

A NECESSARY AND SUFFICIENT CONDITION FOR A RIVER METRIC INEQUALITY ON R^2 OR R^3 TO HOLD

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ABSTRACT. Deeply believe that alluring a math student who is willing to keep striving, dares to take responsibility, excels in both character and academics, and is highly talented into a field of mathematical research he/she is interested, just like publishing an important academic, is also important, the paper focuses on how mathematical research can concretely and as early as possible support its successors and exemplifies the formation process of slightly technical approaches and methods. There is an urgent need for increased support for gifted high school students and exceptional mathematics tuition-free normal students in their critical developmental years in China; this is similar to the way research funding tends to favor young researchers. This can be effectively addressed based on the idea of plausible reasoning (such as the combination of teaching, exemplifying, and training, especially the way of organically combining the two on the psychological level of mathematical conjectures). This paper exemplifies the application of these methods in the learning, training, and creation of mathematics, highlighting their role in discovering new thinking and new methods in the study of metrics or metric spaces and in related non-professional research. Metrics are widely used across many fields, its users include practitioners, beginners, students, and adults. Taking the special metric, called river metric, on the 2-dimensional Euclidean plane, R^2 , as an example, we begin by providing a necessary and sufficient condition for

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the river metric inequality on R^2 to hold, using the relation-mapping-inversion (RMI) principle and starting from the knowledge point in plane geometry; such a study arises from a decision-making problem which is easy to understand for middle school students. With the above as an embryonic form, we illustrate how to define, by using analogy of plausible reasoning, two river-like metrics on the 3-dimensional Euclidean space R^3 and give a necessary and sufficient condition for a metric inequality related to one river-like metric on R^3 to hold, some sufficient conditions for a metric inequality related to another river-like metric on R^3 to hold, and some clues for subsequent closely related deeper study and stretched exploring (from simple ones to complex ones). An example is provided to show the applications of the first metric inequality in decision-making problems. Additionally, we invited gifted high school students and talented undergraduate students—particularly those who are outstanding in mathematics and enrolled in tuition-free normal programs—in this research, and they showed great enthusiasm in both thinking and doing when participating in this research.

1. MOTIVATIONS AND PRELIMINARIES

In China, some gifted middle school students (especially high school students) desire (and even demand) to understand how current active mathematicians, and even other scientists in other fields, to approach and solve problems. They want to learn not only how these experts think, but also to appreciate the processes and methods (including alternative ones and those they thought of but were not so good) by which they solve problems, including those that may seem trivial – which is very similar to the pursuit of multiple solutions to a problem in learning mathematics. We should meet these already relatively scarce needs as early as possible, because it helps to stimulate the increase and flourishing of their intellectual sparks in childhood (rather than letting them be extinguished in their initial or budding state), and the support provided during adolescence can help them advance long-term (even far into the future) in their future academic and career paths (this is very similar to the early training of Olympic medalists). Conversely, if we evade our mission and thus sit idly by (or even turn a blind eye to it), there may fewer young people dedicating themselves to mathematical research in the future. One reason is that some students with exceptional mathematical talent switch to majors that temporarily and frequently use AI (which

does not strengthen the ability to understand mathematics) due to their existing strong interest in gaming and unusual gaming foundation. Another reason, due to the long-term impact of a low birth rate, the number of teenagers in mathematics is decreasing (this may be a manifestation of the reduction in altruists), and the fact that there are more pleasure-seekers than strivers leads parents to spoil their children even more — they do not want their to participate in Math Olympiad training and mathematical competitions, thereby avoiding complex labor — such as engaging in mathematical research. Meanwhile, in China, some outstanding mathematics tuition-free normal students, who have just entered middle schools and started working, are very reluctant to fall permanently far behind their future competitive partners from their middle school days (because their scores for entering university are very higher); one reason for this frustration is that, from the third semester onwards, the enthusiasm for learning among these students (actually the small group or the set they belong to) tends to decline year by year overall (or in terms of mathematical expectation). This trend highlights the importance of intervening early to assist these students in maintaining motivation and overcoming this stagnation. Without proper support and scientifically oriented guidance in the future, they may, starting from the first summer vacation of their university years, turn to other unrelated, under-employing but temporarily high-paying occupations, and gradually fall to a low point in their academic and generally recognized personal development—which will be an indelible mark of failure in their lives and ultimately affect their sense of responsibility for strengthening the country. Therefore, these situation (especially for math majors) requires immediate attention and concrete and actionable support and relief. In fact, these students represent in a degree the future popularity of fields such as mathematics, computer science, data science, physics, engineering, biology, psychology, and economics. By helping them strengthen their sense of responsibility and reach their potential, we are, in fact, contributing to their future long-term endeavor and success and to the advancement of these fields. This paper aims to explore how to concretely address these challenging issues that fundamentally affect the future by incorporating psychologically-assisted mathematics and plausible reasoning-based training [1]. It shows also how these the methods mentioned above can be used in

practical teaching scenarios, particularly in a group setting, to enhance the learning experience.

Plausible reasoning methods, such as induction, analogy, generalization, specialization, and the relation-mapping-inversion (RMI) principle, have been extensively applied in research across various fields, including mathematics, physics, chemistry, biology, law, philosophy, psychology, for many years. Recently, it has been found that plausible reasoning still plays a significant role in advancing mathematical research, especially in promoting educational innovation and talent development resulting from it. For example, knowledge spaces and learning spaces [2–7] have been studied and used in educational research. As both can stimulate interest among researchers in areas like topology [8] and rough set theory [9], enormous notions and results within topological spaces and proximate spaces can analogously be established in knowledge spaces or learning spaces by using plausible reasoning methods (such as analogy and specialization), which in turn can be further applied to practical problems in education and teaching, such as modeling, real-time representation, regulation, data sharing, etc.

Remark 1.1. *A natural question arises: Can plausible reasoning also enable mathematics to contribute more deeply to psychological research (and vice versa)? The answer is positive, because plausible reasoning always concerns the process of a person's—especially a mathematician's—deep psychological activities, and studying how smart people think before they act is essentially psychological. We will expound on this point in the following discussion.*

- (1) *One key feature of plausible reasoning in mathematics is guessing, conjecture, or speculation. This has a direct connection to psychology, because guessing should also belong to the research category of psychology. It is in fact of more down to earth (since scientists are also human), more thought provoking, more far reaching, and more scientific significance for psychologists to study the specific thought processes or details of mathematicians and other scientists who rely more on divergent thinking, avoid gradually falling into the pit of explanations, elaborations, generalizations, summaries, and evaluations aimed at the majority of ordinary people.*
- (2) *Even the thought processes of ordinary intelligent people, particularly when they make conjectures, are worth analyzing, learning, and imitating, and*

when imitation seems too plain, one may draw inferences about other cases from one instance or make a generalization. In fact, many mental activities of intelligent people—especially experts, researchers, inventors, smart students or children—can inspire and stimulate people of similar age, thereby helping them accomplish things that they may have attempted many times without success, if these mental activities are analyzed, expounded upon, and exemplified in such sufficient detail and with a considerable number of examples that people of similar age not only understand them but also can speculate about their motivation, inducement, and other aspects behind them. It is therefore worth analyzing these activities more closely for the purpose of trying, imitating, expanding, extending, disseminating, and discovering further possibilities (as the saying by You Lu goes, “What is learned from books is superficial; to truly master it, one must put it into practice”).

- (3) *In psychologically assisted mathematics-or mathematics taught mainly through autonomic learning, followed by exercising, practicing, aided exploring, guessing, training, discovering, and writing in mathematical language, rather than in a spoon-feeding way-the situation is even more evident. More practically, in view of co-creation or collaborative innovation in mathematics and other subjects, it is advantageous to expound or display in detail the process of the authors’ thinking, including analysis, guessing, verifying, reanalysis, guessing again, reverifying, and so on, while relying step by step on plausible reasoning methods before a mathematics result is obtained. Furthermore, gifted students’ early imitation of mathematicians’ deep thinking will benefit them for their whole life. In a word, detailed explanations of such thinking are valuable.*
- (4) *Unfortunately, teaching in many high schools in China depends completely on the same teaching programme and is directed toward the same target, namely, obtaining a high score in the college entrance examination. Teachers are helpless because there are no more advanced, more scientific, and more specific teaching methods available to replace it, but some high school leaders hope to find and demonstrate some scientific teaching methods. Moreover, some excellent high school students who are interested in mathematics and psychology hope to know in advance which parts of what they study (though*

requiring little effort but consuming considerable time) are practically useful and why. Actually, some part (e.g., the sine and cosine function part) may make a few students know, by analogy, the rest of a kind or draw inferences about other cases from one instance, or make them have strong regenerative capacity. Gifted and smart students among them would also like to know whether the process behind the discovery of a theorem that is not too difficult, once revealed, can be disseminated widely or deeply like a cultural medium (e.g., the specialization of the discovery process and its dissemination among the general public), and may even have the ambition to discover a similar result or to do some subsequent work related to the theorem. The same idea actually appeared in the study of groups and vector spaces (resp., topological spaces), where mathematicians hoped to know a generating set of a given group and a base of a given vector space (resp., a given topological space). This is exactly what we have been looking forward to; that is, high schools should also be a fertile soil for innovative talents to stand out. From a developmental perspective, for the small number of students who are farsighted, have a solid foundation, and possess the perseverance to work hard, the ideas and behaviors mentioned above are of paramount importance—indeed, indispensable. This is because higher education institutions are not merely training ordinary job seekers, but also cultivating the future mathematicians, computer scientists, data scientists, physicists, engineers, biologists, psychologists, economists, and other professionals who will shape society.

Mathematically speaking², this is much like the situation to compare the functions $f(x) = 1 + \sin \pi x$ and $g(x) = \frac{1}{3}x + \sin \pi x$ based on two much

²This illustrates the application of mathematical methods in clarifying certain disputes, disagreements, and conflicts in scenarios such as daily life, court proceedings, and arguments. For example, which is the preferable option for current gifted students: early employment or pursuing a further academic degree? Should current homebuyers adopt a positive or negative stance? Should depositors be optimistic or cautious with their savings? How to clarify the current biases regarding the Fields Medalists, which are overshadowing and should not be attracting so much attention? Some prejudices that shouldn't have attracted so many followers by overshadowing the main focus? By reviewing the historical responses after the first appearance of fashioned tricycles in rural areas, the first appearance of diesel engines in rural areas, the first electrification of rural areas, and the first emergence of calculators (and even computers), we can calm the current heated debates and psychological micro-anxiety over the contributions of AI; design an electronic pulse diagnostic device under the guidance of traditional Chinese medicine (TCM for short) experts by deeply studying TCM techniques and drawing analogies with electronic blood pressure monitors.

different cognitions. The first way is to compare f and g just on $X = [0, 3]$, and the result is that $m(X_{f>g}) = m([0, 3]) = 3 > 0 = m(\emptyset) = m(X_{g>f})$, where $X_{f>g} = \{x \in X \mid f(x) > g(x)\}$ and m is the Lebesgue measure on $\mathcal{L}(R)$ (the set of all Lebesgue measurable sets on the real number set R). The second way is to compare f and g on the bigger set $Y = [0, 100]$, the results in two aspects are (see Figure 1):

- (i) $f > g$ on $[0, 3]$ but $g > f$ on $(3, 100]$ (therefore, $m(X_{f>g}) = m([0, 3]) = 3 < 97 = m((3, 100]) = m(X_{g>f})$).
- (ii) $\int_0^{100} f(x)dx = 100 < \frac{10^4}{6} = \int_0^{100} g(x)dx$ (therefore, the integer averages satisfy $\frac{1}{100} \int_0^{100} f(x)dx = 1 < \frac{10^2}{6} = \frac{1}{100} \int_0^{100} g(x)dx$).

We more appreciate the second way.

- (5) Mathematics indeed needs all the excellent talents who are willing to devote themselves to it. The success of a gifted student is usually attributed to firm youthful aspirations that remain grounded in reality, together with goal-oriented and long-term relentless efforts. Therefore, early cultivation has much to do with the stimulation and promotion of interest, and even with the above-mentioned urgently needed sense of responsibility and dedication. This is because it is well known that even a very small success of a gifted young student in mathematics will surely boost his/her interest and ambition in mathematics. Moreover, several such successes will usually lead to new related issues or scientific problems to be studied or solved, which will further motivate them to voluntarily continue their research. All these can be seen from the classmates and colleagues, as well as from many gifted students whom we and our friends have taught. Please also see [10] and the references herein for related data analysis.
- (6) Aiming at the above requirements, this paper exemplifies the key role of plausible reasoning methods in discovering modes of thinking and methods in the study and non-professional research of metrics, a mathematical notion commonly used by practitioners, beginners, students, and nearly all adults. Taking the river metric on the 2-dimensional Euclidean plane R^2 as an example, we use the RMI principle and begin with the well-known knowledge point of plane geometry to give a necessary and sufficient condition (like a spark that can keep burning) for the river metric inequality on R^2 to hold.

Another original motivation for proving the river metric inequality is to improve related decision-making methods. To illustrate the motivation, we first consider the following decision-making problem.

Example 1. A matrix $A = (a_{ij})_{n \times n}$ ($n > 1$) satisfying $a_{ij} \in [0, 1]$ and $\iota(a_{ij}) = (a_{ij})' = 1 - a_{ji}$ ($1 \leq i, j \leq n$) is called a complementary symmetric matrix, where the mapping $\iota : [0, 1] \rightarrow [0, 1]$, defined by $\iota(a) = a' = 1 - a$ ($\forall a \in [0, 1]$) is called an order-reversing involution on $[0, 1]$. Try to rank the 3-element set $X = \{x_1, x_2, x_3\}$ of alternatives based on the following two comparative advantage judgment matrices (called complementary symmetric matrices mathematically) given by two assessment teams:

$$A^{(1)} = \begin{bmatrix} 0.5 & 0.6439 & 0.41037 \\ 0.3561 & 0.5 & 0.45423 \\ 0.58963 & 0.54577 & 0.5 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0.5 & 0.3812 & 0.6513 \\ 0.6188 & 0.5 & 0.4379 \\ 0.3487 & 0.5621 & 0.5 \end{bmatrix},$$

where 0.5 (resp., 0.6439, 0.41037) in the first matrix $A^{(1)}$ denotes comparative advantage of x_1 over x_1 (resp., over x_2 , over x_3) given by the first assessment team; other elements in $A^{(1)}$ and $A^{(2)}$ may be explained similarly.

