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ON MULTIVARIATE SEGMENTAL INTERPOLATION PROBLEM

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Abstract. In this paper the following problem is introduced, which we call segmental interpolation problem, or briefly segmental problem: Suppose $\mathcal{X}_I = \{\mathbf{x}^{(\nu)} : \nu \in I\}$ is a finite or infinite set of knots in $\mathbb{R}^d$. Suppose also that $\mathcal{S}_I = \{[\alpha_\nu, \beta_\nu] : \nu \in I\}$ is a respective set of any segments. The segmental problem $\{\mathcal{X}, \mathcal{S}\}_I^n$ is to find a polynomial p in d variables and of total degree less than or equal to n, satisfying the conditions

$$\alpha_\nu \le p(\mathbf{x}^{(\nu)}) \le \beta_\nu, \quad \forall \nu \in I.$$

We bring a necessary and sufficient condition for the solvability of the segmental problem. In case when the problem is solvable and the set of knots $\mathcal{X}_I$ is finite, we bring a method to find a solution of the segmental problem.

1. Introduction, the segmental interpolation problem

The univariate polynomial interpolation problem always has a unique solution provided the number of interpolation knots fits the dimension of the polynomial space. In contrast with this, in the multivariate polynomial interpolation the existence and uniqueness of solution of a Lagrange problem essentially depend on the situation of the interpolation knots. A given set of knots, naturally arising from some physical or modeling problem, may not guarantee the solvability of the interpolation problem. Besides, usually it is difficult to modify the knot set. Consideration of this challenging question is a subject of permanent interest in the theory of multivariate interpolation. In this paper a new constructive approach is proposed, where the frame of the solvability of the polynomial interpolation is enlarged essentially, by allowing an error stripe for the data.

Let $\Pi_n := \Pi_n^d$ be the space of all polynomials in d variables and of total degree less than or equal to n. Its dimension is given by

$$N := \dim \Pi_n^d = \binom{n+d}{d}.$$

To present the *segmental interpolation problem*, or briefly *segmental problem*, we need a set of distinct knots:

$$\mathcal{X}_I = \{\mathbf{x}^{(\nu)} = \left(x_1^{(\nu)}, \ldots, x_d^{(\nu)}\right) : \nu \in I\} \subset \mathbb{R}^d.$$

Suppose also that $\mathcal{S}_I = \{[\alpha_\nu, \beta_\nu] : \nu \in I\}$ is a respective set of any segments. The segmental interpolation problem $\{\mathcal{X}, \mathcal{S}\}_I^n$ is to find a polynomial $p \in \Pi_n$, satisfying the conditions

$$\alpha_\nu \le p(\mathbf{x}^{(\nu)}) \le \beta_\nu, \quad \forall \nu \in I. \tag{1.1}$$

It is worth mentioning, that the segmental interpolation problem can be considered equivalently as an interpolation problem with pregiven errors. Namely, for any function f defined on $\mathcal{X}_I$, and any set of errors $\mathcal{E}_I = \{\epsilon_\nu : \nu \in I\}$, find a polynomial $p \in \Pi_n$, satisfying the conditions

$$| p(\mathbf{x}^{(\nu)}) - f(\mathbf{x}^{(\nu)}) | \le \epsilon_\nu, \quad \forall \nu \in I.$$

Denote the set of all polynomials in Π_n satisfying (1.1) by $Sol\{\mathcal{X}, \mathcal{S}\}_I^n$. The problem $\{\mathcal{X}, \mathcal{S}\}_I^n$ is called *solvable*, if $Sol\{\mathcal{X}, \mathcal{S}\}_I^n \ne \emptyset$. Also, a segmental problem $\{\mathcal{X}, \mathcal{S}\}_J^n$ is a *subproblem* of $\{\mathcal{X}, \mathcal{S}\}_I^n$, if $J \subset I$. Evidently, we have

Remark 1.1. *Any subproblem of a solvable segmental problem is solvable.*

By the cardinality of a finite set $\mathcal{X}$, denoted by $\#\mathcal{X}$, we mean the number of elements of the set.

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A segmental problem (subproblem) with finite set of knots is called *finite segmental problem (subproblem).* A finite knot set of cardinality m is denoted by

$$\mathcal{X}_m = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(m)}\} \subset \mathbb{R}^d \tag{1.2}$$

and the respective set of segments by $\mathcal{S}_m = \{[\alpha_i, \beta_i] : i = 1, \ldots, m\}$. We denote by $\{\mathcal{X}, \mathcal{S}\}_m^n$ the corresponding finite segmental problem of cardinality m, i.e., the problem of finding a polynomial $p \in \Pi_n$ satisfying the conditions

$$\alpha_i \le p(\mathbf{x}^{(i)}) \le \beta_i, \quad i = 1, \ldots, m. \tag{1.3}$$

Finite segmental problems we consider in a slightly wider setting. Namely, in this case α_i and β_i may assume also the values $-\infty$ and $+\infty$, respectively. Besides, among the inequalities in (1.3) strict ones are allowed. In this case the finite segmental problem is said to have *mixed conditions.* Note that in a simplest case $n = 1$, i.e., the case of linear polynomials in d variables, the segmental problem $\{\mathcal{X}, \mathcal{S}\}_m^1$ reduces to a general (finite or infinite) system of two-sided linear inequalities:

$$\alpha_\nu \le \eta_0 + \sum_{i=1}^{d} \eta_i x_i^{(\nu)} \le \beta_\nu, \quad \nu \in I.$$

Here η_i and $x_i^{(\nu)}$ are the unknowns and the coefficients, respectively.

Now, to describe briefly how the paper is organized, we need some definitions beforehand. A set of knots $\mathcal{X} \subset \mathbb{R}^d$ is called n-independent, if each its knot has an n-fundamental polynomial. Let $\mathcal{H}_n(\mathcal{X})$ be the Hilbert n-function of a knot set $\mathcal{X}$, which equals the cardinality of the maximal n-independent subset of $\mathcal{X}$. (Later we will see that $\mathcal{H}_n(\mathcal{X}) \le N \ \ \forall \mathcal{X}$.) We call the segmental subproblem $\{\mathcal{X}, \mathcal{S}\}_{\mathfrak{b}}^n$, $\mathfrak{b} \subset I$, basic, if $\mathcal{H}_n(\mathcal{X}_{\mathfrak{b}}) = \#\mathfrak{b} - 1$, and the knot set $\mathcal{X}_{\mathfrak{b}}$ is essentially n-dependent, i.e., no knot of $\mathcal{X}_{\mathfrak{b}}$ has an n-fundamental polynomial.

First, in Section 2 we consider some basic concepts in multivariate polynomial interpolation, such as fundamental polynomials, n-independence, and the Hilbert function of knot sets. In Subsections 3.1 and 3.2 we bring two characterizations for solvability of basic subproblems. Then, based on this, in Section 4, we get a solvability characterization for general segmental problem, in finite and infinite cases. Namely, we prove that the segmental problem $\{\mathcal{X}, \mathcal{S}\}_I^n$ is solvable if and only if all its basic subproblems are solvable. Here, besides the n-independence techniques, we use the Helly theorem on convex sets' intersection (see forthcoming Theorem 4.1). In Subsection 4.1 we bring a method for finding a solution of any finite segmental problem, provided it is solvable. Let us mention that a step of this method is based on a proof of the Helly theorem (Theorem 4.1). In the final Section 5 we present more detailed consideration of the univariate segmental problem, i.e., of the case $d = 1$.

