bc}{\alpha a(b+c)}\right) + \beta f\left(\frac{\eta b^2 + \theta ca}{\beta b(c+a)}\right) + \gamma f\left(\frac{\eta c^2 + \theta ab}{\gamma c(a+b)}\right)$$
$$\geq (\alpha+\beta+\gamma) f\left(\frac{3(\eta+\theta)}{2(\alpha+\beta+\gamma)}\right).$$

The equality holds for $a = b = c$ and $\alpha = \beta = \gamma$, or $\eta = \theta = 0$.

Proof. Adopting the classical cycle $a \to b \to c$ and, where applicable, $\alpha \to \beta \to \gamma$, into the definition of $\sum_{cyc}$, the Jensen inequality applied to the convex function f yields

$$\alpha f\left(\frac{\eta a^2 + \theta bc}{\alpha a(b+c)}\right) + \beta f\left(\frac{\eta b^2 + \theta ca}{\beta b(c+a)}\right) + \gamma f\left(\frac{\eta c^2 + \theta ab}{\gamma c(a+b)}\right)$$
$$= \sum_{cyc} \alpha f\left(\frac{\eta a^2 + \theta bc}{\alpha a(b+c)}\right) = (\alpha+\beta+\gamma) \sum_{cyc} \left(\frac{\alpha}{\alpha+\beta+\gamma}\right) f\left(\frac{\eta a^2 + \theta bc}{\alpha a(b+c)}\right)$$
$$\geq (\alpha+\beta+\gamma) f\left(\frac{\sum_{cyc} \alpha \frac{\eta a^2+\theta bc}{\alpha a(b+c)}}{\alpha+\beta+\gamma}\right) = (\alpha+\beta+\gamma) f\left(\frac{\sum_{cyc} \frac{\eta a^2+\theta bc}{a(b+c)}}{\alpha+\beta+\gamma}\right).$$

We have

$$\sum_{cyc} \frac{\eta a^2 + \theta bc}{a(b+c)} = \frac{\eta a^2 + \theta bc}{a(b+c)} + \frac{\eta b^2 + \theta ca}{b(c+a)} + \frac{\eta c^2 + \theta ab}{c(a+b)}$$
$$= \eta \underbrace{\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)}_{S_1} + \theta \underbrace{\left(\frac{bc}{a(b+c)} + \frac{ca}{b(c+a)} + \frac{ab}{c(a+b)}\right)}_{S_2}.$$

Using the Nesbitt inequality, we directly have

$$S_1 = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

To analyze S_2, we introduce the reciprocal dual variables

$$x = \frac{1}{a}, \quad y = \frac{1}{b}, \quad z = \frac{1}{c}.$$

Expressing the individual term of S_2 in terms of $x, y,$ and z, we obtain

$$\frac{bc}{a(b+c)} = \frac{\frac{1}{yz}}{\frac{1}{x}\left(\frac{1}{y} + \frac{1}{z}\right)} = \frac{\frac{1}{yz}}{\frac{1}{x}\left(\frac{y+z}{yz}\right)} = \frac{x}{y+z}.$$

Substituting this back into S_2 transforms it into the Nesbitt inequality, as follows:

$$S_2 = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}.$$

Combining the inequalities above, we obtain

$$\sum_{cyc} \frac{\eta a^2 + \theta bc}{a(b+c)} \geq \eta\frac{3}{2} + \theta\frac{3}{2} = \frac{3(\eta+\theta)}{2}.$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{\eta a^2+\theta bc}{a(b+c)}}{\alpha+\beta+\gamma}\right) \geq f\left(\frac{\frac{3(\eta+\theta)}{2}}{\alpha+\beta+\gamma}\right) = f\left(\frac{3(\eta+\theta)}{2(\alpha+\beta+\gamma)}\right).$$

Combining the inequalities above, we obtain

$$\begin{aligned}&\alpha f\left(\frac{\eta a^2+\theta bc}{\alpha a(b+c)}\right) + \beta f\left(\frac{\eta b^2+\theta ca}{\beta b(c+a)}\right) + \gamma f\left(\frac{\eta c^2+\theta ab}{\gamma c(a+b)}\right)\\ &\geq (\alpha+\beta+\gamma) f\left(\frac{3(\eta+\theta)}{2(\alpha+\beta+\gamma)}\right).\end{aligned}$$

Clearly, the equality holds for $a = b = c$ and $\alpha = \beta = \gamma$, with both sides equal to $3\alpha f\left((\eta+\theta)/(2\alpha)\right)$, or for $\eta = \theta = 0$, with both sides equal to $(\alpha+\beta+\gamma)f(0)$. This completes the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Some examples of this theorem are presented below.