(1) Usually, this ranking is finished in the following three steps:

Step 1. Compute $A_j^{(i)}$ (i.e., aggregation of line j of matrix $A^{(i)}$) simply relying on the weight vector $\vec{w} = (w_1, w_2, w_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ($i = 1, 2; j = 1, 2, 3$). We obtain $A_1^{(1)} = \frac{1}{3}(0.5 + 0.6439 + 0.41037) = 0.51809$, $A_2^{(1)} = \frac{1}{3}(0.3561 + 0.5 + 0.45423) = 0.436777$, $A_3^{(1)} = \frac{1}{3}(0.58963 + 0.54577 + 0.5) = 0.545133$; $A_1^{(2)} = 0.510833$, $A_2^{(2)} = 0.5189$, and $A_3^{(2)} = 0.470267$.

Step 2. Compute score A_j (i.e., aggregation of $A_j^{(1)}$ and $A_j^{(2)}$) relying on the weight vector $\vec{w} = (w_1, w_2) = (0.6, 0.4)$ ($j = 1, 2, 3$). We obtain $A_1 = 0.6 \times 0.51809 + 0.4 \times 0.51833 = 0.515187$, $A_2 = 0.6 \times 0.436777 + 0.4 \times 0.5189 = 0.46962$, and $A_3 = 0.6 \times 0.545133 + 0.4 \times 0.470267 = 0.515187$.

Step 3. Obtain the ranking result: $x_2 \prec x_1 = x_3$ (since $A_2 < A_1 = A_3$). This is an invalid result.

As $\min_{i \neq j} |A_i - A_j| = A_1 - A_3 = 0$ (denoted by ϵ_1), x_1 and x_3 are incomparable. Therefore, the above method need to be improved.

(2) Reduce the weight of each element in principle diagonal line of each matrix.

Step 1. $A_1^{(1)} = 0.5 \times 0.2 + 0.6439 \times 0.4 + 0.41037 \times 0.4 = 0.521708$, $A_2^{(1)} = 0.5 \times 0.2 + 0.3561 \times 0.4 + 0.45423 \times 0.4 = 0.424132$, $A_3^{(1)} = 0.5 \times 0.2 + 0.58963 \times 0.4 + 0.54577 \times 0.4 = 0.55416$; $A_1^{(2)} = 0.513$, $A_2^{(2)} = 0.52268$, and $A_3^{(2)} = 0.46432$.

Step 2. Compute score A_j relaying on weight vector $\vec{w} = (w_1, w_2) = (0.6, 0.4)$ ($j = 1, 2, 3$). $A_1 = 0.6 \times 0.521708 + 0.4 \times 0.513 = 0.518225$, $A_2 = 0.6 \times 0.424132 + 0.4 \times 0.52268 = 0.463551$, and $A_3 = 0.6 \times 0.55416 + 0.4 \times 0.46432 = 0.518224$.

Step 3. Obtain the ranking result: $x_2 \prec x_1 \prec x_3$ (since $A_2 < A_1 < A_3$). This is a valid result.

As $\min_{i \neq j} |A_i - A_j| = A_1 - A_3 = 0.000001$ (denoted by ϵ_2), we guess from the inequality $\epsilon_1 < \epsilon_2$ that the following method is improved more (it really is!).

(3) Reduce the weight of each element in diagonal line of each matrix to 0.

Step 1. $A_1^{(1)} = 0.6439 \times 0.5 + 0.41037 \times 0.5 = 0.527135$, $A_2^{(1)} = 0.3561 \times 0.5 + 0.45423 \times 0.5 = 0.405165$, $A_3^{(1)} = 0.58963 \times 0.5 + 0.54577 \times 0.5 = 0.5677$; $A_1^{(2)} = 0.51625$, $A_2^{(2)} = 0.52835$, and $A_3^{(2)} = 0.4554$.

Step 2. Compute score A_j relaying on weight vector $\vec{w} = (w_1, w_2) = (0.6, 0.4)$ ($j = 1, 2, 3$). $A_1 = 0.6 \times 0.527135 + 0.4 \times 0.51625 = 0.522781$, $A_2 = 0.6 \times 0.405165 + 0.4 \times 0.52835 = 0.454439$, and $A_3 = 0.6 \times 0.5677 + 0.4 \times 0.4554 = 0.52278$.

Step 3. Obtain the ranking result: $x_2 \prec x_1 \prec x_3$ (since $A_2 < A_1 < A_3$). This is a valid result.

The above Example 1 on decision-making problem is thought-provoking. Actually, as $\min_{i \neq j} |A_i - A_j| = A_1 - A_3 = 0.000001$ (denoted by ϵ_3), $\epsilon_3 > \epsilon_2$ (actually, $\epsilon_2 = 0.0000008$), which, together (1), urges us to guess the following inequality:

$$(1.1) \quad \rho\left(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j, \frac{1}{n+1} \sum_{j=0}^n \vec{b}_j\right) \leq \rho\left(\frac{1}{n} \sum_{j=1}^n \vec{a}_j, \frac{1}{n} \sum_{j=1}^n \vec{b}_j\right),$$

where $\vec{a}_0 = \vec{b}_0 \in R^m$ (the m -dimensional Euclidean space, $m \geq 1$), $\{\vec{a}_1, \vec{b}_1, \dots, \vec{a}_n, \vec{b}_n\} \subseteq R^m$, and ρ is a metric [2] on R^m , i.e., $\rho : R^m \times R^m \rightarrow R$ is a mapping satisfying the following conditions ($\{\vec{x}, \vec{y}, \vec{z}\} \subseteq R^m$):

- (i) $\rho(\vec{x}, \vec{y}) \geq 0$, and $\rho(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$.
- (ii) $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$.
- (iii) $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$.

Addition $+$, subtraction $-$, and scalar multiplication on the m -dimensional vector (or linear) space R^m are defined by $\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_m + y_m)$, $\vec{x} - \vec{y} = (x_1 -$

$y_1, \dots, x_m - y_m$), and $k\vec{x} = (kx_1, \dots, kx_m)$ ($\vec{x} = (x_1, \dots, x_m)$, $\vec{y} = (y_1, \dots, y_m)$, $k \in R$) respectively, and the ordinary metric (called also the Euclidean metric) ρ_m on R^m is defined by $\rho_m(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_m - y_m)^2}$. Recall again that the Chebyshev metric $\varrho_{1,m}$ and the Manhattan metric or the city block metric $\varrho_{2,m}$ on R^m are defined in [2] by ($\vec{x} = (x_1, \dots, x_m)$, $\vec{y} = (y_1, \dots, y_m)$, and $m > 1$): $\varrho_{1,m}(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_m - y_m|\}$, $\varrho_{2,m}(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2| + \dots + |x_m - y_m|$. The river metric ϱ_3 on R^2 are defined in [2] by ($\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$):

$$\varrho_3(\vec{x}, \vec{y}) = \begin{cases} |x_2 - y_2|, & x_1 = y_1, \\ |x_2| + |x_1 - y_1| + |y_2|, & x_1 \neq y_1. \end{cases}$$

The river metric ϱ_3 measures distances in a wooded country with a river $\{(z_1, z_2) \in R^2 \mid z_2 = 0\}$. The inhabitants of that country, in order to get access to the water, have cut out paths perpendicular to the river. If someone wishes to go from (x_1, x_2) to (y_1, y_2) (where (x_1, x_2) and (y_1, y_2) are not in the same path), he would find out that the best solution is to go to straight to the river, swim to get nearest to (y_1, y_2) and then walk again through the wood.

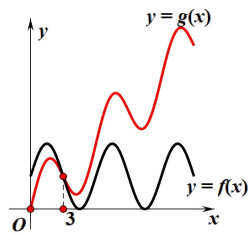


FIGURE 1. For Remark 1.1(4)

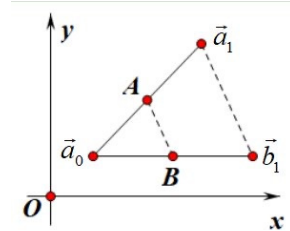


FIGURE 2. For Step 1 of Remark 1.2

After the river metric ϱ_3 being described and expounded geometrically, some interesting, easy-to-understand, and worth-to-study problems are proposed based on plausible reasoning methods [1] (mainly, RMI principle, generalization, and analogy) and are considered by our gifted high school students and outstanding mathematics tuition-free normal students:

Problem 1.1. *What are the similar situations? Can the river be replaced by an arbitrary line (even ...)?*

Problem 1.2. Can a metric ϱ on $X \neq \emptyset$ induce a river metric ρ_ϱ on X^2 or X ?

Problem 1.3. Try to make some technical suggestions through deeper comparative case studies of diminishing mountain torrent disasters in China.

This deliberative thinking process will result in some new applications of the river metric inequalities (the old or the new ones).

Remark 1.2. It can be easily verified that the inequality (1.1) holds if $\rho = \rho_1$. It can also be shown that the inequality (1.1) holds for metric spaces (R^m, ρ_m) , $(R^m, \varrho_{1,m})$, and $(R^m, \varrho_{2,m})$ with $m = 2$ (actually, with $m \geq 2$). We guided our high school students to guess the inequality (1.1) beforehand for the ordinary Euclidean metric ρ_2 on R^2 in the following steps:

Step 1. For $n = 1$, we have $\rho_2\left(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j, \frac{1}{n+1} \sum_{j=0}^n \vec{b}_j\right) = \rho_2\left(\frac{\vec{a}_0 + \vec{a}_1}{2}, \frac{\vec{b}_0 + \vec{b}_1}{2}\right) = \rho_2(A, B) = \frac{1}{2} \rho_2(\vec{a}_1, \vec{b}_1) \leq \rho(\vec{a}_1, \vec{b}_1)$ (see Figure 2), thus the inequality (1.1) holds.

Step 2. For $n = 2$, we find the inequality (1.1) holds after trying several cases (see Figure 3).

Step 3. Based on Steps 1–2 and many times trying, we guess the inequality (1.1) holds for each positive integer n .

Step 4. It is absolutely possible for some gifted high school students to prove the inequality (1.1) for the m -dimensional Euclidean metric space (R^m, ρ_m) ($m > 2$), even for $(R^m, \varrho_{1,m})$ and $(R^m, \varrho_{2,m})$ ($m > 1$).

Naturally, the above analysis urges us to consider the following:

Question 1.1. Does the inequality (1.1) hold for metric space (R^2, ϱ_3) ?

This is the nearest motivation of the present paper in which we give a positive answer and more: i.e., give a necessary and sufficient condition for inequality (1.1) holds for metric space (R^2, ϱ_3) and also some sufficient conditions (even a necessary and sufficient condition) for inequality (1.1) holds for two river-like metrics on R^3 which initiates the further extending study. We provide an example to illustrate applications of inequality (1.1) for metric space (R^2, ϱ_3) in decision-making problems. We also give (based on this inequality) three suggestions to the future decision-making.

Now we present other preliminary notions and results required in this paper. An interval number [11] can be regarded as a special point $\mathfrak{a} = \langle \dot{a}, \ddot{a} \rangle$ in the Euclidean plane R^2 which satisfies $\dot{a} \leq \ddot{a}$. The set of all interval numbers is denoted

by $\mathbb{R}$ (which is the closed upper left semi-infinite plane); notice that addition $+$ and nonactive scalar multiplication are defined as above (since $\mathbb{R} \subseteq R^2$). Moreover, an linear order or a total order $\leq \subseteq \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ (i.e., $\leq$ is a reflexive [12], antisymmetric, and transitive relation on $\mathbb{R}$ satisfying $a \leq b$ or $b \leq a$ holds for all $a, b \in \mathbb{R}$) on $\mathbb{R}$ was defined (cf. [13–15]) by $\langle \dot{a}, \ddot{a} \rangle \leq \langle \dot{b}, \ddot{b} \rangle$ iff $\dot{a} + \ddot{a} < \dot{b} + \ddot{b}$ or $\dot{a} + \ddot{a} = \dot{b} + \ddot{b}$ but $\ddot{a} - \dot{a} \leq \ddot{b} - \dot{b}$ (see Figure 4, where $a < b$ means $a \leq b$ but $a \neq b$), and an order-reversing involutions $\iota : \mathbb{R} \rightarrow \mathbb{R}$ on $\mathbb{R}$ is defined by $\iota(\langle \dot{a}, \ddot{a} \rangle) = \langle \dot{a}, \ddot{a} \rangle' = \langle 1 - \ddot{a}, 1 - \dot{a} \rangle$ ($\forall \langle \dot{a}, \ddot{a} \rangle \in \mathbb{R}$).