2. Multivariate interpolation, n-independence

Next we consider some basic concepts of multivariate polynomial interpolation (see [1]–[6], [8]–[16], [18]–[23]). Let a finite set of knots $\mathcal{X}_m \subset \mathbb{R}^d$ be given by (1.2) and $(c_1, \ldots, c_m) \in \mathbb{R}$ be any data. The problem of finding a d-variate polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(\mathbf{x}^{(i)}) = c_i, \quad i = 1, \ldots, m, \tag{2.1}$$

is called *interpolation problem.*

Definition 2.1. *The set of knots $\mathcal{X}_m$ is called n-poised, if for any data $(c_1, \ldots, c_m)$ there is a* unique *polynomial $p \in \Pi_n$ satisfying the conditions* (2.1).

By a Linear Algebra argument, a necessary condition for n-poisedness is

$$m = \#\mathcal{X}_m = \dim \Pi_n = N. \tag{2.2}$$

In other words, the number of interpolation knots has to match the dimension of the polynomial space.

The condition (2.2) is both necessary and sufficient for the n-poisedness in the univariate case ($d = 1$), while in the multivariate case ($d \ge 2$), which is much more involved, this condition is not anymore sufficient, unless $n = 0$. And even (2.2) is the case, the multivariate interpolation problem does not always have a solution or the solution is not necessarily unique.

There are several approaches to overcome this problem. In the Kergin and Hakopian interpolations (see [18], [12], [23], [19]) the pointwise interpolation conditions are replaced by mean-value ones. In the least choice and minimal degree interpolations, the former introduced by C. de Boor and A. Ron (see [2], [3]), and the latter by T. Sauer (see [21], [9]), the total degree spaces of polynomials Π_n are replaced by their appropriate subspaces. The present paper will approach the question of finding proper interpolating polynomial for any given knot set by allowing certain (small) errors for the data.

In the theory of polynomial interpolation the concept of fundamental polynomial is crucial. A polynomial $p \in \Pi_n$ is called *n-fundamental polynomial* of a knot $A = \mathbf{x}^{(i)} \in \mathcal{X}_m$, if

$$p(A) = 1 \qquad \text{and} \qquad p\big|_{\mathcal{X}_m \setminus \{A\}} = 0,$$

where $p\big|_{\mathcal{X}}$ means the restriction of p to $\mathcal{X}$. This polynomial is denoted by $p_A^\star := p_i^\star := p_{i,\mathcal{X}_m}^\star := p_{A,\mathcal{X}_m}^\star$. Sometimes we call n-fundamental also a polynomial from Π_n vanishing at all the knots of $\mathcal{X}_m$ but A, since such a polynomial is a nonzero constant multiple of $p_A^\star$.

Next we consider an important concept of n-independence and n-dependence of knot sets (see [7], [13], [14], [15]).

Definition 2.2. *A set of knots $\mathcal{X} \subset \mathbb{R}^d$ is called* Π_n-independent*, or briefly* n-independent*, if each its knot has an n-fundamental polynomial. Otherwise, if at least one of its knots does not have an n-fundamental polynomial, $\mathcal{X}$ is called* n-dependent*. Furthermore, it is called* essentially n-dependent*, if no its knot has an n-fundamental polynomial.*

Since fundamental polynomials are linearly independent we obtain that a necessary condition for n-independence is

$$\#\mathcal{X} \leq \dim \Pi_n = N\,.$$

Note that this condition is also sufficient for the n-independence in the univariate case. Suppose a knot set $\mathcal{X}_m$ is n-independent. Then by Lagrange formula we obtain a polynomial

$$p = \sum_{i=1}^{m} c_i p_{i,\mathcal{X}_m}^\star,$$

satisfying the interpolation conditions (2.1). In view of this formula, we readily get that n-independence of $\mathcal{X}_m$ is equivalent to the *solvability* of the interpolation problem (2.1), meaning that for any data $\{c_1, \dots, c_m\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (2.1).

We call a segmental problem $\{\mathcal{X}, S\}_m^n$ *n-independent* if its knot set $\mathcal{X}_m$ is n-independent. From what was said above we conclude easily

Lemma 2.1. *Any n-independent segmental problem $\{\mathcal{X}, S\}_m^n$ is solvable.*

Indeed, one can find a solution of n-independent segmental interpolation problem $\{\mathcal{X}, S\}_m^n$ given by (1.3) by solving the interpolation problem (2.1), where c_i are any intermediate values between α_i and β_i, $i = 1, \dots, m$.

For knot set $\mathcal{X}_m$ with $m = N$ the n-independence means n-poisedness. Furthermore, we have the following well-known (see, e.g., [13], Lemma 1)

Lemma 2.2. *Any n-independent set of knots $\mathcal{X}_m$ with $m < N$ can be enlarged to an n-poised set $\mathcal{X}_N$.*

2.1. **Some properties of n-independence.**

Lemma 2.3. *Suppose that a knot A of a finite knot set $\mathcal{X}$ has n-fundamental polynomial with respect to $\mathcal{X}$ and all the knots of a finite set $\mathcal{Y}$ have n-fundamental polynomials with respect to the set $\mathcal{X} \cup \mathcal{Y}$. Then the knot A has an n-fundamental polynomial with respect to the set $\mathcal{X} \cup \mathcal{Y}$, too.*

Proof. Suppose p_0 is an n-fundamental polynomial of A with respect to $\mathcal{X}$. Next, suppose that $\mathcal{Y} = \{B_i\}_{i=1}^k$ and $p_i^\star := p_{B_i,\mathcal{X}\cup\mathcal{Y}}^\star$, $i = 1, \dots, k$, are n-fundamental polynomials. Now one can readily verify that the polynomial

$$q_0 := p_0 - \sum_{i=1}^{k} p_0(B_i) p_i^\star$$

is an n-fundamental polynomial of A with respect to $\mathcal{X} \cup \mathcal{Y}$. □

From Lemma 2.3 we get immediately the following (see Lemma 2.2, [15]):

Corollary 2.1. *Suppose that a knot set $\mathcal{X}$ is n-independent and each knot of a set $\mathcal{Y}$ has n-fundamental polynomial with respect to the set $\mathcal{X} \cup \mathcal{Y}$. Then the latter knot set is n-independent, too.*

Let us remove from a knot set all the knots that have n-fundamental polynomials. Next, we prove that the remaining set is essentially n-dependent, i.e., no its knot has a fundamental polynomial.