- Setting $\alpha = \beta = \gamma = 1$, $\eta = 1$ and $\theta = 0$, and keeping $f : (0, \infty) \to \mathbb{R}$ as a convex and increasing function, Theorem 2.5 gives

$$f\left(\frac{a}{b+c}\right) + f\left(\frac{b}{c+a}\right) + f\left(\frac{c}{a+b}\right) \geq 3f\left(\frac{1}{2}\right).$$

- Setting $\alpha=\beta=\gamma=1$, $\eta=1$ and $\theta=0$, and $f(x)=x$, Theorem 2.5 yields
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2},$$
which corresponds to the Nesbitt inequality.
- Setting $\alpha=\beta=\gamma=1$, $\eta=0$ and $\theta=1$, and keeping $f:(0,\infty)\to\mathbb{R}$ as a convex and increasing function, Theorem 2.5 gives
$$f\left(\frac{bc}{a(b+c)}\right)+f\left(\frac{ca}{b(c+a)}\right)+f\left(\frac{ab}{c(a+b)}\right)\geq 3f\left(\frac{1}{2}\right).$$
- Setting $\alpha=\beta=\gamma=1$, $\eta=0$ and $\theta=1$, and $f(x)=x$, Theorem 2.5 yields
$$\frac{bc}{a(b+c)}+\frac{ca}{b(c+a)}+\frac{ab}{c(a+b)}\geq\frac{3}{2},$$
which can be considered as a counterpart of the Nesbitt inequality.
- Setting $\eta=1$ and $\theta=1$, and keeping $f:(0,\infty)\to\mathbb{R}$ as a convex and increasing function, Theorem 2.5 gives
$$\alpha f\left(\frac{a^2+bc}{\alpha a(b+c)}\right)+\beta f\left(\frac{b^2+ca}{\beta b(c+a)}\right)+\gamma f\left(\frac{c^2+ab}{\gamma c(a+b)}\right)$$
$$\geq(\alpha+\beta+\gamma)f\left(\frac{3}{\alpha+\beta+\gamma}\right).$$
- Setting $\eta=1$ and $\theta=1$, and $f(x)=x^2$, Theorem 2.5 yields
$$\frac{1}{\alpha}\left(\frac{a^2+bc}{a(b+c)}\right)^2+\frac{1}{\beta}\left(\frac{b^2+ca}{b(c+a)}\right)^2+\frac{1}{\gamma}\left(\frac{c^2+ab}{c(a+b)}\right)^2\geq\frac{9}{\alpha+\beta+\gamma}.$$
- Setting $\eta=1$ and $\theta=1$, and $f(x)=e^x$, Theorem 2.5 gives
$$\alpha e^{(a^2+bc)/(\alpha a(b+c))}+\beta e^{(b^2+ca)/(\beta b(c+a))}+\gamma e^{(c^2+ab)/(\gamma c(a+b))}$$
$$\geq(\alpha+\beta+\gamma)e^{3/(\alpha+\beta+\gamma)}.$$
- Note that replacing the parameters α, β, and γ, with the cyclic variables a, b, and c, is admissible in our framework. Accordingly, setting $\eta=1$, $\theta=1$, $\alpha=a$, $\beta=b$, and $\gamma=c$, and keeping $f:(0,\infty)\to\mathbb{R}$ as a convex and increasing function, Theorem 2.5 gives

$$af\left(\frac{a^2+bc}{a^2(b+c)}\right)+bf\left(\frac{b^2+ca}{b^2(c+a)}\right)+cf\left(\frac{c^2+ab}{c^2(a+b)}\right)$$
$$\geq (a+b+c)f\left(\frac{3}{a+b+c}\right).$$