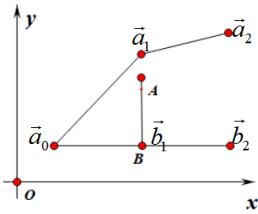


FIGURE 3. For Step 2 of Remark 1.2

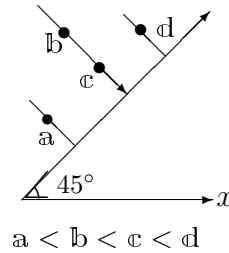


FIGURE 4. The linear order $\leq$ on $\mathbb{R}$

2. A NECESSARY AND SUFFICIENT CONDITION FOR A RIVER METRIC INEQUALITY ON R^2 TO HOLD

The river metric inequality $\delta_0 \leq \delta_1$ (if true) is useful in decision-making (see Example 2 below), where $\delta_0 = \varrho_3(\vec{c}, \vec{d})$, $\delta_1 = \varrho_3(\vec{a}, \vec{b})$, $\vec{a} = \frac{1}{n} \sum_{j=1}^n \vec{a}_j = (\dot{a}, \ddot{a})$, $\vec{b} = \frac{1}{n} \sum_{j=1}^n \vec{b}_j = (\dot{b}, \ddot{b})$, $\vec{c} = \frac{1}{n+1} \sum_{j=0}^n \vec{a}_j = (\dot{c}, \ddot{c})$, $\vec{d} = \frac{1}{n+1} \sum_{j=0}^n \vec{b}_j = (\dot{d}, \ddot{d})$, $\vec{a}_0 = (\dot{a}_0, \ddot{a}_0) = (\dot{b}_0, \ddot{b}_0) = \vec{b}_0$, $\dot{b}_0 = \ddot{b}_0 = 0.5$, $\vec{a}_1 = (\dot{a}_1, \ddot{a}_1)$, $\vec{a}_2 = (\dot{a}_2, \ddot{a}_2)$, $\dots$, $\vec{a}_n = (\dot{a}_n, \ddot{a}_n)$, $\vec{b}_1 = (\dot{b}_1, \ddot{b}_1)$, $\vec{b}_2 = (\dot{b}_2, \ddot{b}_2)$, $\dots$, $\vec{b}_n = (\dot{b}_n, \ddot{b}_n)$ (n is a positive integer). In this section we will provide a necessary and sufficient condition for inequality $\delta_0 \leq \delta_1$ on R^2 to hold for the river metric ϱ_3 (see the following Theorem 2.1). As the RMI principle is used (precisely, by using geometric figures), Theorem 2.1 and its proof may be understood by gifted high school students, making some analogies and generalizations (as in Section 3) possible. This is of profound significance (see Remark 1.1(5)).

Theorem 2.1. Let $\vec{p} = (\dot{p}, \ddot{p}) = 0.5(\vec{a} + \vec{b})$, $\vec{q} = (\dot{q}, \ddot{q}) = 0.5(\vec{c} + \vec{d})$, $R^+ = [0, +\infty)$, and $R^- = (-\infty, 0)$. Then inequality $\delta_0 \leq \delta_1$ holds if and only if one of the following conditions is satisfied:

- (1) $\dot{a} = \dot{b}$ (see Figure 5).
- (2) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \subseteq R^+$, $\ddot{p} \geq 0.5$ (see Figure 6).
- (3) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \subseteq R^+$, $0 < \ddot{p} < 0.5$, and $\ddot{a} + \ddot{b} + |\dot{a} - \dot{b}| \geq 1$ (i.e., $\ddot{p} + 0.5|\dot{a} - \dot{b}| \geq 0.5$, see Figure 7).
- (4) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}\} \cap R^-$ is a singleton, $\{\ddot{c}, \ddot{d}\} \subseteq R^+$, $|\ddot{a}| + |\ddot{b}| + |\dot{a} - \dot{b}| \geq 1 + 2n \min\{\ddot{a}, \ddot{b}\}$ (see Figure 8).
- (5) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ \in \{\{\ddot{a}, \ddot{c}\}, \{\ddot{b}, \ddot{d}\}\}$ (see Figure 9). Notice that $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ \neq \{\ddot{a}, \ddot{b}\}$ because $\ddot{a} \geq 0$ and $\ddot{b} \geq 0$ imply $\ddot{c} = \frac{\ddot{a} + \ddot{a}_0}{n+1} \geq 0$ and $\ddot{d} = \frac{\ddot{b} + \ddot{a}_0}{n+1} \geq 0$; $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ \neq \{\ddot{a}, \ddot{d}\}$ because $\ddot{a} \geq 0$ implies $\ddot{c} = \frac{\ddot{a} + \ddot{a}_0}{n+1} \geq 0$; $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ \neq \{\ddot{b}, \ddot{c}\}$ because $\ddot{b} \geq 0$ implies $\ddot{d} = \frac{\ddot{b} + \ddot{a}_0}{n+1} \geq 0$.
- (6) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ = \{\ddot{c}, \ddot{d}\}$, $|\ddot{a}| + |\ddot{b}| + |\dot{a} - \dot{b}| \geq 1 + 2n(\ddot{a} + \ddot{b})$ (see Figure 10).
- (7) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ \in \{\{\ddot{c}\}, \{\ddot{d}\}\}$ (see Figure 11). Notice that $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ \neq \{\ddot{a}\}$ because $\ddot{a} \geq 0$ implies $\ddot{c} = \frac{\ddot{a} + \ddot{a}_0}{n+1} \geq 0$; $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ \neq \{\ddot{b}\}$ because $\ddot{b} \geq 0$ implies $\ddot{d} = \frac{\ddot{b} + \ddot{a}_0}{n+1} \geq 0$.
- (8) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ = \emptyset$ (see Figure 12).

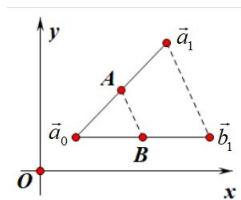


FIGURE 5. For Theorem 2.1(1)

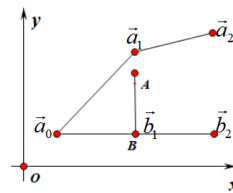


FIGURE 6. For Theorem 2.1(2)

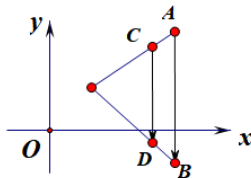


FIGURE 7. For Theorem 2.1(3)

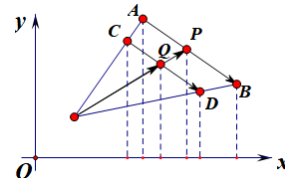


FIGURE 8. For Theorem 2.1(4)

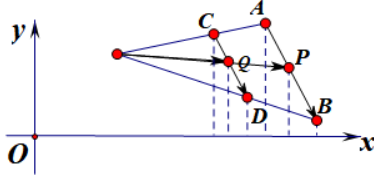


FIGURE 9. For Theorem 2.1(5)

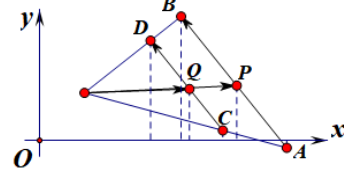


FIGURE 10. For Theorem 2.1(6)

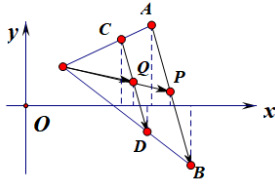


FIGURE 11. For Theorem 2.1(7)

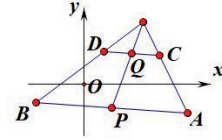


FIGURE 12. For Theorem 2.1(8)

To prove Theorem 2.1, we need the following two lemmas (where $\vec{b} \neq \vec{a}$ and the tilt angle θ of vector $\vec{b} - \vec{a}$ is not $\frac{\pi}{2}$):

Lemma 2.1. $\vec{d} - \vec{c} = \frac{n}{n+1}(\vec{b} - \vec{a})$ and $\vec{q} - \vec{a}_0 = \frac{n}{n+1}(\vec{p} - \vec{a}_0)$, and thus $\|\vec{d} - \vec{c}\| = \frac{n}{n+1}\|\vec{b} - \vec{a}\|$ and $\vec{q} = \frac{n\vec{p} + 0.5}{n+1}$, where $\|\vec{b} - \vec{a}\|$ is the norm of vector $\vec{b} - \vec{a}$. Furthermore, since $\vec{c} = \frac{1}{n+1}(n\vec{a} + \vec{a}_0)$, the line segments $\overline{OM}$ (i.e., the line segment with O and M as endpoints), $\overline{OC}$, and $\overline{OA}$ are included in the same straight line, and the length of $\overline{MC}$ is $\frac{1}{n}$ of the length of $\overline{CA}$. Similarly, $\overline{OM}$, $\overline{OD}$, and $\overline{OB}$ are included in the same straight line, and the length of $\overline{MD}$ is $\frac{1}{n}$ of the length of $\overline{CB}$ (see Figure 13).

Proof. It can be seen from definitions of $\{\vec{a}_0, \vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{p}, \vec{q}\}$ that $\vec{d} - \vec{c} = \frac{n}{n+1}(\vec{b} - \vec{a})$ and $\vec{q} - \vec{a}_0 = \frac{n}{n+1}(\vec{p} - \vec{a}_0)$. Thus $\vec{q} = \frac{n}{n+1}\vec{p} + \frac{1}{n+1}\vec{a}_0$, which implies $\vec{q} = \frac{n\vec{p} + 0.5}{n+1}$. $\square$

Lemma 2.2.

- (1) If $\ddot{a}\ddot{b} \geq 0$, then $\delta_1 = 2|\vec{p}| + |\cos \theta| \|\vec{b} - \vec{a}\|$ and $2|\vec{p} - \vec{a}| = 2|\vec{p} - \vec{b}| = |\sin \theta| \|\vec{b} - \vec{a}\|$.
- (2) If $\ddot{a}\ddot{b} \leq 0$, then $\delta_1 = (|\sin \theta| + |\cos \theta|) \|\vec{b} - \vec{a}\|$.
- (3) $\delta_0 = 2|\vec{q}| + |\cos \theta| \|\vec{d} - \vec{c}\|$ if $\ddot{c}\ddot{d} \geq 0$, and $\delta_0 = (|\sin \theta| + |\cos \theta|) \|\vec{d} - \vec{c}\|$ if $\ddot{c}\ddot{d} \leq 0$ (see Figure 14).

Proof. (1) As $\theta \neq \frac{\pi}{2}$, $\dot{a} \neq \dot{b}$ and $|\dot{a} - \dot{b}| = |\cos \theta| \|\vec{b} - \vec{a}\|$. As $\ddot{a}\ddot{b} \geq 0$ and $P = (\dot{p}, \ddot{p})$ is the midpoint of line segment $\overline{AB}$, $|\dot{a}| + |\dot{b}| = |\dot{a} + \dot{b}| = |2\dot{p}| = 2|\dot{p}|$. Therefore,

$\delta_1 = |\ddot{a}| + |\ddot{b}| + |\dot{a} - \dot{b}| = 2|\ddot{p}| + |\cos \theta| \|\vec{b} - \vec{a}\|$ by definition of δ_1 . Without losing generality we assume $\ddot{a} \geq 0$ and $\ddot{b} \geq 0$. Then $\ddot{p} = \frac{1}{2}(\ddot{a} + \ddot{b}) \geq 0$ and thus $|\sin \theta| \|\vec{b} - \vec{a}\| = |\dot{a} - \dot{b}| = 2|\ddot{p} - \ddot{a}| = 2|\ddot{p} - \ddot{b}|$ (because $\vec{p} - \vec{a} = \vec{b} - \vec{p} = \frac{1}{2}(\vec{b} - \vec{a})$).

(2) As $\theta \neq \frac{\pi}{2}$, $\dot{a} \neq \dot{b}$ and $|\dot{a} - \dot{b}| = |\cos \theta| \|\vec{b} - \vec{a}\|$. If $\ddot{a}\ddot{b} < 0$, then $|\dot{a}| + |\dot{b}| = |\dot{a} - \dot{b}| = |\sin \theta| \|\vec{b} - \vec{a}\|$. If $\ddot{a}\ddot{b} = 0$ (without loss generality we assume $\ddot{a} = 0$), then $|\sin \theta| \|\vec{b} - \vec{a}\| = |\dot{b}| = |\dot{a}| + |\dot{b}|$. It follows that $\delta_1 = |\dot{a} - \dot{b}| + |\dot{a}| + |\dot{b}| = |\cos \theta| \|\vec{b} - \vec{a}\| + |\sin \theta| \|\vec{b} - \vec{a}\| = (|\sin \theta| + |\cos \theta|) \|\vec{b} - \vec{a}\|$.

(3) Similar to the proofs of (1) and (2). $\square$

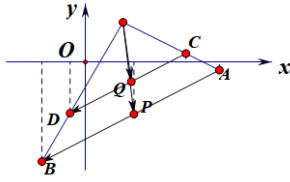


FIGURE 13. For Lemma 2.1

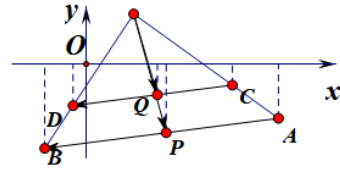


FIGURE 14. For Lemma 2.2

Proof of Theorem 2.1 (for the case $\vec{a} \neq \vec{b}$). (1) As $\dot{a} = \dot{b}$, $\delta_1 = \rho(\vec{a}, \vec{b}) = |\ddot{a} - \ddot{b}| = |\frac{1}{n} \sum_{j=1}^n \ddot{a}_j - \frac{1}{n} \sum_{j=1}^n \ddot{b}_j| = \frac{1}{n} |\sum_{j=1}^n \ddot{a}_j - \sum_{j=1}^n \ddot{b}_j|$ and $\delta_0 = \rho(\vec{c}, \vec{d}) = |\ddot{c} - \ddot{d}| = \frac{1}{n+1} |\sum_{j=0}^n \ddot{a}_j - \sum_{j=0}^n \ddot{b}_j| = \frac{1}{n+1} |\sum_{j=1}^n \ddot{a}_j - \sum_{j=1}^n \ddot{b}_j|$, which implies $\delta_0 \leq \delta_1$.