Corollary 2.2. *Suppose a knot set $\mathcal{Z}$ is given. Denote by $\mathcal{Y}$ the set of knots of $\mathcal{Z}$ that have n-fundamental polynomials with respect to $\mathcal{Z}$. Then the knot set $\mathcal{Z} \setminus \mathcal{Y}$ is essentially n-dependent.*

Proof. Indeed, assume to the contrary that $\mathcal{X} := \mathcal{Z} \setminus \mathcal{Y}$ is not essentially n-dependent, i.e., there is a knot $A \in \mathcal{X}$ which has an n-fundamental polynomial with respect to $\mathcal{X}$. Then, since the knots of $\mathcal{Y}$ have n-fundamental polynomials with respect to $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$, we get from Lemma 2.3, that A has n-fundamental polynomial with respect to $\mathcal{Z}$, which is a contradiction. □

Corollary 2.3. *Suppose that a set of knots $\mathcal{X}$ is n-independent, $A \notin \mathcal{X}$ and the set $\mathcal{X} \cup \{A\}$ is n-dependent. Then we have that*

$$p \in \Pi_n \ \text{ and } \ p\big|_{\mathcal{X}} = 0 \quad \Longrightarrow \quad p(A) = 0.$$

Proof. Indeed, assume to the contrary that there is a polynomial $p \in \Pi_n$ that vanishes on $\mathcal{X}$ and does not vanish at A. This means that A has an n-fundamental polynomial with respect to the set $\mathcal{X} \cup \{A\}$. Then, by Corollary 2.1, the set $\mathcal{X} \cup \{A\}$ is n-independent, which is a contradiction. □

2.2. **The space $\mathcal{P}_{n,\mathcal{Z}}$ and the Hilbert function.** Denote the linear space of polynomials of total degree at most n vanishing on $\mathcal{Z}$ by

$$\mathcal{P}_{n,\mathcal{Z}} := \left\{p \in \Pi_n : p\big|_{\mathcal{Z}} = 0\right\}.$$

The following result is well-known (see e.g. [13], Section 1)

Proposition 2.1. *For any knot set $\mathcal{Z}$ we have that*

$$\dim \mathcal{P}_{n,\mathcal{Z}} \geq N - \#\mathcal{Z}.$$

Moreover, equality takes place here if and only if the set $\mathcal{Z}$ is n-independent.

Corollary 2.4. *Let $\mathcal{X}$ be a maximal n-independent subset of $\mathcal{Z}$, i.e., $\mathcal{X}$ is n-independent and $\mathcal{X} \cup \{A\}$ is n-dependent for any $A \in \mathcal{Z} \setminus \mathcal{X}$. Then we have that*

$$\mathcal{P}_{n,\mathcal{Z}} = \mathcal{P}_{n,\mathcal{X}}. \tag{2.3}$$

Proof. Indeed, we have that $\mathcal{P}_{n,\mathcal{Z}} \subset \mathcal{P}_{n,\mathcal{X}}$, since $\mathcal{X} \subset \mathcal{Z}$. Now, suppose that $p \in \Pi_n$, $p\big|_{\mathcal{X}} = 0$ and A is any knot of $\mathcal{Z}$. Then $\mathcal{X} \cup \{A\}$ is dependent and therefore, in view of Corollary 2.3, $p\big|_A = 0$. □

From (2.3) and Proposition 2.1 (part "moreover") we have that

$$\dim \mathcal{P}_{n,\mathcal{Z}} = N - \#\mathcal{X}, \tag{2.4}$$

where $\mathcal{X}$ is any maximal n-independent subset of $\mathcal{Z}$. Thus, all the maximal n-independent subsets of $\mathcal{Z}$ have the same cardinality, which is denoted by $\mathcal{H}_n(\mathcal{Z})$ – *the Hilbert n-function* of $\mathcal{Z}$. Hence, according to (2.4), we have

$$\dim \mathcal{P}_{n,\mathcal{Z}} = N - \mathcal{H}_n(\mathcal{Z}). \tag{2.5}$$

Now, let us extend slightly Lemma 2.2:

Lemma 2.4. *Let $\mathcal{X}_I$ be a knot set. Then any n-independent subset $\mathcal{X} \subset \mathcal{X}_I$, with $\mathcal{H}_n(\mathcal{X}) < \mathcal{H}_n(\mathcal{X}_I)$ can be enlarged to a maximal n-independent subset of $\mathcal{X}_I$.*

Proof. Indeed, it suffices to find a knot $A \in \mathcal{X}_I$, such that the set $\mathcal{X} \cup \{A\}$ is n-independent. We have that $\mathcal{P}_{n,\mathcal{X}_I} \subset \mathcal{P}_{n,\mathcal{X}}$. On the other hand, by (2.5), these linear spaces do not coincide. Therefore there is $p \in \mathcal{P}_{n,\mathcal{X}}$ such that $p(A) \neq 0$ for some $A \in \mathcal{X}_I$. Now, in view of Corollary 2.1, A is the desired knot. □

At the end of Section 2 let us present

Lemma 2.5. *Suppose a knot set $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ is given and each knot of $\mathcal{Y}$ has n-fundamental polynomial with respect to the set $\mathcal{Z}$. Then any segmental interpolation problem with the knot set $\mathcal{Z}$ is solvable if and only if the respective subproblem with the knot set $\mathcal{X}$ is solvable.*

Proof. Indeed, the direction "only if" is obvious. For the direction "if" notice that the polynomial

$$q = \sum_{A \in \mathcal{Y}} c_A p^{\star}_{A,\mathcal{Z}}$$

vanishes on $\mathcal{X}$ and assumes any values c_A at the knots $A \in \mathcal{Y}$. Hence, by adding to a solution of the segmental problem with the knot set $\mathcal{X}$ an appropriate polynomial q we will get a solution of the segmental problem with the knot set $\mathcal{Z}$. □

3. The basic segmental interpolation problem

Consider a segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_I$. Let us set $h_0 := \mathcal{H}_n(\mathcal{X}_I)$. Next, we bring the definition of quasi-basic and basic interpolation subproblem:

Definition 3.1. *Suppose $\mathfrak{b} \subset I$ and $\sigma := \mathcal{H}_n(\mathcal{X}_\mathfrak{b}) = \#\mathfrak{b} - 1$. Then the subproblem $\{\mathcal{X}, \mathcal{S}\}^n_\mathfrak{b}$ is called σ-*quasi-basic*, or briefly* quasi-basic. *If, in addition the knot set $\mathcal{X}_\mathfrak{b}$ is essentially n-dependent, then $\{\mathcal{X}, \mathcal{S}\}^n_\mathfrak{b}$ is called σ-*basic*, or briefly* basic.

Obviously we have that $\sigma \leq h_0$ for any σ-basic or σ-quasi-basic subproblem.
By using Corollary 2.2 and Lemma 2.5 one can reduce the solvability of any quasi-basic subproblem to the solvability of a basic subproblem:

Corollary 3.1. *Let a quasi-basic subproblem $\{\mathcal{X}, \mathcal{S}\}^n_{\tilde{\mathfrak{b}}}$, $\tilde{\mathfrak{b}} \subset I$, be given. Let also $\mathcal{X}_J$, $J \subset \tilde{\mathfrak{b}}$ be the set of knots of $\mathcal{X}_{\tilde{\mathfrak{b}}}$ that have n-fundamental polynomials. Then the quasi-basic segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_{\tilde{\mathfrak{b}}}$ is equivalent to the basic segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_\mathfrak{b}$, where $\mathfrak{b} = \tilde{\mathfrak{b}} \setminus J$, meaning that one of these problems is solvable if and only if the other is solvable.*

Next, in the following two subsections, we present two different characterizations for the solvability of quasi-basic, and hence basic, problems.