- Setting $\eta=1$, $\theta=1$, $\alpha=a$, $\beta=b$, $\gamma=c$, and $f(x)=\sqrt{x^2+1}$, Theorem 2.5 gives

$$a\sqrt{\left(\frac{a^2+bc}{a^2(b+c)}\right)^2+1}+b\sqrt{\left(\frac{b^2+ca}{b^2(c+a)}\right)^2+1}$$
$$+c\sqrt{\left(\frac{c^2+ab}{c^2(a+b)}\right)^2+1}\geq (a+b+c)\sqrt{\left(\frac{3}{a+b+c}\right)^2+1}.$$

- Setting $\eta=1$, $\theta=1$, $\alpha=a$, $\beta=b$, $\gamma=c$, and $f(x)=x\ln(1+x)$, Theorem 2.5 yields

$$\frac{a^2+bc}{a(b+c)}\ln\left(1+\frac{a^2+bc}{a^2(b+c)}\right)+\frac{b^2+ca}{b(c+a)}\ln\left(1+\frac{b^2+ca}{b^2(c+a)}\right)$$
$$+\frac{c^2+ab}{c(a+b)}\ln\left(1+\frac{c^2+ab}{c^2(a+b)}\right)\geq 3\ln\left(1+\frac{3}{a+b+c}\right),$$

so that

$$\left(1+\frac{a^2+bc}{a^2(b+c)}\right)^{(a^2+bc)/(a(b+c))}\left(1+\frac{b^2+ca}{b^2(c+a)}\right)^{(b^2+ca)/(b(c+a))}$$
$$\times\left(1+\frac{c^2+ab}{c^2(a+b)}\right)^{(c^2+ab)/(c(a+b))}\geq\left(1+\frac{3}{a+b+c}\right)^3.$$

Many additional examples of this type can be constructed similarly, including by varying the parameters η and θ, which were set to 0 or 1 in the preceding examples.

3. Conclusion

In this paper, we have established five general theorems for cyclic inequalities, providing a unified framework for generating, extending, and analyzing a wide class of functional and algebraic bounds. In particular, the first four theorems involve two functions, f and g, and take the following forms:

-: [Form 1]

$$f(a)g\left(\frac{a}{b+c}\right)+f(b)g\left(\frac{b}{c+a}\right)+f(c)g\left(\frac{c}{a+b}\right)$$
$$\geq (f(a)+f(b)+f(c))\,g\left(\frac{1}{2}\right).$$

-: [Form 2]

$$f(a)g(b+c)+f(b)g(c+a)+f(c)g(a+b)$$
$$\geq (f(a)+f(b)+f(c))\,g\left(\frac{2(a+b+c)}{3}\right).$$

-: [Form 3]

$$f\left(\frac{a}{b+c}\right)g(b+c)+f\left(\frac{b}{c+a}\right)g(c+a)+f\left(\frac{c}{a+b}\right)g(a+b)$$
$$\geq 3f\left(\frac{1}{2}\right)g\left(\frac{2(a+b+c)}{3}\right).$$

-: [Form 4]

$$\frac{f(a)}{g(b)+g(c)}+\frac{f(b)}{g(c)+g(a)}+\frac{f(c)}{g(a)+g(b)}\geq \frac{3\,(f(a)+f(b)+f(c))}{2\,(g(a)+g(b)+g(c))}.$$

The fifth theorem involves a single function f together with multiple parameters. It takes the following form:

$$\alpha f\left(\frac{\eta a^2+\theta bc}{\alpha a(b+c)}\right)+\beta f\left(\frac{\eta b^2+\theta ca}{\beta b(c+a)}\right)+\gamma f\left(\frac{\eta c^2+\theta ab}{\gamma c(a+b)}\right)$$
$$\geq (\alpha+\beta+\gamma)f\left(\frac{3(\eta+\theta)}{2(\alpha+\beta+\gamma)}\right).$$

Each of these forms offers a unified approach to generalizing Nesbitt's inequality, typically under suitable monotonicity and convexity conditions on f and g. Furthermore, this framework yields new families of cyclic inequalities under flexible hypotheses while naturally recovering several known results from the literature. A promising direction for future research is to extend these cyclic functional principles to higher-dimensional settings (that is, with more than three variables) or to investigate analogous generalizations under relaxed convexity conditions such as s-convexity or quasi-convexity.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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