(2) By Lemma 2.1, $0 < \ddot{q} = \frac{n\ddot{p} + \ddot{a}_0}{n+1} \leq \ddot{p}$ (since $0 < \ddot{a}_0 \leq \ddot{p}$) and $\|\vec{d} - \vec{c}\| \leq \|\vec{b} - \vec{a}\|$; by Lemma 2.2(1), $\delta_1 = 2|\ddot{p}| + |\cos \theta| \|\vec{b} - \vec{a}\|$ (since $\ddot{a}\ddot{b} \geq 0$); by Lemma 2.2(3), $\delta_0 = 2|\ddot{q}| + |\cos \theta| \|\vec{d} - \vec{c}\| = 2\ddot{q} + |\cos \theta| \|\vec{d} - \vec{c}\| \leq 2\ddot{p} + |\cos \theta| \|\vec{d} - \vec{c}\| = 2|\ddot{p}| + |\cos \theta| \|\vec{d} - \vec{c}\|$ (since $\ddot{c}\ddot{d} \geq 0$). Therefore, $\delta_0 < \delta_1$.

(3) By Lemma 2.2(1), $\ddot{q} = \frac{n\ddot{p} + \ddot{a}_0}{n+1} \geq 0$. Without losing generality we assume $\ddot{a} \leq \ddot{b}$. By Lemma 2.2(1), $\delta_1 = 2\ddot{p} + |\cos \theta| \|\vec{b} - \vec{a}\| = 2\ddot{p} + |\dot{a} - \dot{b}|$ (since $\ddot{b} \geq \ddot{a} \geq 0$); by Lemmas 2.1 and 2.2(3), $\delta_0 = 2|\ddot{q}| + |\cos \theta| \|\vec{d} - \vec{c}\| = 2\ddot{q} + |\cos \theta| \|\vec{d} - \vec{c}\| = 2\ddot{q} + \frac{n}{n+1} |\cos \theta| \|\vec{b} - \vec{a}\| = 2\ddot{q} + \frac{n}{n+1} |\dot{a} - \dot{b}| = 2\frac{n\ddot{p} + \ddot{a}_0}{n+1} + \frac{n}{n+1} |\dot{a} - \dot{b}|$ (since $\ddot{q} = \frac{n\ddot{p} + \ddot{a}_0}{n+1}$). To prove $\delta_0 \leq \delta_1$, it suffices to prove that $\delta_1 - \delta_0 = \frac{1}{n+1} |\dot{a} - \dot{b}| + 2\frac{\ddot{p} - \ddot{a}_0}{n+1} \geq 0$ (equivalently, $|\dot{a} - \dot{b}| + 2\ddot{p} - 2\ddot{a}_0 \geq 0$). This holds because $2\ddot{p} = |\dot{a}| + |\dot{b}| = \ddot{a} + \ddot{b}$, $\ddot{a}_0 = 0.5$, and $\ddot{a} + \ddot{b} + |\dot{a} - \dot{b}| \geq 1$.

(4) We only prove the first case. Without losing generality we assume $\ddot{a} < 0, \ddot{b} \geq 0, \ddot{c} \geq 0, \ddot{d} \geq 0$. By Lemma 2.2, $\delta_1 = (|\sin \theta| + |\cos \theta|) \|\vec{b} - \vec{a}\| = |\dot{b} - \dot{a}| + |\dot{b} - \dot{a}| = |\dot{b} - \dot{a}| + \ddot{b} - \ddot{a}$; by Lemma 2.1, $\delta_0 = 2\ddot{q} + |\cos \theta| \|\vec{d} - \vec{c}\| = 2\frac{n\ddot{p} + \ddot{a}_0}{n+1} + \frac{n}{n+1} |\cos \theta| \|\vec{b} - \vec{a}\| =$

$2\frac{n\ddot{p}+\ddot{a}_0}{n+1} + \frac{n}{n+1}|\dot{b}-\dot{a}|$ (since $\ddot{q} \geq 0$). As

$$\begin{aligned}\delta_1 - \delta_0 &= \frac{1}{n+1}|\dot{b}-\dot{a}| + (\ddot{b}-\ddot{a}) - \frac{n}{n+1}(\ddot{b}+\ddot{a}) - \frac{2\ddot{a}_0}{n+1} \\ &= \frac{1}{n+1}|\dot{b}-\dot{a}| + \frac{1}{n+1}\ddot{b} - n\frac{2n+1}{n+1}\ddot{a} - \frac{2\ddot{a}_0}{n+1} \\ &= \frac{1}{n+1}|\dot{b}-\dot{a}| + \frac{1}{n+1}(\ddot{b}-\ddot{a}) - \frac{2n}{n+1}\ddot{a} - \frac{2\ddot{a}_0}{n+1} \\ &= \frac{1}{n+1}(|\dot{b}-\dot{a}| + |\ddot{b}-\ddot{a}|) - 2\frac{n\ddot{a}+\ddot{a}_0}{n+1},\end{aligned}$$

it suffices to prove that $\frac{1}{n+1}(|\dot{b}-\dot{a}| + |\ddot{b}-\ddot{a}|) - 2\frac{n\ddot{a}+\ddot{a}_0}{n+1} \geq 0$ (i.e., $|\dot{a}| + |\ddot{b}| + |\dot{a}-\dot{b}| \geq 2\ddot{a}_0 + 2n\ddot{a}$). This holds because $\ddot{a}_0 = 0.5$, $\ddot{a} < \ddot{b}$, and $|\dot{a}| + |\ddot{b}| + |\dot{a}-\dot{b}| \geq 1 + 2n \min\{\ddot{a}, \ddot{b}\}$.

(5) Without losing generality we assume $\ddot{a} < 0, \ddot{b} \geq 0, \ddot{c} < 0, \ddot{d} \geq 0$. By Lemma 2.2, $\delta_1 = (|\sin \theta| + |\cos \theta|)\|\vec{b}-\vec{a}\|$, $\delta_0 = (|\sin \theta| + |\cos \theta|)\|\vec{d}-\vec{c}\|$. It follows that $\delta_1 - \delta_0 = \frac{1}{n+1}(|\sin \theta| + |\cos \theta|)\|\vec{b}-\vec{a}\| \geq 0$, i.e., $\delta_0 \leq \delta_1$ holds.

(6) By Lemma 2.2, $\delta_1 = 2|\ddot{p}| + |\cos \theta|\|\vec{b}-\vec{a}\| = 2|\ddot{p}| + |\dot{b}-\dot{a}| = |\dot{b}-\dot{a}| - 2\ddot{p}$ (since $\ddot{p} = \frac{1}{2}(\ddot{a} + \ddot{b}) < 0$); $\delta_0 = 2|\ddot{q}| + |\cos \theta|\|\vec{d}-\vec{c}\| = 2|\ddot{q}| + |\dot{d}-\dot{c}| = 2\ddot{q} + |\dot{d}-\dot{c}| = 2\frac{n\ddot{p}+\ddot{a}_0}{n+1} + \frac{n}{n+1}|\dot{b}-\dot{a}|$ (since $\ddot{q} = \frac{n\ddot{p}+\ddot{a}_0}{n+1} \geq 0$). As

$$\begin{aligned}\delta_1 - \delta_0 &= \frac{1}{n+1}|\dot{b}-\dot{a}| - (\ddot{a} + \ddot{b}) - \frac{n(\ddot{a} + \ddot{b}) + 2\ddot{a}_0}{n+1} \\ &= \frac{1}{n+1}[|\dot{b}-\dot{a}| - (2n+1)(\ddot{a} + \ddot{b}) - 2\ddot{a}_0] \\ &= \frac{1}{n+1}[|\dot{b}-\dot{a}| - \ddot{b} - \ddot{a} - 2n(\ddot{a} + \ddot{b}) - 2\ddot{a}_0] \\ &= \frac{1}{n+1}[|\dot{b}-\dot{a}| + |\ddot{b}| + |\ddot{a}| - 2n(\ddot{a} + \ddot{b}) - 2\ddot{a}_0],\end{aligned}$$

it suffices to prove that $\frac{1}{n+1}[|\dot{b}-\dot{a}| + |\ddot{b}| + |\ddot{a}| - 2n(\ddot{a} + \ddot{b}) - 2\ddot{a}_0] \geq 0$ (i.e., $|\dot{b}-\dot{a}| + |\ddot{b}| + |\ddot{a}| \geq 2n(\ddot{a} + \ddot{b}) + 2\ddot{a}_0$). This holds because $\ddot{a}_0 = 0.5$ and $|\dot{b}-\dot{a}| + |\ddot{b}| + |\ddot{a}| \geq 1 + 2n(\ddot{a} + \ddot{b})$.

(7) Without losing generality we assume $\ddot{a} < 0, \ddot{b} < 0, \ddot{c} \geq 0, \ddot{d} < 0$. By Lemma 2.2, $\delta_0 = |\ddot{c}| + |\ddot{d}| + |\cos \theta|\|\vec{d}-\vec{c}\| = \ddot{c} - \ddot{d} + |\dot{d}-\dot{c}| = \frac{n\ddot{a}+\ddot{a}_0}{n+1} - \frac{n\ddot{b}+\ddot{a}_0}{n+1} + |\dot{d}-\dot{c}| = \frac{n}{n+1}(\ddot{a}-\ddot{b}) + \frac{n}{n+1}|\dot{b}-\dot{a}|$ (since $\ddot{c} \geq 0, \ddot{d} < 0$); $\delta_1 = |\ddot{a}| + |\ddot{b}| + |\cos \theta|\|\vec{b}-\vec{a}\| = |\dot{b}-\dot{a}| - \ddot{a} - \ddot{b}$ (since $\ddot{a} < 0, \ddot{b} < 0$). As $\delta_1 - \delta_0 = \frac{1}{n+1}|\dot{b}-\dot{a}| - \frac{(2n+1)\ddot{a}+\ddot{b}}{n+1} = \frac{1}{n+1}[|\dot{b}-\dot{a}| - ((2n+1)\ddot{a} + \ddot{b})]$, it suffices to prove that $|\dot{b}-\dot{a}| - ((2n+1)\ddot{a} + \ddot{b}) \geq 0$ (i.e., $|\dot{b}-\dot{a}| \geq (2n+1)\ddot{a} + \ddot{b}$). This holds because $\ddot{a} < 0, \ddot{b} < 0$, and $|\dot{b}-\dot{a}| \geq 0 > (2n+1)\ddot{a} + \ddot{b}$.

(8) By Lemma 2.2, $\delta_1 = |\cos \theta| \|\vec{b} - \vec{a}\| + |\ddot{a}| + |\ddot{b}| = |\dot{b} - \dot{a}| + |\ddot{a}| + |\ddot{b}|$;

$$\begin{aligned} \delta_0 &= |\cos \theta| \|\vec{d} - \vec{c}\| + |\ddot{c}| + |\ddot{d}| = |\dot{d} - \dot{c}| + |\ddot{c}| + |\ddot{d}| \\ &= \frac{n}{n+1} |\dot{b} - \dot{a}| + \left| \frac{n\ddot{a} + \ddot{a}_0}{n+1} \right| + \left| \frac{n\ddot{b} + \ddot{a}_0}{n+1} \right| < |\dot{b} - \dot{a}| + |\ddot{a}| + |\ddot{b}| \end{aligned}$$

(since $\ddot{a} < 0$, $\ddot{c} < 0$, $\ddot{b} < 0$, $\ddot{d} < 0$). Therefore, $\delta_0 < \delta_1$.

Apart from the above 8 cases, there are 3 cases that need to be considered:

(9) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \subseteq R^+$, $0 < \ddot{p} < 0.5$, $\ddot{a} + \ddot{b} + |\dot{a} - \dot{b}| < 1$.

(10) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}\} \cap R^-$ is a singleton, $\{\ddot{c}, \ddot{d}\} \subseteq R^+$, $|\ddot{a}| + |\ddot{b}| + |\dot{a} - \dot{b}| < 1 + 2n \min\{\ddot{a}, \ddot{b}\}$.

(11) $\dot{a} \neq \dot{b}$, $\{\ddot{a}, \ddot{b}, \ddot{c}, \ddot{d}\} \cap R^+ = \{\ddot{c}, \ddot{d}\}$, $|\ddot{a}| + |\ddot{b}| + |\dot{a} - \dot{b}| < 1 + 2n(\ddot{a} + \ddot{b})$.

It can be proven that $\delta_1 < \delta_0$ in each of the three cases. Next, we will prove this only for case (9). Without losing generality we assume $\ddot{a} \leq \ddot{b}$. By Lemma 2.2(1), $\delta_1 = |\cos \theta| \|\vec{b} - \vec{a}\| + 2\ddot{p}$ and $\delta_0 = |\cos \theta| \|\vec{d} - \vec{c}\| + 2\ddot{q}$. It follows that

$$\begin{aligned} \delta_1 - \delta_0 &= \frac{1}{n+1} |\cos \theta| \|\vec{b} - \vec{a}\| + 2\ddot{p} - 2\ddot{q} = \frac{1}{n+1} |\dot{b} - n\dot{a}| + \ddot{a} + \ddot{b} - 2\frac{n\ddot{p} + \ddot{a}_0}{n+1} \\ &= \frac{1}{n+1} (|\dot{b} - \dot{a}| + \ddot{a} + \ddot{b} - 2\ddot{a}_0) < 0 \end{aligned}$$

(since $2\ddot{p} = \ddot{a} + \ddot{b}$, $\ddot{a}_0 = 0.5$, and $\ddot{q} = \frac{n\ddot{p} + \ddot{a}_0}{n+1}$), i.e., $\delta_1 < \delta_0$. $\square$

Remark 2.1. 11 cases in Theorem 2.1 are obtained in the following steps:

Step 1. Denote the vectors or points $\vec{a}_0, \vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ by M, A, B, C , and D respectively. Since C and D can be determined by M, A and B , the primary focus should be on the locations of M, A and B in R^2 . Furthermore, since M is fixed, we should give more priority to considering locations of A and B in R^2 . To do this, we should distinguish $\dot{a} = \dot{b}$ and $\dot{a} \neq \dot{b}$.