3.1. **The solvability of quasi-basic problem, I.** Suppose we have a σ-quasi-basic problem $\{\mathcal{X}, \mathcal{S}\}^n_\mathfrak{b} = \{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$ with the set of knots

$$\mathcal{X}_{\sigma+1} = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(\sigma+1)}\} \subset \mathbb{R}^d, \tag{3.1}$$

i.e., the problem of finding a polynomial $p \in \Pi_n$ satisfying the conditions

$$\alpha_i \leq p(\mathbf{x}^{(i)}) \leq \beta_i, \quad i = 1, \ldots, \sigma + 1. \tag{3.2}$$

According to Definition 3.1 the set $\mathcal{X}_{\sigma+1}$ is n-dependent and there is k_0, $1 \leq k_0 \leq \sigma + 1$, such that the set $\mathcal{X}_{\sigma+1} \setminus \{\mathbf{x}^{(k_0)}\}$ is n-independent. Suppose, without loss of generality, that $k_0 = \sigma + 1$. Hence, the following hold:

(3.3) $\mathcal{X}_{\sigma+1}$ is n-dependent, $\mathcal{X}_\sigma$ is n-independent.

In this subsection we are going to characterize the solvability of the σ-quasi-basic segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$ by determining the set of values of solutions of its n-independent subproblem $\{\mathcal{X}, \mathcal{S}\}^n_\sigma$ at the knot $\mathbf{x}^{(\sigma+1)}$:

$$\mathcal{A} := \left\{p(\mathbf{x}^{(\sigma+1)}),\ p \in Sol\{\mathcal{X}, \mathcal{S}\}^n_\sigma\right\}.$$

We will readily determine the maximal and minimal values in $\mathcal{A}$. On the other hand the set $Sol\{\mathcal{X}, \mathcal{S}\}^n_\sigma$ is convex. Hence, we have that the set of values $\mathcal{A}$ actually is a segment $[\alpha, \beta]$. Therefore, a necessary and sufficient condition for the solvability of the basic segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$ becomes the condition

$$[\alpha, \beta] \cap [\alpha_{\sigma+1}, \beta_{\sigma+1}] \neq \emptyset.$$

To determine the (endpoints of the) segment $[\alpha, \beta]$ we first enlarge, in view of Lemma 2.2, the n-independent set $\mathcal{X}_\sigma$ with a knot set $\mathcal{Y} = \{\mathbf{y}^{(i)}\}^N_{i=\sigma+1}$ till an n-poised set $\mathcal{Z} = \mathcal{X}_\sigma \cup \mathcal{Y}$. Then we use the Lagrange formula, according to which, we have for any polynomial $p \in \Pi_n$

$$p(\mathbf{x}) = \sum_{k=1}^{\sigma} p(\mathbf{x}^{(k)}) p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}) + \sum_{k=\sigma+1}^{N} p(\mathbf{y}^{(k)}) p^\star_{\mathbf{y}^{(k)}}(\mathbf{x}).$$

Next, by taking into account (3.3), let us use Corollary 2.3, with $\mathcal{X} = \mathcal{X}_\sigma$ and $A = \mathbf{x}^{(\sigma+1)}$. Then we get that all the fundamental polynomials of second sum above vanish at $\mathbf{x}^{(\sigma+1)}$, since all they vanish on $\mathcal{X}_\sigma$. Therefore, for any polynomial $p \in \Pi_n$ we have

$$p(\mathbf{x}^{(\sigma+1)}) = \sum_{k=1}^{\sigma} p(\mathbf{x}^{(k)}) p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}^{(\sigma+1)}). \tag{3.4}$$

In other words, the value of any polynomial from Π_n at the knot $\mathbf{x}^{(\sigma+1)}$ is determined by its values at the remaining knots of $\mathcal{X}_{\sigma+1}$, provided (3.3) holds.

Now, having the signs of the values of the fundamental polynomials in (3.4), we can easily determine the interval of values $[\alpha, \beta]$. Indeed, we get the minimal (maximal) value of polynomials $p \in Sol\{\mathcal{X}, \mathcal{S}\}^n_\sigma$, i.e., α, $(\beta,)$ by replacing the value $p(\mathbf{x}^{(k)})$ with α_k (β_k) in the expression in the right hand side of (3.4), if $p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}^{(\sigma+1)})$ is positive, and with β_k (α_k), otherwise. Therefore, we get

(3.5)
$$\alpha = \sum_{k=1}^{\sigma} \gamma_k p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}^{(\sigma+1)}), \qquad \beta = \sum_{k=1}^{\sigma} \gamma'_k p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}^{(\sigma+1)}),$$

where $\gamma_k = \alpha_k, \gamma'_k = \beta_k$ if $\operatorname{sgn} p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}^{(\sigma+1)})) = 1$, and $\gamma_k = \beta_k, \gamma'_k = \alpha_k$, otherwise. Let us mention, that α (β) equals to $-\infty$ $(+\infty)$, if $\gamma_k p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}^{(\sigma+1)})$ $(\gamma'_k p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}^{(\sigma+1)}))$ equals to $-\infty$ $(+\infty)$, for some k, $k = 1, \dots, \sigma$.

Thus, we obtain finally

Theorem 3.1. *Suppose we have a σ-quasi-basic problem $\{\mathcal{X}, \mathcal{S}\}^n_{\mathfrak{b}} = \{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$ with the set of knots $\mathcal{X}_{\sigma+1}$ satisfying the condition* (3.3). *Then it is solvable if and only if*

$$[\alpha, \beta] \cap [\alpha_{\sigma+1}, \beta_{\sigma+1}] \neq \emptyset, \tag{3.6}$$

where the endpoints of the first interval are given by (3.5).

Remark 3.1. *Consider a σ-quasi-basic segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$ with mixed conditions. Let us call the quantities α_i or β_i, $i = 1, \dots \sigma + 1$, "missing" if the neighboring inequality sign in* (3.2) *is strict. Then Theorem 3.1 still holds with the following possible changes in* (3.6)*: From the interval $[\alpha, \beta]$ the left endpoint α (the right endpoint β) is removed, if in* (3.5) *a coefficient γ_i (γ'_i) assumes a "missing" value: α_i or β_i. Hence, the interval $[\alpha, \beta]$ in* (3.6) *is replaced with $(\alpha, \beta]$, $[\alpha, \beta)$, (α, β), or remains unchanged.*

At the end of this subsection let us point out how one can find a solution of the σ-quasi-basic problem $\{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$, provided it is solvable. For this end we first choose a number $\xi \in [\alpha, \beta] \cap [\alpha_{\sigma+1}, \beta_{\sigma+1}]$. Then we present ξ as a convex combination of α and β : $\xi = \lambda_0 \alpha + (1 - \lambda_0)\beta$, $(0 \le \lambda_0 \le 1)$. Now one can verify readily that the polynomial

$$p(\mathbf{x}) = \sum_{k=1}^{\sigma} c_k p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}),$$

where $c_k = \lambda_0 \gamma_k + (1 - \lambda_0)\gamma'_k$, is a solution of $\{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$.