Step 2. The case $\dot{a} = \dot{b}$ corresponds exactly to (1) of Theorem 2.1.

Step 3. For $\dot{a} \neq \dot{b}$, it can be divided into the following three cases by considering locations of A and B in R^2 :

Case 1. Neither A and B is in the lower half-plane $R_-^2 = \{(x, y) \in R^2 \mid y < 0\}$ of R^2 , i.e., $\ddot{a} \geq 0$ and $\ddot{b} \geq 0$. In this case, neither C and D is in R_-^2 , i.e., both $\ddot{c} \geq 0$ and $\ddot{d} \geq 0$ hold. It corresponds exactly to (2), (3), and (9) of Theorem 2.1.

Case 2. Just one of A and B is in R_-^2 , i.e., just one of $\ddot{a} < 0$ and $\ddot{b} < 0$ holds. Without losing of generality we assume that A is in R_-^2 but B is not, i.e., $\ddot{a} < 0$ and

$\ddot{b} \geq 0$ hold. Then it can be divided into the following two subcases by considering locations of C and D in R^2 :

Subcase 2.1. Neither C and D is in R_-^2 , i.e., $\ddot{c} \geq 0$ and $\ddot{d} \geq 0$. It corresponds exactly to (4) and (10) of Theorem 2.1.

Subcase 2.2. Either C or D is in R_-^2 . As A is in R_-^2 but B is not, C is in R_-^2 but D is not, i.e., $\ddot{c} < 0$ but $\ddot{d} \geq 0$ hold. It corresponds exactly to (5) of Theorem 2.1.

Case 3. Both A and B are in R_-^2 , i.e., both $\ddot{a} < 0$ and $\ddot{b} < 0$ hold. This can be divided into the following three subcases by considering locations of C and D in R^2 :

Subcase 3.1. Neither C and D is in R_-^2 , i.e., $\ddot{c} \geq 0$ and $\ddot{d} \geq 0$. It corresponds exactly to (6) of Theorem 2.1.

Subcase 3.2. Just one of C and D is in R_-^2 , i.e., $\ddot{c} < 0$ but $\ddot{d} \geq 0$ or $\ddot{d} < 0$ but $\ddot{c} \geq 0$ hold. It corresponds exactly to (7) and (11) of Theorem 2.1.

Subcase 3.3. Both C and D are in R_-^2 . As $\{A, B\} \subseteq R_-^2$ (i.e., $\ddot{a} < 0$ and $\ddot{b} < 0$ hold), it corresponds exactly to (8) of Theorem 2.1.

Remark 2.2. When we begin this study, some high school students in Yuci No.1 Middle School of Shanxi Province have been invited or encouraged to participate in this research process. They witnessed almost the entire process of our paper writing. Quit a few of them can understand Remark 2.1 and can roughly understand the origin and formation process of Theorem 2.1. Some of them can understand the content and proof of Theorem 2.1. Unexpectedly, student Chu Fan Zheng can explain Theorem 2.1 and its proof in detail, and ask if she can continue this writing. This urges us to put forward the following questions:

Question 2.1. Can $\vec{a}_0 = (0.5, 0.5)$ (written also as $\langle 0.5, 0.5 \rangle$ or **0.5**) in Theorem 2.1 be replaced by an arbitrary point in R^2 ? And can you define another river metric $\tilde{\rho}_3$ on R^2 by replacing the “river” $\{(x, 0) \mid x \in R\}$ by another “river” $\{(x, \sin x) \mid x \in R\}$?

Question 2.2. What is the analogous of Theorem 2.1 in R^3 ?

The first half of Question 2.1 was solved by high school students at Yuci No.1 Middle School of Shanxi Province. Their conclusion states: “From its proof, we can see that $\vec{a}_0 = (0.5, 0.5)$ in Theorem 2.1 can be replaced by an arbitrary point $(r, r) \in R^2$ satisfying $r > 0$.”; this suggests that they may have proposed a new question based on the observation that the first half of Question 2.1 is just one possible generalization of Theorem 2.1. The second half of Question 2.1 was

solved by an undergraduate student, named Yang Yang Feng, in Shaanxi Normal University. His conclusion states much more: definition of the river metric on R^2 can be generalized by replacing the “river” $\{(x, 0) \mid x \in R\}$ by another “river” $\{(x, f(x)) \mid x \in R\}$, where f is an arbitrary continuous function on R such that $\widehat{P_i P_{i+1}}$ is rectifiable curve ($P_i = (i, f(i))$) for each integer i ; other generalizations of “river” (and thus “river” metric and “river” metric inequality) will be considered by other students when they pick up a package, design roads or internet. Question 2.2 will be discussed in the next section, where additional problems will be introduced and solved.

Remark 2.3. *While high school students should not rush into solving big scientific problems prematurely, it is beneficial for-especially gifted high school students-to participate in innovation training within their capabilities. This allows them to achieve visible results early. Such engagement is similar to the developmental process of athletes. The positive impact of Theorem 2.1 on these students lies in providing opportunities for training and growth. It not only helps them develop their skills but also fosters a collaborative learning environment where they can teach others (naturally, it is also conducive to cultivating altruists):*

- (1) *It motivated Zheng, a student from Yuci No.1 Middle School, to actively participate in the research process, leading to the development of research ideas and even a certain prototype of research.*
- (2) *It enabled other students at Yuci No.1 Middle School to witness the formation of Theorem 2.1 and its proof. They saw how Zheng, through this experience, developed the idea of research, creating an environment of mutual learning.*

3. AN ANALOGOUS OF THEOREM 2.1 IN R^3

The intuitiveness and other goodness of the river metric ϱ_3 on R^2 lead us to define its analogous in R^3 , which is a crucial step toward extending Theorem 2.1 to the case of R^m ($m > 2$).

Definition 3.1. *The first river metric $\rho_{\perp}$ and the second river metric $\rho_{\parallel}$ on R^3 may be defined by $(\vec{a} = (\dot{a}, \ddot{a}, \bar{a}), \vec{b} = (\dot{b}, \ddot{b}, \bar{b}))$*

$$\rho_{\perp}(\vec{a}, \vec{b}) = \begin{cases} \rho_1(\bar{a}, \bar{b}), & (\dot{a}, \ddot{a}) = (\dot{b}, \ddot{b}), \\ \rho_1(\bar{a}, 0) + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + \rho_1(\bar{b}, 0), & (\dot{a}, \ddot{a}) \neq (\dot{b}, \ddot{b}), \end{cases}$$

and

$$\rho_{\top}(\vec{a}, \vec{b}) = \begin{cases} \rho_2((\ddot{a}, \bar{a}), (\ddot{b}, \bar{b})), & \dot{a} = \dot{b}, \\ \rho_2((\ddot{a}, \bar{a}), (0, 0)) + \rho_1(\dot{a}, \dot{b}) + \rho_2((\ddot{b}, \bar{b}), (0, 0)), & \dot{a} \neq \dot{b}. \end{cases}$$

Remark 3.1. $\rho_1(a, b)$ (resp., $\rho_2((a_1, b_1), (0, 0))$) can also be written as $|a - b|$ (resp., $\|(a_1, b_1)\|$).

It is natural to seek an analogous result of Theorem 2.1 in R^3 ; and this can be seen in Theorem 3.1, which is a partial embryonic form for further generalization of Theorem 2.1.

Theorem 3.1. Let $\vec{a}_0 = (\dot{a}_0, \ddot{a}_0, \bar{a}_0) = (\dot{b}_0, \ddot{b}_0, \bar{b}_0) = \vec{b}_0$, $\dot{b}_0 = \ddot{b}_0 = \bar{b}_0 = 0.5$, $\vec{a}_1 = (\dot{a}_1, \ddot{a}_1, \bar{a}_1)$, $\vec{a}_2 = (\dot{a}_2, \ddot{a}_2, \bar{a}_2)$, $\dots$, $\vec{a}_n = (\dot{a}_n, \ddot{a}_n, \bar{a}_n)$, $\vec{b}_1 = (\dot{b}_1, \ddot{b}_1, \bar{b}_1)$, $\vec{b}_2 = (\dot{b}_2, \ddot{b}_2, \bar{b}_2)$, $\dots$, $\vec{b}_n = (\dot{b}_n, \ddot{b}_n, \bar{b}_n)$, $\vec{a} = \frac{1}{n} \sum_{j=1}^n \vec{a}_j = (\dot{a}, \ddot{a}, \bar{a})$, $\vec{b} = \frac{1}{n} \sum_{j=1}^n \vec{b}_j = (\dot{b}, \ddot{b}, \bar{b})$, $\vec{c} = \frac{1}{n+1} \sum_{j=0}^n \vec{a}_j = (\dot{c}, \ddot{c}, \bar{c})$, $\vec{d} = \frac{1}{n+1} \sum_{j=0}^n \vec{b}_j = (\dot{d}, \ddot{d}, \bar{d})$, $\vec{p} = (\dot{p}, \ddot{p}, \bar{p}) = 0.5(\vec{a} + \vec{b})$, $\vec{q} = (\dot{q}, \ddot{q}, \bar{q}) = 0.5(\vec{c} + \vec{d})$, $\delta_0 = \rho_{\perp}(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j, \frac{1}{n+1} \sum_{j=0}^n \vec{b}_j)$, and $\delta_1 = \rho_{\perp}(\frac{1}{n} \sum_{j=1}^n \vec{a}_j, \frac{1}{n} \sum_{j=1}^n \vec{b}_j)$. Then $\rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = \frac{n}{n+1} \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))$, and the river metric inequality $\delta_0 \leq \delta_1$ holds if and only if one of the following conditions is satisfied:

- (1) $(\dot{a}, \ddot{a}) = (\dot{b}, \ddot{b})$.
- (2) $(\dot{a}, \ddot{a}) \neq (\dot{b}, \ddot{b})$, $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\} \subseteq R^+$, $\bar{p} \geq 0.5$.
- (3) $(\dot{a}, \ddot{a}) \neq (\dot{b}, \ddot{b})$, $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\} \subseteq R^+$, $0 < \bar{p} < 0.5$, $\bar{a} + \bar{b} + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \geq 1$.
- (4) $(\dot{a}, \ddot{a}) \neq (\dot{b}, \ddot{b})$, $\{\bar{a}, \bar{b}\} \cap R^-$ (resp., $\{\bar{c}, \bar{d}\} \cap R^-$) is a singleton, $\{\bar{c}, \bar{d}\} \subseteq R^+$ (resp., $\{\bar{a}, \bar{b}\} \subseteq R^+$), $|\bar{a}| + |\bar{b}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \geq 1 + 2n \min\{\bar{a}, \bar{b}\}$ (resp., $|\bar{c}| + |\bar{d}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) \geq 1 + 2n \min\{\bar{c}, \bar{d}\}$).
- (5) $(\dot{a}, \ddot{a}) \neq (\dot{b}, \ddot{b})$, $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\} \cap R^+ \in \{\{\bar{a}, \bar{c}\}, \{\bar{b}, \bar{d}\}\}$.
- (6) $(\dot{a}, \ddot{a}) \neq (\dot{b}, \ddot{b})$, $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\} \cap R^+ = \{\bar{c}, \bar{d}\}$, $|\bar{a}| + |\bar{b}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \geq 1 + 2n(\bar{a} + \bar{b})$.
- (7) $(\dot{a}, \ddot{a}) \neq (\dot{b}, \ddot{b})$, $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\} \cap R^+ \in \{\{\bar{c}\}, \{\bar{d}\}\}$.
- (8) $(\dot{a}, \ddot{a}) \neq (\dot{b}, \ddot{b})$, $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\} \cap R^+ = \emptyset$.

We need to prove first that Definition 3.1 is well-defined. Let $\vec{x} = (\dot{x}, \ddot{x}, \bar{x})$, $\vec{y} = (\dot{y}, \ddot{y}, \bar{y})$, and $\vec{z} = (\dot{z}, \ddot{z}, \bar{z})$ be arbitrary points taken from R^3 . We only prove $\rho_i(\vec{x}, \vec{z}) \leq \rho_i(\vec{x}, \vec{y}) + \rho_i(\vec{y}, \vec{z})$ ($i = \perp, \top$).

Firstly, we prove $\rho_{\perp}(\vec{x}, \vec{z}) \leq \rho_{\perp}(\vec{x}, \vec{y}) + \rho_{\perp}(\vec{y}, \vec{z})$.

Case 1. $(\dot{x}, \ddot{x}) = (\dot{z}, \ddot{z})$. Then

$$\begin{aligned} \rho_{\perp}(\vec{x}, \vec{z}) &= |\bar{x} - \bar{z}| \leq \min \{ |\bar{x} - \bar{y}|, |\bar{x}| + \rho_2((\dot{x}, \ddot{x}), (\dot{y}, \ddot{y})) + |\bar{y}| \} \\ &\quad + \min \{ |\bar{y} - \bar{z}|, |\bar{y}| + \rho_2((\dot{y}, \ddot{y}), (\dot{z}, \ddot{z})) + |\bar{z}| \} \leq \rho_{\perp}(\vec{x}, \vec{y}) + \rho_{\perp}(\vec{y}, \vec{z}). \end{aligned}$$

Case 2. $(\dot{x}, \ddot{x}) \neq (\dot{z}, \ddot{z})$. We consider the following 3 subcases:

Subcase 1. $(\dot{x}, \ddot{x}) = (\dot{y}, \ddot{y})$. Then $(\dot{y}, \ddot{y}) \neq (\dot{z}, \ddot{z})$. As $\rho_2((\dot{x}, \ddot{x}), (\dot{y}, \ddot{y})) = 0$,

$$\begin{aligned} \rho_{\perp}(\vec{x}, \vec{z}) &= \rho_2((\dot{x}, \ddot{x}), (\dot{z}, \ddot{z})) + |\bar{x}| + |\bar{z}| \\ &\leq \rho_2((\dot{x}, \ddot{x}), (\dot{y}, \ddot{y})) + |\bar{x}| + |\bar{y}| + \rho_2((\dot{y}, \ddot{y}), (\dot{z}, \ddot{z})) + |\bar{y}| + |\bar{z}| \\ &= \rho_{\perp}(\vec{x}, \vec{y}) + \rho_{\perp}(\vec{y}, \vec{z}). \end{aligned}$$

Subcase 2. $(\dot{x}, \ddot{x}) \neq (\dot{y}, \ddot{y})$ and $(\dot{z}, \ddot{z}) = (\dot{y}, \ddot{y})$. Analogously to Subcase 1, we can prove $\rho_{\perp}(\vec{x}, \vec{z}) \leq \rho_{\perp}(\vec{x}, \vec{y}) + \rho_{\perp}(\vec{y}, \vec{z})$.

Subcase 3. $(\dot{x}, \ddot{x}) \neq (\dot{y}, \ddot{y})$ and $(\dot{z}, \ddot{z}) \neq (\dot{y}, \ddot{y})$. Then

$$\begin{aligned} \rho_{\perp}(\vec{x}, \vec{z}) &= \rho_2((\dot{x}, \ddot{x}), (\dot{z}, \ddot{z})) + |\bar{x}| + |\bar{z}| \\ &\leq \rho_2((\dot{x}, \ddot{x}), (\dot{y}, \ddot{y})) + |\bar{x}| + |\bar{y}| + \rho_2((\dot{y}, \ddot{y}), (\dot{z}, \ddot{z})) + |\bar{y}| + |\bar{z}| \\ &= \rho_{\perp}(\vec{x}, \vec{y}) + \rho_{\perp}(\vec{y}, \vec{z}). \end{aligned}$$

Secondly, we prove $\rho_{\top}(\vec{x}, \vec{z}) \leq \rho_{\top}(\vec{x}, \vec{y}) + \rho_{\top}(\vec{y}, \vec{z})$.

Case 1. $\dot{x} = \dot{z}$. Then

$$\begin{aligned} \rho_{\top}(\vec{x}, \vec{z}) &= \rho_2((\ddot{x}, \bar{x}), (\ddot{z}, \bar{z})) \\ &\leq \min \{ \rho_2((\ddot{x}, \bar{x}), (\ddot{y}, \bar{y})), |\dot{x} - \dot{y}| + \rho_2((\ddot{x}, \bar{x}), (0, 0)) + \rho_2((\ddot{y}, \bar{y}), (0, 0)) \} \\ &\quad + \min \{ \rho_2((\ddot{y}, \bar{y}), (\ddot{z}, \bar{z})), |\dot{y} - \dot{z}| + \rho_2((\ddot{y}, \bar{y}), (0, 0)) + \rho_2((\ddot{z}, \bar{z}), (0, 0)) \} \\ &\leq \rho_{\top}(\vec{x}, \vec{y}) + \rho_{\top}(\vec{y}, \vec{z}). \end{aligned}$$

Case 2. $\dot{x} \neq \dot{z}$. We consider the following 3 subcases:

Subcase 1. $\dot{x} = \dot{y}$. Then $\dot{y} \neq \dot{z}$. As $|\dot{x} - \dot{y}| = 0$,

$$\begin{aligned} \rho_{\top}(\vec{x}, \vec{z}) &= |\dot{x} - \dot{z}| + \rho_2((\ddot{x}, \bar{x}), (0, 0)) + \rho_2((\ddot{z}, \bar{z}), (0, 0)) \\ &= |\dot{x} - \dot{y}| + |\dot{y} - \dot{z}| + \rho_2((\ddot{x}, \bar{x}), (0, 0)) + \rho_2((\ddot{z}, \bar{z}), (0, 0)) \\ &\leq \rho_2((\ddot{x}, \bar{x}), (\ddot{y}, \bar{y})) + |\dot{y} - \dot{z}| + \rho_2((\ddot{y}, \bar{y}), (0, 0)) + \rho_2((\ddot{z}, \bar{z}), (0, 0)) \\ &= \rho_{\top}(\vec{x}, \vec{y}) + \rho_{\top}(\vec{y}, \vec{z}). \end{aligned}$$

Subcase 2. $\dot{x} \neq \dot{y}$ and $\dot{z} = \dot{y}$. Analogously to Subcase 1, we can prove $\rho_{\uparrow}(\vec{x}, \vec{z}) \leq \rho_{\uparrow}(\vec{x}, \vec{y}) + \rho_{\uparrow}(\vec{y}, \vec{z})$.

Subcase 3. $\dot{x} \neq \dot{y}$ and $\dot{y} \neq \dot{z}$. Then

$$\begin{aligned} \rho_{\uparrow}(\vec{x}, \vec{z}) &= |\dot{x} - \dot{z}| + \rho_2((\ddot{x}, \bar{x}), (0, 0)) + \rho_2((\ddot{z}, \bar{z}), (0, 0)) \\ &\leq |\dot{x} - \dot{y}| + \rho_2((\ddot{x}, \bar{x}), (0, 0)) + \rho_2((\ddot{y}, \bar{y}), (0, 0)) + |\dot{y} - \dot{z}| \\ &\quad + \rho_2((\ddot{y}, \bar{y}), (0, 0)) + \rho_2((\ddot{z}, \bar{z}), (0, 0)) \\ &= \rho_{\uparrow}(\vec{x}, \vec{y}) + \rho_{\uparrow}(\vec{y}, \vec{z}). \end{aligned}$$

To prove Theorem 3.1, we also need the following Lemma 3.1 (which can be proved as Lemma 2.2):

Lemma 3.1. (1) If $\bar{a}\bar{b} \geq 0$, then $\delta_1 = 2|\bar{p}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))$ and $2|\bar{p} - \bar{a}| = 2|\bar{p} - \bar{b}| = |\bar{b} - \bar{a}|$.
 (2) If $\bar{a}\bar{b} \leq 0$, then $\delta_1 = |\bar{b} - \bar{a}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))$.
 (3) $\delta_0 = 2|\bar{q}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d}))$ if $\bar{c}\bar{d} \geq 0$, and $\delta_0 = |\bar{d} - \bar{c}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d}))$ if $\bar{c}\bar{d} \leq 0$.

Proof. (1) As $\bar{a}\bar{b} \geq 0$ and $P = (\dot{p}, \ddot{p}, \bar{p})$, $|\bar{a}| + |\bar{b}| = |\bar{a} + \bar{b}| = |2\bar{p}| = 2|\bar{p}|$. Therefore, $\delta_1 = |\bar{a}| + |\bar{b}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) = 2|\bar{p}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))$ by definition of δ_1 . Without losing of generality we assume $\bar{a} \geq 0$ and $\bar{b} \geq 0$. Then $\bar{p} = \frac{1}{2}(\bar{a} + \bar{b}) \geq 0$ and thus $|\bar{a} - \bar{b}| = 2|\bar{p} - \bar{a}| = 2|\bar{p} - \bar{b}|$ (because $\vec{p} - \vec{a} = \vec{b} - \vec{p} = \frac{1}{2}(\vec{b} - \vec{a})$).

(2) If $\bar{a}\bar{b} < 0$, then $|\bar{a}| + |\bar{b}| = |\bar{a} - \bar{b}|$. If $\bar{a}\bar{b} = 0$ (without loss generality we assume $\bar{a} = 0$), then $|\bar{b}| = |\bar{a}| + |\bar{b}|$. It follows that

$$\begin{aligned} \delta_1 &= \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{a}| + |\bar{b}| = \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b}| \\ &= \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b} - \bar{a}|. \end{aligned}$$

(3) Similar to the proofs of (1) and (2). □

Proof of Theorem 3.1 (for the case $\vec{a} \neq \vec{b}$). (1) As $(\dot{a}, \ddot{a}) = (\dot{b}, \ddot{b})$, $\delta_1 = \rho_{\downarrow}(\vec{a}, \vec{b}) = |\bar{a} - \bar{b}| = |\frac{1}{n} \sum_{j=1}^n \bar{a}_j - \frac{1}{n} \sum_{j=1}^n \bar{b}_j| = \frac{1}{n} |\sum_{j=1}^n \bar{a}_j - \sum_{j=1}^n \bar{b}_j|$ and $\delta_0 = \rho_{\downarrow}(\vec{c}, \vec{d}) = |\bar{c} - \bar{d}| = \frac{1}{n+1} |\sum_{j=0}^n \bar{a}_j - \sum_{j=0}^n \bar{b}_j| = \frac{1}{n+1} |\sum_{j=1}^n \bar{a}_j - \sum_{j=1}^n \bar{b}_j|$, which implies $\delta_0 \leq \delta_1$.

(2) By Proof of Lemma 2.1, $0 < \bar{q} = \frac{n\bar{p} + \bar{a}_0}{n+1} \leq \bar{p}$ (since $0 < \bar{a}_0 \leq \bar{p}$) and $\|\vec{d} - \vec{c}\| \leq \|\vec{b} - \vec{a}\|$; by Lemma 3.1(1), $\delta_1 = 2|\bar{p}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))$ (since $\bar{a}\bar{b} \geq 0$); by Lemma

3.1(3),

$$\begin{aligned}\delta_0 &= 2|\bar{q}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = 2\bar{q} + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) \\ &\leq 2\bar{p} + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = 2|\bar{p}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d}))\end{aligned}$$

(since $\bar{c}\bar{d} \geq 0$). Therefore, $\delta_0 < \delta_1$.

(3) By Lemma 3.1(1), $\bar{q} = \frac{n\bar{p} + \bar{a}_0}{n+1} \geq 0$. Without losing generality we assume $\bar{a} \leq \bar{b}$. By Lemma 3.1(1), $\delta_1 = 2\bar{p} + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) = 2\bar{p} + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))$ (since $\bar{b} \geq \bar{a} \geq 0$); by Proof of Lemma 2.1 and Lemma 3.1(3),

$$\begin{aligned}\delta_0 &= 2|\bar{q}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = 2\bar{q} + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = 2\bar{q} + \frac{n}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \\ &= 2\bar{q} + \frac{n}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) = 2\frac{n\bar{p} + \bar{a}_0}{n+1} + \frac{n}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))\end{aligned}$$

(since $\bar{q} = \frac{n\bar{p} + \bar{a}_0}{n+1}$). To prove $\delta_0 \leq \delta_1$, it suffices to prove that $\delta_1 - \delta_0 = \frac{1}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + 2\frac{\bar{p} - \bar{a}_0}{n+1} \geq 0$ (equivalently, $\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + 2\bar{p} - 2\bar{a}_0 \geq 0$). This holds because $2\bar{p} = |\bar{a}| + |\bar{b}| = \bar{a} + \bar{b}$, $\bar{a}_0 = 0.5$, and $\bar{a} + \bar{b} + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \geq 1$.

(4) We only prove the first case. Without losing generality we assume $\bar{a} < 0, \bar{b} \geq 0, \bar{c} \geq 0, \bar{d} \geq 0$. By Lemma 3.1, $\delta_1 = \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b} - \bar{a}| = \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + \bar{b} - \bar{a}$; by Proof of Lemma 2.1,

$$\begin{aligned}\delta_0 &= 2\bar{q} + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = 2\frac{n\bar{p} + \bar{a}_0}{n+1} + \frac{n}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \\ &= 2\frac{n\bar{p} + \bar{a}_0}{n+1} + \frac{n}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))\end{aligned}$$

(since $\bar{q} \geq 0$). As

$$\begin{aligned}\delta_1 - \delta_0 &= \frac{1}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + (\bar{b} - \bar{a}) - \frac{n}{n+1}(\bar{b} + \bar{a}) - \frac{2\bar{a}_0}{n+1} \\ &= \frac{1}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + \frac{1}{n+1}\bar{b} - \frac{2n+1}{n+1}\bar{a} - \frac{2\bar{a}_0}{n+1} \\ &= \frac{1}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + \frac{1}{n+1}(\bar{b} - \bar{a}) - \frac{2n}{n+1}\bar{a} - \frac{2\bar{a}_0}{n+1} \\ &= \frac{1}{n+1}(\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b} - \bar{a}|) - 2\frac{n\bar{a} + \bar{a}_0}{n+1},\end{aligned}$$

it suffices to prove that $\frac{1}{n+1}(\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b} - \bar{a}|) - 2\frac{n\bar{a} + \bar{a}_0}{n+1} \geq 0$ (i.e., $|\bar{a}| + |\bar{b}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \geq 2\bar{a}_0 + 2n\bar{a}$). This holds because $\bar{a}_0 = 0.5$, $\bar{a} < \bar{b}$, and $|\bar{a}| + |\bar{b}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \geq 1 + 2n \min\{\bar{a}, \bar{b}\}$.

(5) Without losing generality we assume $\bar{a} < 0, \bar{b} \geq 0, \bar{c} < 0, \bar{d} \geq 0$. By Lemma 3.1, $\delta_1 = (\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))) + |\bar{b} - \bar{a}|$, $\delta_0 = (|\sin \theta| + |\cos \theta|)\|\vec{d} - \vec{c}\|$. It follows that $\delta_1 - \delta_0 = \frac{1}{n+1}(\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b}))) + |\bar{b} - \bar{a}| \geq 0$, i.e., $\delta_0 \leq \delta_1$ holds.