3.2. **The solvability of quasi-basic problem, II.** Suppose we have a σ-quasi-basic problem $\{\mathcal{X}, \mathcal{S}\}^n_{\mathfrak{b}} = \{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$ with the set of knots $\mathcal{X}_{\sigma+1}$ given by (3.1). Now we are going to present a solution of quasi-basic segmental problem, where all the $\sigma + 1$ knots take part in a same way.

Below we use standard multivariate notation. Set for $\mathrm{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d_+$ and $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$: $|\mathrm{i}| = i_1 + \cdots + i_d$, $\mathbf{x}^{\mathrm{i}} = x_1^{i_1} \cdots x_d^{i_d}$.

Consider the following polynomial space, where $|\mathrm{i}| = n + 1$:

$$\Pi_{n,\mathrm{i}} := \Pi^d_{n,\mathrm{i}} := \left\{q(\mathbf{x}) + c\mathbf{x}^{\mathrm{i}} : q \in \Pi_n,\ c \in \mathbb{R}\right\}.$$

Denote by $[\mathrm{i}]p$ the leading i-coefficient of $p(\mathbf{x}) = q(\mathbf{x}) + c\mathbf{x}^{\mathrm{i}} \in \Pi_{n,\mathrm{i}}$, i.e., $[\mathrm{i}]p := c$.

We first show that if $\{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$ is a quasi-basic problem then there is a multiindex i, $|\mathrm{i}| = n+1$, such that the interpolation problem with the knot set $\mathcal{X}_{\sigma+1}$ and the polynomial space $\Pi_{n,\mathrm{i}}$ is solvable, or equivalently, the knot set $\mathcal{X}_{\sigma+1}$ is $\Pi_{n,\mathrm{i}}$-independent.

Proposition 3.1. *Suppose we have a set of knots $\mathcal{X}_{\sigma+1}$ satisfying the condition* (3.3). *Then there is a multiindex i, $|i| = n+1$, such that the interpolation problem with the knot set $\mathcal{X}_{\sigma+1}$ and the polynomial space $\Pi_{n,i}$ is solvable.*

Proof. Fix a multiindex i, $|\mathrm{i}| = n+1$. Since the knot set $\mathcal{X}_\sigma$ is n-independent we have that the interpolation problem with the knot set $\mathcal{X}_\sigma$ and the polynomial space Π_n is solvable. In particular there is a polynomial $p_\mathrm{i} \in \Pi_n$ such that

$$p_\mathrm{i}(\mathbf{x}) = \mathbf{x}^\mathrm{i} \text{ for each } \mathbf{x} = \mathbf{x}^{(k)},\ k = 1, \ldots, \sigma.$$

In other words the polynomial $q_\mathrm{i} := \mathbf{x}^\mathrm{i} - p_\mathrm{i}(\mathbf{x})$ vanishes on $\mathcal{X}_\sigma$. Now we have an alternative: Either q_i vanishes at the knot $\mathbf{x}^{(\sigma+1)}$ and hence on $\mathcal{X}_{\sigma+1}$, or q_i does not vanish at $\mathbf{x}^{(\sigma+1)}$ and hence it is a fundamental polynomial of this knot with respect to the polynomial space $\Pi_{n,\mathrm{i}}$. In the later case, we get in view of Corollary 2.1, that the knot set $\mathcal{X}_{\sigma+1}$ is independent regarding the space $\Pi_{n,\mathrm{i}}$, and hence the interpolation problem with the knot set $\mathcal{X}_{\sigma+1}$ and the polynomial space $\Pi_{n,\mathrm{i}}$ is solvable. Assume to the contrary that we have the first alternative for all multiindexes i, $|\mathrm{i}| = n+1$. Then we get that the knot set $\mathcal{X}_{\sigma+1}$ is a solution of the following polynomial system:

$$\mathbf{x}^\mathrm{i} - p_\mathrm{i}(\mathbf{x}) = 0,\ \forall i,\ |i| = n+1. \tag{3.7}$$

But according to a known result (see Theorem 2.6, Chapter 1, [20]), the set of solutions of type (3.7) systems are necessarily n-independent, which contradicts the condition (3.3). □

Next, let us fix a multiindex i, satisfying the conditions of Proposition 3.1, and consider all polynomials from $\Pi_{n,\mathrm{i}}$, satisfying the conditions (3.2) of the σ-quasi-basic segmental problem. Denote the set of all such polynomials by $Sol\{\mathcal{X}, \mathcal{S}\}_{\sigma+1}^{n,\mathrm{i}}$. This time we are going to characterize the solvability of the σ-quasi-basic segmental problem $\{\mathcal{X}, \mathcal{S}\}_{\sigma+1}^{n}$ by determining the set of values of the leading coefficients of polynomials from $Sol\{\mathcal{X}, \mathcal{S}\}_{\sigma+1}^{n,\mathrm{i}}$:

$$\mathcal{B} := \left\{ [\mathrm{i}]p :\ p \in Sol\{\mathcal{X}, \mathcal{S}\}_{\sigma+1}^{n,\mathrm{i}} \right\}.$$

We will readily determine the maximal and minimal values of the leading coefficients in $\mathcal{B}$. On the other hand the latter set is convex. Hence, we obtain that the set $\mathcal{B}$ actually is a segment $[a, b]$. Therefore, a necessary and sufficient condition for the solvability of the basic segmental problem $\{\mathcal{X}, \mathcal{S}\}_{\sigma+1}^{n}$ becomes the condition

$$0 \in [a, b], \text{ or in other words, } a \le 0, b \ge 0.$$

To determine the segment $[a, b]$ we again are going to use the Lagrange interpolation formula. For this end we first enlarge, in view of Lemma 2.2, the $\Pi_{n,\mathrm{i}}$-independent set $\mathcal{X}_{\sigma+1}$ with a knot set $\mathcal{Y} = \{\mathbf{y}^{(i)}\}_{i=\sigma+2}^{N+1}$ till a $\Pi_{n,\mathrm{i}}$-poised set $\mathcal{Z} = \mathcal{X}_{\sigma+1} \cup \mathcal{Y}$. Now, according to the Lagrange formula, we have for any polynomial $p \in \Pi_{n,\mathrm{i}}$

$$p(\mathbf{x}) = \sum_{k=1}^{\sigma+1} p(\mathbf{x}^{(k)}) p^\star_{\mathbf{x}^{(k)}}(\mathbf{x}) + \sum_{k=\sigma+2}^{N+1} p(\mathbf{y}^{(k)}) p^\star_{\mathbf{y}^{(k)}}(\mathbf{x}).$$