(6) By Lemma 3.1,

$$\delta_1 = 2|\bar{p}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) = 2|\bar{p}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) = \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) - 2\bar{p}$$

(since $\bar{p} = \frac{1}{2}(\bar{a} + \bar{b}) < 0$);

$$\begin{aligned} \delta_0 &= 2|\bar{q}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = 2|\bar{q}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = 2\bar{q} + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) \\ &= 2\frac{n\bar{p} + \bar{a}_0}{n+1} + \frac{n}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \end{aligned}$$

(since $\bar{q} = \frac{n\bar{p} + \bar{a}_0}{n+1} \geq 0$). As

$$\begin{aligned} \delta_1 - \delta_0 &= \frac{1}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) - (\bar{a} + \bar{b}) - \frac{n(\bar{a} + \bar{b}) + 2\bar{a}_0}{n+1} \\ &= \frac{1}{n+1}[\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) - (2n+1)(\bar{a} + \bar{b}) - 2\bar{a}_0] \\ &= \frac{1}{n+1}[\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) - \bar{b} - \bar{a} - 2n(\bar{a} + \bar{b}) - 2\bar{a}_0] \\ &= \frac{1}{n+1}[\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b}| + |\bar{a}| - 2n(\bar{a} + \bar{b}) - 2\bar{a}_0], \end{aligned}$$

it suffices to prove that $\frac{1}{n+1}[\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b}| + |\bar{a}| - 2n(\bar{a} + \bar{b}) - 2\bar{a}_0] \geq 0$ (i.e., $\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b}| + |\bar{a}| \geq 2n(\bar{a} + \bar{b}) + 2\bar{a}_0$). This holds because $\bar{a}_0 = 0.5$ and $\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{b}| + |\bar{a}| \geq 1 + 2n(\bar{a} + \bar{b})$.

(7) Without losing generality we assume $\bar{a} < 0, \bar{b} < 0, \bar{c} \geq 0, \bar{d} < 0$. By Lemma 3.1,

$$\begin{aligned} \delta_0 &= |\bar{c}| + |\bar{d}| + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = \bar{c} - \bar{d} + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) \\ &= \frac{n\bar{a} + \bar{a}_0}{n+1} - \frac{n\bar{b} + \bar{a}_0}{n+1} + \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) = \frac{n}{n+1}(\bar{a} - \bar{b}) + \frac{n}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \end{aligned}$$

(since $\bar{c} \geq 0, \bar{d} < 0$); $\delta_1 = |\bar{a}| + |\bar{b}| + \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) = \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) - \bar{a} - \bar{b}$ (since $\bar{a} < 0, \bar{b} < 0$). As

$$\begin{aligned} \delta_1 - \delta_0 &= \frac{1}{n+1}\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) - \frac{(2n+1)\bar{a} + \bar{b}}{n+1} \\ &= \frac{1}{n+1}[\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) - ((2n+1)\bar{a} + \bar{b})], \end{aligned}$$

it suffices to prove that $\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) - ((2n+1)\bar{a} + \bar{b}) \geq 0$ (i.e., $\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \geq (2n+1)\bar{a} + \bar{b}$). This holds because $\bar{a} < 0$, $\bar{b} < 0$, and $\rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) \geq 0 > (2n+1)\bar{a} + \bar{b}$.

(8) By Lemma 3.1, $\delta_1 = \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{a}| + |\bar{b}| = \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{a}| + |\bar{b}|$;

$$\begin{aligned} \delta_0 &= \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) + |\bar{c}| + |\bar{d}| = \rho_2((\dot{c}, \ddot{c}), (\dot{d}, \ddot{d})) + |\bar{c}| + |\bar{d}| \\ &= \frac{n}{n+1} \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + \left| \frac{n\bar{a} + \bar{a}_0}{n+1} \right| + \left| \frac{n\bar{b} + \bar{a}_0}{n+1} \right| \\ &< \rho_2((\dot{a}, \ddot{a}), (\dot{b}, \ddot{b})) + |\bar{a}| + |\bar{b}| \end{aligned}$$

(since $\bar{a} < 0$, $\bar{c} < 0$, $\bar{b} < 0$, $\bar{d}$). Therefore, $\delta_0 < \delta_1$. $\square$

We aim to guide gifted students to start from well-known plane geometry knowledge concepts, practicing, thinking, and repeating this cycle until they produce relatively difficult and novel mathematical results. It is a nature next step to propose an analogous form of Theorem 2.1 in R^3 (with respect to the second metric ρ_r on R^3).

Theorem 3.2. Let $\{\vec{a}_0, \vec{a}_1, \vec{b}_1, \dots, \vec{a}_n, \vec{b}_n\} \subseteq R^3$, $\delta = \varrho_3(\frac{1}{n} \sum_{j=1}^n \vec{a}_j, \frac{1}{n} \sum_{j=1}^n \vec{b}_j)$, $\delta_0 = \varrho_3(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j, \frac{1}{n+1} \sum_{j=0}^n \vec{b}_j)$ ($\vec{a}_0 = \vec{b}_0$), $\dot{a} = p_1(\frac{1}{n} \sum_{j=1}^n \vec{a}_j)$, $\vec{a} = (p_2(\frac{1}{n} \sum_{j=1}^n \vec{a}_j), p_3(\frac{1}{n} \sum_{j=1}^n \vec{a}_j))$, $\dot{b} = p_1(\frac{1}{n} \sum_{j=1}^n \vec{b}_j)$, $\vec{b} = (p_2(\frac{1}{n} \sum_{j=1}^n \vec{b}_j), p_3(\frac{1}{n} \sum_{j=1}^n \vec{b}_j))$, $\vec{a}_0 = (p_2(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j), p_3(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j))$, $\vec{b}_0 = (p_2(\frac{1}{n+1} \sum_{j=0}^n \vec{b}_j), p_3(\frac{1}{n+1} \sum_{j=0}^n \vec{b}_j))$, and $\vec{\vartheta} = (p_2(\vec{a}_0), p_3(\vec{a}_0))$, where $p_i : R^3 \rightarrow R$ is the i -th projection ($i = 1, 2, 3$). Then $\delta_0 \leq \delta$ holds if $\|\vec{\vartheta}\| = \rho_2(\vec{\vartheta}, (0, 0))$, $\dot{a}, \dot{b}$, $\|\vec{a}\| = \rho_2(\vec{a}, (0, 0))$, and $\|\vec{b}\| = \rho_2(\vec{b}, (0, 0))$ satisfy one of the following conditions:

- (1) $\dot{a} = \dot{b}$.
- (2) $\dot{a} \neq \dot{b}$ but $2\|\vec{\vartheta}\| \leq \|\vec{a}\| + \|\vec{b}\| + |\dot{a} - \dot{b}| = \delta$ (particularly, $2\|\vec{\vartheta}\| \leq \|\vec{a}\| + \|\vec{b}\|$). This means that $(\vec{a}, \vec{b}) \in [R^2 - B((0, 0), \delta)] \times [R^2 - B((0, 0), 2\delta)]$ or $(\vec{b}, \vec{a}) \in [R^2 - B((0, 0), \delta)] \times [R^2 - B((0, 0), 2\delta)]$ for some $\delta \in (0, \|\vec{\vartheta}\|]$ if $\|\vec{\vartheta}\| > 0$, where $B((0, 0), \delta) = \{\vec{e} \in R^2 \mid \|\vec{e}\| < \delta\}$. In other words, neither $(\dot{a}, \vec{a})$ is in the hole $R \times B((0, 0), \delta)$ nor $(\dot{b}, \vec{b})$ is in the hole $R \times B((0, 0), 2\delta)$, or else neither $(\dot{b}, \vec{b})$ is in the hole $R \times B((0, 0), \delta)$ nor $(\dot{a}, \vec{a})$ is in the hole $R \times B((0, 0), 2\delta)$ for some $\delta \in (0, \|\vec{\vartheta}\|]$ if $\|\vec{\vartheta}\| > 0$.

³and thus $(\dot{a}, \vec{a}) = \frac{1}{n} \sum_{j=1}^n \vec{a}_j$ and $(\dot{b}, \vec{b}) = \frac{1}{n} \sum_{j=1}^n \vec{b}_j$.

Proof. (1) As

$$p_1\left(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j\right) = \frac{1}{n+1} \left[p_1(\vec{a}_0) + np_1\left(\frac{1}{n} \sum_{j=1}^n \vec{a}_j\right) \right],$$

and

$$p_1\left(\frac{1}{n+1} \sum_{j=0}^n \vec{b}_j\right) = \frac{1}{n+1} \left[p_1(\vec{b}_0) + np_1\left(\frac{1}{n} \sum_{j=1}^n \vec{b}_j\right) \right],$$

$p_1\left(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j\right) = p_1\left(\frac{1}{n+1} \sum_{j=0}^n \vec{b}_j\right)$ if and only if $p_1\left(\frac{1}{n} \sum_{j=1}^n \vec{a}_j\right) = p_1\left(\frac{1}{n} \sum_{j=1}^n \vec{b}_j\right)$ (i.e., $\dot{a} = \dot{b}$). It follows that $\delta_0 = \rho_2(\vec{a}_0, \vec{b}_0) = \frac{n}{n+1} \rho_2(\vec{a}, \vec{b}) \leq \rho_2(\vec{a}, \vec{b}) = \delta$ when $\dot{a} = \dot{b}$.

(2) When $\dot{a} \neq \dot{b}$,

$$\begin{aligned} \delta_0 &= \rho_2(\vec{a}_0, (0, 0)) + \left| p_1\left(\frac{1}{n+1} \sum_{j=0}^n \vec{a}_j\right) - p_1\left(\frac{1}{n+1} \sum_{j=0}^n \vec{b}_j\right) \right| + \rho_2(\vec{b}_0, (0, 0)) \\ &= \|\vec{a}_0\| + \frac{n}{n+1} |\dot{a} - \dot{b}| + \|\vec{b}_0\|, \\ \delta &= \rho_2(\vec{a}, (0, 0)) + \left| p_1\left(\frac{1}{n} \sum_{j=1}^n \vec{a}_j\right) - p_1\left(\frac{1}{n} \sum_{j=1}^n \vec{b}_j\right) \right| + \rho_2(\vec{b}, (0, 0)) \\ &= \|\vec{a}\| + |\dot{a} - \dot{b}| + \|\vec{b}\|, \end{aligned}$$

and $\delta - \delta_0 = \|\vec{a}\| + \|\vec{b}\| - \|\vec{a}_0\| - \|\vec{b}_0\| + \frac{1}{n+1} |\dot{a} - \dot{b}|$. To prove $\delta - \delta_0 \geq 0$ (i.e., $\delta_0 \leq \delta$), it suffices to verify $\|\vec{a}\| + \|\vec{b}\| \geq \|\vec{a}_0\| + \|\vec{b}_0\| - \frac{1}{n+1} |\dot{a} - \dot{b}|$. As $\|\vec{a}_0\| = \frac{1}{n+1} \|\vec{a}\| + n\|\vec{a}\|$ and $\|\vec{b}_0\| = \frac{1}{n+1} \|\vec{b}\| + n\|\vec{b}\|$,

$$\begin{aligned} \|\vec{a}_0\| + \|\vec{b}_0\| - \frac{1}{n+1} |\dot{a} - \dot{b}| &\leq \frac{1}{n+1} \left[2\|\vec{a}\| + n(\|\vec{a}\| + \|\vec{b}\|) \right] - \frac{1}{n+1} |\dot{a} - \dot{b}| \\ &\leq \|\vec{a}\| + \|\vec{b}\| \end{aligned}$$

since $2\|\vec{a}\| \leq \|\vec{a}\| + \|\vec{b}\| + |\dot{a} - \dot{b}|$. □

The following Problem 3.1 is left to the readers:

Problem 3.1. Give a necessary and sufficient condition under which the metric inequality on R^3 for the metric ρ to hold.

4. AN EXAMPLE OF APPLICATIONS

In this section, we explore the application of the metric inequality (1.1) in decision-making problems.

Example 2. *Over the past 20 years, private enterprises in some regions of a country have been developing rapidly, contributing significantly to the national GDP, particularly in these regions. However, instances of corruption have also emerged. For example, many business owners have offered money (as rewards) to helpful technicians or officials. These actions are illegal and need to be addressed in order to maintain the healthy development of the country's economy. Several methods can be used to deal with this issue, and three potential schemes are outlined below:*

Scheme 1 (denoted as x_1): Most business owners and money receivers are jailed for 1 to 3 months (undergo labor reform for 3 to 12 months), provided that all (or most) of the corruption money is returned to the national treasury.

Scheme 2 (denoted as x_2): A few selected business owners or money receivers are jailed for many years and have most of their fortune confiscated. This serves as a warning to others, and the consequences are made publicly known across the country. The blanket approach taken by some magazines when withdrawing a large number of papers is similar to this; in fact, it is easier with the help of AI (but unaware of the consequences of popularity).

Scheme 3 (denoted as x_3): All business owners and money receivers who have engaged in illegal activities are reported. Each individual involved is jailed for 1 to 10 months, provided that most of the corruption money is returned to the national treasury and they contribute significantly to the country.

The local government now needs to rank these schemes- $\{x_1, x_2, x_3\}$ -based on two comparative advantage judgment interval matrixes given by two assessment teams. The elements of these matrices are interval numbers, with the matrixes being complementary symmetric matrixes that incorporate order-reversing involutions on $\mathbb{R}$ as defined in Section 1:

$$A^{(1)} = \begin{bmatrix} \mathbf{0.5} & \langle 0.48, 0.5974 \rangle & \langle 0.55, 0.56 \rangle \\ \langle 0.4026, 0.52 \rangle & \mathbf{0.5} & \langle 0.3, 0.4102 \rangle \\ \langle 0.44, 0.45 \rangle & \langle 0.5898, 0.7 \rangle & \mathbf{0.5} \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} \mathbf{0.5} & \langle 0.3, 0.4895 \rangle & \langle 0.5, 0.7 \rangle \\ \langle 0.5105, 0.7 \rangle & \mathbf{0.5} & \langle 0.3, 0.5 \rangle \\ \langle 0.3, 0.5 \rangle & \langle 0.5, 0.7 \rangle & \mathbf{0.5} \end{bmatrix},$$

where $\mathbf{0.5}$ (resp., $\langle 0.48, 0.5974 \rangle$, $\langle 0.55, 0.56 \rangle$) is an interval number, standing for comparative advantage judgment (given by the first assessment team) of x_1 over x_1 (resp., over x_2 , over x_3); the meanings of other elements of the two matrixes can be explained similarly.

(1) Usually, this rank is finished in the following three steps:

Step 1. Compute $A_j^{(i)}$ (i.e., aggregate line j of matrix $A^{(i)}$) simply relying on weight vector $\vec{w} = (w_1, w_2, w_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ $i = 1, 2; j = 1, 2, 3$. We obtain $A_1^{(1)} = \frac{1}{3}(\mathbf{0.5} + \langle 0.48, 0.5974 \rangle + \langle 0.55, 0.56 \rangle) = \langle 0.51, 0.5525 \rangle$, $A_2^{(1)} = \frac{1}{3}(\langle 0.4026, 0.52 \rangle + \mathbf{0.5} + \langle 0.3, 0.4102 \rangle) = \langle 0.4009, 0.4767 \rangle$, $A_3^{(1)} = \frac{1}{3}(\langle 0.44, 0.45 \rangle + \langle 0.5898, 0.7 \rangle + \mathbf{0.5}) = \langle 0.5099, 0.5500 \rangle$; $A_1^{(2)} = \langle 0.4333, 0.5632 \rangle$, $A_2^{(2)} = \langle 0.4368, 0.5667 \rangle$, $A_3^{(2)} = \langle 0.4333, 0.5632 \rangle$.

Step 2. Compute the score s_j by aggregating the results from both $A_j^{(1)}$ and $A_j^{(2)}$ by using the weight vector $\vec{w} = (w_1, w_2) = (0.6, 0.4)$ ($j = 1, 2, 3$). We obtain $s_1 = \langle 0.4793, 0.5567 \rangle$, $s_2 = \langle 0.4153, 0.5127 \rangle$, $s_3 = \langle 0.4793, 0.5567 \rangle$.

Step 3. Rank the schemes based on the computed scores. This process results in the initial ranking: $x_2 \prec x_1 = x_3$ (since $s_2 < s_1 = s_3$), which is invalid because x_1 and x_3 are incomparable. To improve this, we modify the method.

(2) Motivated by inequality (1.1) and Example 1, we will reduce weight of each element in diagonal line of each matrix to 0 (so as to make any two in the scheme set $\{x_1, x_2, x_3\}$ are comparable).

Step 1. $A_1^{(1)} = \frac{1}{2}(\langle 0.48, 0.5974 \rangle + \langle 0.55, 0.56 \rangle) = \langle 0.515, 0.5787 \rangle$, $A_2^{(1)} = \frac{1}{2}(\langle 0.4026, 0.52 \rangle + \langle 0.3, 0.4102 \rangle) = \langle 0.3513, 0.4651 \rangle$, $A_3^{(1)} = \frac{1}{2}(\langle 0.44, 0.45 \rangle + \langle 0.5898, 0.7 \rangle) = \langle 0.5149, 0.575 \rangle$; $A_1^{(2)} = \langle 0.4, 0.5948 \rangle$, $A_2^{(2)} = \langle 0.4053, 0.6 \rangle$, and $A_3^{(2)} = \langle 0.4, 0.6 \rangle$.

Step 2. Compute score s_j relying on weight vector $\vec{w} = (w_1, w_2) = (0.6, 0.4)$ ($j = 1, 2, 3$). $s_1 = 0.6\langle 0.515, 0.5787 \rangle + 0.4\langle 0.4, 0.5948 \rangle = \langle 0.469, 0.5851 \rangle$, $s_2 = \langle 0.3729, 0.5191 \rangle$, and $s_3 = \langle 0.4689, 0.585 \rangle$.

Step 3. The new ranking result is $x_2 \prec x_3 \prec x_1$, which is valid since $s_2 < s_3 < s_1$.

Now we design a question (i.e., the following Question 4.1) for capable students. Recall that an mapping $\mathfrak{t} = \langle a, \frac{a+b}{2}, b \rangle : [0, 1] \longrightarrow [0, 1]$ ($0 \leq a \leq b \leq 1$),

defined by $\mathfrak{t}(x) = \langle a, \frac{a+b}{2}, b \rangle(x) = \max \left\{ \min \left\{ \frac{2(x-a)}{b-a}, 1, \frac{2(b-x)}{b-a} \right\}, 0 \right\}$, is called an isosceles triangle fuzzy number [16] on $[0, 1]$.

Question 4.1. *What is the ranking result of Example 2 if the two comparative advantage judgment matrixes are given as follows?*

$$A^{(1)} = \begin{bmatrix} \langle 0.5, 0.5, 0.5 \rangle & \langle 0.48, 0.5387, 0.5974 \rangle & \langle 0.55, 0.555, 0.56 \rangle \\ \langle 0.4026, 0.4613, 0.52 \rangle & \langle 0.5, 0.5, 0.5 \rangle & \langle 0.3, 0.3551, 0.4102 \rangle \\ \langle 0.44, 0.445, 0.45 \rangle & \langle 0.5898, 0.6449, 0.7 \rangle & \langle 0.5, 0.5, 0.5 \rangle \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} \langle 0.5, 0.5, 0.5 \rangle & \langle 0.3, 0.39475, 0.4895 \rangle & \langle 0.5, 0.6, 0.7 \rangle \\ \langle 0.5105, 0.60525, 0.7 \rangle & \langle 0.5, 0.5, 0.5 \rangle & \langle 0.3, 0.4, 0.5 \rangle \\ \langle 0.3, 0.4, 0.5 \rangle & \langle 0.5, 0.6, 0.7 \rangle & \langle 0.5, 0.5, 0.5 \rangle \end{bmatrix}.$$

Notice that the two matrixes are also complementary symmetric matrixes in which the order-reversing involutions on its elements is defined by $\langle a, \frac{a+b}{2}, b \rangle' = \langle 1-b, 1-\frac{a+b}{2}, 1-a \rangle$.

Remark 4.1.

- (1) *The above results suggest that using inequalities with river metrics can enhance the discriminability between different schemes, avoid incomparable situations, and thereby provide clearer guidance for-makers.*
- (2) *Inspired by Example 1 and Example 2, to avoid information loss, we propose three suggestions for future decision-making: First, try not approximate two or more real data points (such as finite hesitation fuzzy numbers, interval numbers, etc.) into a single real number through arithmetic averaging or other methods too early. Second, in decision-making problems like the one shown above, do not let the diagonal elements in the data matrix participate in the arithmetic averaging operation in Step 1. Third, adopting a metric suitable for the problem background (such as the river metric) may significantly improve the stability of the ranking.*
- (3) *This example also illustrates that even for non-mathematics decision-makers, as long as they understand the basic geometric meaning, they can apply the criterion of this inequality. Therefore, we should allow (and even support) gifted students of all kinds to realize their dream of participating in scientific research as early as.*

5. CONCLUDING REMARKS

This article highlights the importance of nurturing the next generation of capable mathematicians, who will largely emerge from today's (or in the coming years') gifted middle school students with an interest in both mathematics and technological innovations and are willing to work hard. The urgency of developing their mathematical conjecture and discovery capabilities – including awareness of AI utilization, a special case of the RMI principle utilization – motivates this paper's exploration of how gifted students can be guided in collaborative training in conjecturing, discovering, and proving new mathematical theorems. In this regard, if one is willing, there might be a contribution. For examples, when conducting mathematical research, we can take into account the needs of successors, users, and disseminators, as well as the interests of their followers – that is, to be considerate of successors and other readers. We can also consider to design an AI-assisted motivational (or red-envelope reward) system for adolescents, which is based on the principle that “those near vermilion become red, while those near ink become black”. Moreover, the paper exemplifies how mathematics can be applied to interpret or clarify issues in daily life, court disputes, and more, which are complex, controversial, and hard to make clear; in the future, let us not forget to exemplify that “mathematics is useful” and “mathematicians also have visible donations”.

While the education of ordinary high school and college students accounts for a large proportion of research and efforts – because it directly impacts the overall literacy of a country – there is a lack of research focused on optimizing the education of gifted students⁴, possibly due to their smaller proportion. This gap, combined with the prevalent use of rote teaching methods, poses a significant challenge. Gifted students are fundamental to determining the relative development level and trajectory of a country. Their cultivation is just as important as the role of senior mathematicians, as their potential can greatly influence the future. China's successful strategy of allowing certain regions or individuals to prosper first – proposed by Xiaoping Deng in 1978 – has proven effective in eradicating poverty. A similar strategy should be applied to gifted students to prevent stagnation and foster the development of top-tier talent; indeed, the most worthwhile

⁴It is the same for struggling students.

thing for good schools/universities to do is to try every possible way to keep those with excellent character, a willingness to strive, great potential from stagnating halfway. Such experience can provide useful reference for underdeveloped countries, particularly major powers like India and China. In China, people already have plans to guide primary and secondary school students to think like scientists and practice like engineers (notice that large enterprises probably recruit directly from high school students). The main aim of this paper is to address the shortcomings of the current educational system, using a mathematical and non-cramming teaching approach. It is designed to help gifted students not to be delayed or misdirected (and thus develop their mathematical skills in a timely manner), fostering an environment where they can independently discover and prove new concepts, results, and methods.

The mental health and self-confidence of gifted high school students – particularly their creative ability and related achievements – play a pivotal role in shaping their development. This confidence should be nurtured early and in a practical, matched way, akin to the growth process of athletes. This paper exemplifies how gifted students can be guided to increase their self-confidence in terms of technology through a series of training sessions in mathematical conjecture and discovery, following Polya's plausible reasoning model. By starting from well-known mathematical knowledge points, these students can develop their skills to the point of producing (and even publishing) a research paper.

Motivated by the need for such training and decision-making challenges, this paper examines inequality (1.1) in metric space (R^2, ϱ_3) , where ϱ_3 is the river metric on R^2 . The main result, presented in Theorem 2.1, provides a necessary and sufficient condition for inequality (1.1) to hold in metric space (R^2, ϱ_3) . This result can benefit excellent beginners and capable practitioners and mathematics users (who hope autarky) because Theorem 2.1 is guessed and proved using Polya's plausible reasoning methods [1] (mainly, relation-mapping-inversion principle) and relaying just on well-known knowledge of plane geometry, and thus is easy to understand, imitate, and disseminate; they can generalize at least Theorems 3.1 and 3.2 to the case R^n ($n \geq 2$) with a generalized river metric to be defined. Actually, Theorem 2.1 contains both a problem which is worth to solve

and can be worked out (as Theorems 3.1 and 3.2) and technique useful to beginners. Some students, even high school students from Yuci No.1 Middle School in Shanxi Province, witnessed (and even participated in) the discovery process with great interest, subconsciously revealing a sense of achievement (since it is generally considered impossible even for diligent and gifted high school students to engage in mathematical research).

Reflecting on the history of science (especially, the experiences of mathematicians), it is clear that developing a proactive mindset is crucial for high school students to become great scientists in the future. Most future scientists are already among the high school and college students we are currently (or in the coming years) cultivating. In the next 10 to 20 years, the existing impacts of the pandemic and children's mobile games on employment and birth rates may appear to be getting increasingly severe. Therefore, research focusing on student innovation is becoming increasingly urgent, practical, and necessary. This is consistent with some practices and approaches in the fields of sports, music, and the arts where visible results can be seen; in these fields early development and effort are always regarded as positive actions (and are therefore encouraged early on). Polya's plausible reasoning methods can be helpful for such training. For students who are patient, interested, and capable of achieving results, it is essential to identify suitable content for them and develop appropriate training techniques. By doing so, we can help them reach their full potential (at least avoid delays and going astray), ultimately becoming outstanding talents, especially in the field of mathematics.

Since assisting the development of future scientists – including mathematicians, computer scientists, data scientists, physicists, engineers, biologists, psychologists, economists, and others – is one of our crucial tasks (cf. [17]), we aim to focus our efforts on five key areas moving forward:

- a) Guiding and practicing with students: We will continuously guide gifted students, along with their friends or peers, in practicing (and even rehearsing) key components of the discovery and proof process for Theorem 2.1 (and Theorems 3.1 and 3.2, along with Question 4.1). This includes solving related simpler problems to reinforce their learning and progress.

- b) Studying properties of metric spaces: We plan to study the u -connectedness, $\diamond$ -connectedness, and Ω -connectedness [18, 19] of the metric space (R^2, ϱ_3) .
- c) Generalizing theorems and metrics: The goal is to further generalize (using Polya's plausible reasoning methods) the river metric ϱ_3 on R^2 and generalize Theorem 2.1 to the case R^n ($n \geq 2$).
- d) Encouraging mathematics teaching and mentorship: We will encourage outstanding mathematics students to take on roles as part-time instructors or advisers, or even teachers, for younger gifted students. This measure aims to partially compensate for the long-standing flaws of speech competitions in many primary and secondary schools, colleges, and universities (because the direction certain experiences are extremely important to the future of gifted students).
- e) Make every effort to ensure that those exceptional mathematics tuition-free normal students, who are willing to explore and strive in the field of mathematics together with their gifted high school students, do not stagnated in their critical developmental years in China: Encourage and support some voluntary gifted mathematics tuition-free normal university students for mathematics paper writing training at some normal university (or on line) during holidays (as current graduate summer schools for scenius) and $\frac{1}{7}$ to $\frac{2}{7}$ of each semester, call for some flexible reward requirements for a few gifted mathematics tuition-free normal university students under which both the service period of prizewinners may be reduced and their work school may also be awarded heavily, etc.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data available on request from the authors.

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