Let us verify that the leading i-coefficients of all the fundamental polynomials of the second sum above vanish. Indeed, assume to the contrary that $[\mathrm{i}]p^\star_{\mathbf{y}^{(k_0)}} \neq 0$ for some k_0. Then let us choose constants c_k such that $[\mathrm{i}]q_k = 0$, where $q_k = p^\star_{\mathbf{x}^{(k)}} - c_k p^\star_{\mathbf{y}^{(k_0)}}$, $k = 1, \ldots, \sigma+1$. Now notice that $q_k \in \Pi_n$ are fundamental polynomials of knots of the set $\mathcal{X}_{\sigma+1}$. Therefore, the latter set is n-independent, which contradicts the condition (3.3).
Thus, we have for any polynomial $p \in \Pi_{n,\mathrm{i}}$

$$[\mathrm{i}]p = \sum_{k=1}^{\sigma+1} p(\mathbf{x}^{(k)}) [\mathrm{i}] p^\star_{\mathbf{x}^{(k)}}. \tag{3.8}$$

Now, having the signs of [i]-leading coefficients of the fundamental polynomials in above sum, we can easily determine the interval $[a, b]$. Indeed, we get the minimal value a (maximal value b) by replacing the value $p(\mathbf{x}^{(k)})$ with α_k (β_k) in the expression in the right hand side of (3.8), if $[\mathrm{i}]p^\star_{\mathbf{x}^{(k)}}$ is positive and with β_k (α_k), otherwise. Thus, we have

$$a = \sum_{k=1}^{\sigma+1} \gamma_k [\mathrm{i}] p^\star_{\mathbf{x}^{(k)}}, \qquad b = \sum_{k=1}^{\sigma+1} \gamma'_k [\mathrm{i}] p^\star_{\mathbf{x}^{(k)}}, \tag{3.9}$$

where $\gamma_k = \alpha_k, \gamma'_k = \beta_k$ if $\mathrm{sgn}[\mathrm{i}]p^\star_{\mathbf{x}^{(k)}} = 1$, and $\gamma_k = \beta_k, \gamma'_k = \alpha_k$, otherwise. Let us mention, that a (b) equals to $-\infty$ ($+\infty$), if $\gamma_k[\mathrm{i}]p^\star_{\mathbf{x}^{(k)}}$ ($\gamma'_k[\mathrm{i}]p^\star_{\mathbf{x}^{(k)}}$) equals to $-\infty$ ($+\infty$), for some k, $k = 1, \ldots, \sigma+1$. Hence, we get finally

Theorem 3.2. *A σ-quasi-basic problem $\{\mathcal{X}, \mathcal{S}\}_{\mathfrak{b}}^{n} = \{\mathcal{X}, \mathcal{S}\}_{\sigma+1}^{n}$ with the set of knots $\mathcal{X}_{\sigma+1} \subset \mathbb{R}^d$ is solvable if and only if*

$$0 \in [a, b],\ \textit{i.e.,}\quad a \le 0 \textit{ and } b \ge 0,$$

where a and b are given by (3.9).

Let us mention that the analog of Remark 3.1 holds in this case for the segmental problem with mixed conditions. At the end let us point out how one can find a solution of the σ-quasi-basic problem $\{\mathcal{X}, \mathcal{S}\}_{\sigma+1}^{n}$, provided it is solvable. For this end we first present 0 as a convex combination of a and b: $0 = \lambda_0 a + (1-\lambda_0) b$, $(\lambda_0 = b/(b-a))$. Now one can verify readily that the polynomial

$$p(\mathbf{x}) = \sum_{k=1}^{\sigma} c_k p^\star_{\mathbf{x}^{(k)}}(\mathbf{x})),$$

where $c_k = \lambda_0\gamma_k + (1-\lambda_0)\gamma'_k$, is a solution of $\{\mathcal{X}, \mathcal{S}\}^n_{\sigma+1}$.

4. The solvability of general segmental interpolation problem

In this section we will present two characterizations for the solvability of segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_I$: in terms of its basic subproblems and in terms of quasi-basic subproblems of certain cardinality (see forthcoming Theorem 4.2 and Corollary 4.1). In the proof we will use the Helly theorem (see Theorem 2.1.6, Chapter 2, [17]):

Theorem 4.1 (Helly). *Let $\mathbb{U}$ be a real linear space with $\dim \mathbb{U} = h$ and $\{\mathcal{U}_i,\ i = 1,\dots,m\}$ be a collection of m convex subsets of $\mathbb{U}$, with $m \geq h+2$. If the intersection of every $h+1$ of these sets is nonempty, then the whole collection has a nonempty intersection: $\bigcap_{i=1}^m \mathcal{U}_i \neq \emptyset$. Moreover, this remains true for an infinite collection $\{\mathcal{U}_i,\ i \in I\}$, if, in addition, all $\mathcal{U}_i$ are closed and intersection of some finite subcollection is compact.*

First let us prove

Lemma 4.1. *Let $\mathcal{X}_m$ be a finite knot set with $\mathcal{H}_{n,\mathcal{X}_m} = h_0$ and $\mathcal{X} \subset \mathcal{X}_m$ be a maximal n-independent subset: $\#\mathcal{X} = h_0$. Next, suppose that $\mathcal{X}$ is enlarged with a knot set $\mathcal{Y}$ till an n-poised set $\mathcal{X} \cup \mathcal{Y}$, where $\#\mathcal{Y} = N - h_0$. Then the segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_m$ or any its subproblem is solvable (within Π_n) if and only if it is solvable within $\mathcal{P}_{n,\mathcal{Y}}$.*

Proof. The "if" direction follows from the inclusion $\mathcal{P}_{n,\mathcal{Y}} \subset \Pi_n$. For the direction "only if" suppose, without loss of generality, that the subproblem $\{\mathcal{X}, \mathcal{S}\}^n_k$ of the segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_m$, with $k \leq m$, is solvable (within Π_n), i.e., there is a polynomial $p_0 \in Sol\{\mathcal{X}, \mathcal{S}\}^n_k$. Assume that $\mathcal{Y} = \{\mathbf{y}^{(i)}\}^N_{i=h_0+1}$ and $p^\star_i := p^\star_{\mathbf{y}^{(i)},\mathcal{X}\cup\mathcal{Y}}$, $i = h_0+1,\dots,N$, are the n-fundamental polynomials. Consider the polynomial

$$q_0 := p_0 - \sum_{k=h_0+1}^{N} p_0(\mathbf{y}^{(i)}) p^\star_i. \tag{4.1}$$

We have that $q_0 \in \mathcal{P}_{n,\mathcal{Y}}$, i.e., $q_0\big|_{\mathcal{Y}} = 0$. Now notice that the fundamental polynomials in the right hand side of (4.1) vanish on $\mathcal{X}$. Therefore, in view of Corollary 2.4, they vanish also on $\mathcal{X}_m$. Thus, we obtain that $q_0\big|_{\mathcal{X}_m} = p\big|_{\mathcal{X}_m}$. Hence, q_0 is a solution of the segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_k$, too. □

Now we present the main result of this section.

Theorem 4.2. *The segmental interpolation problem $\{\mathcal{X}, \mathcal{S}\}^n_I$ is solvable if and only if all its basic subproblems are solvable.*

Before we prove Theorem let us verify that it yields the following:

Corollary 4.1. *The problem $\{\mathcal{X}, \mathcal{S}\}^n_I$ is solvable if and only if all its quasi-basic subproblems of cardinality h_0+1 are solvable, where $h_0 = \mathcal{H}_n(\mathcal{X}_I)$.*

Proof. Indeed, if all basic subproblems of $\{\mathcal{X}, \mathcal{S}\}^n_I$ are solvable then, by Corollary 3.1, all quasi-basic subproblems of $\{\mathcal{X}, \mathcal{S}\}^n_I$ are solvable, too. For the reverse implication it suffices to show that for any basic subproblem $\{\mathcal{X}, \mathcal{S}\}^n_{\mathfrak{b}}$, with $\mathfrak{b} \subset I$, $\#\mathfrak{b} < h_0$, there is an equivalent h_0-quasi-basic subproblem $\{\mathcal{X}, \mathcal{S}\}^n_{\tilde{\mathfrak{b}}}$, with $\mathfrak{b} \subset \tilde{\mathfrak{b}} \subset I$, $\#\tilde{\mathfrak{b}} = h_0+1$. To show this suppose that the knot set $\mathcal{X} = \mathcal{X}_{\mathfrak{b}} \setminus \{A\}$, where $A \in \mathcal{X}_{\mathfrak{b}}$, is n-independent. Suppose also, in view of Lemma 2.4, that an enlarged set $\mathcal{Z} := \mathcal{X} \cup \mathcal{Y}$ is a maximal n-independent subset of $\mathcal{X}_I$, hence $\#\mathcal{Z} = h_0$. Let us show that as a desired set we can take $\tilde{\mathfrak{b}} := \mathcal{X}_{\mathfrak{b}} \cup \mathcal{Y} = \mathcal{Z} \cup \{A\}$. Indeed, in view of Corollary 2.3, we have that the fundamental polynomials of the knots of $\mathcal{Y}$ with respect to the knot set $\mathcal{Z}$ are fundamental also with respect to the knot set $\mathcal{Z} \cup \{A\}$. Therefore, in view of Corollary 3.1, the basic subproblem $\{\mathcal{X}, \mathcal{S}\}^n_{\mathfrak{b}}$ is equivalent to the segmental subproblem with the knot set $\mathcal{Z} \cup \{A\}$. □

Proof of Theorem 4.2. Let us divide the proof into two parts, where the cases of finite and infinite knot sets are discussed, respectively.

Part 1. Consider first the finite segmental problem: $\{\mathcal{X}, \mathcal{S}\}^n_m$. We are going to use Theorem 4.1 (the Helly theorem) for spaces $\mathbb{U} := \mathcal{P}_{n,\mathcal{Y}}$, with various point sets $\mathcal{Y}$, to show that $Sol\{\mathcal{X}, \mathcal{S}\}^n_m \neq \emptyset$. We will carry out the proof in Part 1 in two steps.

Step 1. Let us show that the segmental problem $\{\mathcal{X}, \mathcal{S}\}^n_m$ is solvable by assuming that all its subproblems of cardinality h_0+1 are solvable, where $h_0 := \mathcal{H}_n(\mathcal{X}_m)$. From latter equality we have that $m \geq h_0$. Note that if $m = h_0$ then the segmental problem is n-independent. In this case of course it is solvable (Lemma 2.1). Also if $m = h_0+1$ then the segmental problem is solvable by the assumption of Step 1. Hence, assume that $m \geq h_0+2$.
Suppose that the knot set $\mathcal{X}_J$, where $J = \{j_1,\dots,j_{h_0}\} \subset \{1,\dots,m\}$, is a maximal n-independent subset of $\mathcal{X}_m$. Suppose also, in view of

Lemma 2.2, that an enlarged set $\mathcal{X}_J \cup \mathcal{Y}$ is n-poised, where $\#\mathcal{Y} = N - h_0$.
By Lemma 4.1 the segmental problem $\{\mathcal{X}, \mathcal{S}\}_m^n$ and any its subproblem is solvable (within Π_n) if and only if it is solvable within $\mathcal{P}_{n,\mathcal{Y}}$.
According to Proposition 2.1 (part "moreover"), we have that

$$\dim \mathcal{P}_{n,\mathcal{Y}} = N - \#\mathcal{Y} = \#\mathcal{X}_J = h_0. \tag{4.2}$$

Denote for $i = 1, \ldots, m$,

$$\mathcal{U}_i := \left\{ p \in \mathcal{P}_{n,\mathcal{Y}} : \alpha_i \le p(\mathbf{x}^{(i)}) \le \beta_i \right\}.$$

It is easily seen that the sets $\mathcal{U}_i \subset \mathcal{P}_{n,\mathcal{Y}}$ here are convex. Then let us verify that the intersection of any h_0+1 sets of $\mathcal{U}_i$ is nonempty. Indeed, $\bigcap_{\ell=1}^{h_0+1} \mathcal{U}_{k_\ell} = \left\{ p \in \mathcal{P}_{n,\mathcal{Y}} : \ \alpha_{k_\ell} \le p(\mathbf{x}^{(k_\ell)}) \le \beta_{k_\ell}, \ \ell = 1, \ldots, h_0+1 \right\}$. In view of the assumption of Step 1 and Lemma 4.1, all subproblems of cardinality h_0+1 of $\{\mathcal{X}, \mathcal{S}\}_J^n$ are solvable within $\mathcal{P}_{n,\mathcal{Y}}$. Therefore, $\bigcap_{\ell=1}^{h_0+1} \mathcal{U}_{k_\ell} \neq \emptyset$. Now, by Theorem 4.1, the whole collection has a nonempty intersection: $\bigcap_{i=1}^{m} \mathcal{U}_i \neq \emptyset$. Thus, the problem $\{\mathcal{X}, \mathcal{S}\}_m^n$ is solvable within $\mathcal{P}_{n,\mathcal{Y}}$, and hence it is solvable.

Step 2. What remains to show in this step is that all subproblems of cardinality h_0+1 of $\{\mathcal{X}, \mathcal{S}\}_m^n$ are solvable. For this we use complete induction on the cardinality of subproblems. Note that the subproblems with one knot are evidently solvable. Suppose that all subproblems of cardinality at most k of $\{\mathcal{X}, \mathcal{S}\}_m^n$ are solvable and let us prove that the subproblems of cardinality $k+1$ are solvable, too.
Thus, consider any subproblem $\{\mathcal{X}, \mathcal{S}\}_J^n$ of cardinality $\#J = k+1$, where $J = \{j_1, \ldots, j_{k+1}\} \subset \{1, \ldots, m\}$. Set $h_1 := \mathcal{H}_n(\mathcal{X}_J)$. If $h_1 = k+1$, then $\{\mathcal{X}, \mathcal{S}\}_J^n$ is an independent subproblem and hence it is solvable by Lemma 2.1. If $h_1 = k$, then either it is a basic subproblem and is solvable by assumption of Theorem, or it is a quasi-basic subproblem. In the latter case, by Corollary 3.1, $\{\mathcal{X}, \mathcal{S}\}_J^n$ is equivalent to a basic subproblem and hence is solvable, too.

Now, assume that $h_1 \le k-1$. Let us apply Step 1 to the subproblem $\{\mathcal{X}, \mathcal{S}\}_J^n$, considered as a problem. In view of the induction hypothesis we have that all subproblems of cardinality at most k of $\{\mathcal{X}, \mathcal{S}\}_J^n$ are solvable. On the other hand we have that $k \ge h_1+1$. Thus, we have that all subproblems of cardinality h_1+1 of $\{\mathcal{X}, \mathcal{S}\}_J^n$ are solvable. Therefore, according to Step 1, the segmental problem $\{\mathcal{X}, \mathcal{S}\}_J^n$ is solvable.

Part 2. Consider now the case of segmental problem with infinite knot set: $\{\mathcal{X}, \mathcal{S}\}_I^n$. Set $h_2 := \mathcal{H}_n(\mathcal{X}_I)$. Suppose that the knot set $\mathcal{X}_J$, where $J = \{j_1, \ldots, j_{h_2}\} \subset I$, is a maximal n-independent subset of $\mathcal{X}_I$. Suppose also, in view of Lemma 2.2, that an enlarged set $\mathcal{Z} := \mathcal{X}_J \cup \mathcal{Y}$ is an n-poised set, where $\#\mathcal{Y} = N - h_2$. We are going to use Theorem 4.1 (the Helly theorem) with $\mathbb{U} := \mathcal{P}_{n,\mathcal{Y}}$. By using the Lagrange formula we get readily

$$p \in \mathcal{P}_{n,\mathcal{Y}} \Leftrightarrow p(\mathbf{x}) = \sum_{\ell=1}^{h_2} c_\ell p^\star_{\mathbf{x}^{(j_\ell)}}(\mathbf{x}), \tag{4.3}$$

where $c_\ell = p(\mathbf{x}^{(j_\ell)}) \in \mathbb{R}$ and $p^\star_{\mathbf{x}^{(j_\ell)}} = p^\star_{\mathbf{x}^{(j_\ell)},\mathcal{Z}}$. Denote for $\nu \in I$:

$$\mathcal{U}_\nu := \left\{ p \in \mathcal{P}_{n,\mathcal{Y}} : \alpha_\nu \le p(\mathbf{x}^{(\nu)}) \le \beta_\nu \right\}.$$

Let us show that the segmental problem $\{\mathcal{X}, \mathcal{S}\}_I^n$ is solvable: $\bigcap_{\nu \in I} \mathcal{U}_\nu \neq \emptyset$.

According to Part 1, we have that any finite subproblem of $\{\mathcal{X}, \mathcal{S}\}_I^n$ is solvable. Hence, any finite intersection of sets $\{\mathcal{U}_\nu : \nu \in I\}$ is not empty. As was mentioned above the sets $\mathcal{U}_\nu \subset \mathcal{P}_{n,\mathcal{Y}}$ are convex. Let us verify that they are also closed. Indeed, assume, in view of (4.3), that $\alpha_\nu \le p_s(\mathbf{x}^{(\nu)}) \le \beta_\nu$, where

$$p_s(\mathbf{x}) = \sum_{\ell=1}^{h_2} c_\ell^{(s)} p^\star_{\mathbf{x}^{(j_\ell)}}(\mathbf{x}) \text{ and } c_\ell^{(s)} \to c_\ell.$$

Then we get readily that

$$\alpha_\nu \le p(\mathbf{x}^{(\nu)}) \le \beta_\nu, \text{ where } p(\mathbf{x}) = \sum_{\ell=1}^{h_2} c_\ell p^\star_{\mathbf{x}^{(j_\ell)}}(\mathbf{x}).$$

Now, denote by $\mathcal{U}^\diamond := \bigcap_{\ell=1}^{h_2} \mathcal{U}_{j_\ell} =$

$$\left\{ p \in \mathcal{P}_{n,\mathcal{Y}} : \alpha_{j_\ell} \le p(\mathbf{x}^{(j_\ell)}) \le \beta_{j_\ell}, \ \ell = 1, \ldots, h_2 \right\}.$$

In view of (4.3) the set $\mathcal{U}^\diamond \subset \mathcal{P}_{n,\mathcal{Y}}$ is bounded. On the other hand it is closed as an intersection of closed sets. Hence, $\mathcal{U}^\diamond$ is a compact set. Thus, according to Theorem 4.1 (the Helly theorem, part "moreover") we have that $\bigcap_{\nu \in I} \mathcal{U}_\nu \neq \emptyset$.

Remark 4.1. *Theorem 4.2 remains valid, in the case of finite segmental problems, if some of α_ν and β_ν assume extended values: $+\infty$ and $-\infty$, respectively, or some of the inequalities in* (1.1) *are strict (the case of mixed conditions). In the latter case the solvability of basic subproblems must be verified according to Remark 3.1.*

Indeed, the weakness of the inequalities and the finiteness of the mentioned values in (1.1) were used only in the case of infinite knot sets to show that the sets $\mathcal{U}_\nu$ are closed and the set $\mathcal{U}^\diamond$ is compact.

4.1. **A method of solving finite segmental problems.** Consider a segmental problem $\{\mathcal{X}, \mathcal{S}\}_m^n$, with a set of knots $\mathcal{X}_m$ given by (1.2). Assume that $\{\mathcal{X}, \mathcal{S}\}_m^n$ is solvable, i.e., the hypotheses of Theorem 4.2 hold, and let us bring a method for finding a solution. The method is inductive, with respect to the cardinality of the knot set. Let us mention that a step of the method is based on a proof of Theorem 4.1, (see the proof of Theorem 2.1.6, Chapter 2, [17]). To start, note that for a subproblem with one knot we can easily choose a solution - just an intermediate constant. Suppose that we have solutions of all subproblems of $\{\mathcal{X},\mathcal{S}\}_m^n$ with set of knots of cardinality k and let us find a solution of any given subproblem of cardinality $k+1$. Assume, without loss of generality, that the given subproblem of cardinality $k+1$ is $\{\mathcal{X},\mathcal{S}\}_{k+1}^n$.

Set $h_0 := \mathcal{H}_n(\mathcal{X}_{k+1})$. Of course we have that $h_0 \le k+1$ and $h_0 \le h$, where $h := \mathcal{H}_n(\mathcal{X}_m)$. Suppose that the knot set $\mathcal{X}_J$, where $J = \{j_1,\dots,j_{h_0}\} \subset \{1,\dots,k+1\}$, is a maximal n-independent subset of $\mathcal{X}_{k+1}$. Suppose also, in view of Lemma 2.2, that an enlarged set $\mathcal{X}_J \cup \mathcal{Y}_0$ is n-poised, where $\#\mathcal{Y}_0 = N - h_0$. We are going to find a polynomial $p \in \mathcal{P}_{n,\mathcal{Y}_0}$ satisfying the conditions

$$\alpha_i \le p(\mathbf{x}^{(i)}) \le \beta_i, \quad i = 1,\dots,k+1. \tag{4.4}$$

If $h_0 = k+1$, then $\{\mathcal{X},\mathcal{S}\}_{k+1}^n$ is an independent subproblem and to find a solution it suffices to solve an interpolation problem with any intermediate values (Lemma 2.1). If $h_0 = k$, then $\{\mathcal{X},\mathcal{S}\}_{k+1}^n$ is a quasi-basic (or basic) subproblem. We know that quasi-basic problems are solvable if the basic problems are such (Corollary 3.1). Also, at the ends of Subsections 3.1 and 3.2, we have descriptions of how to find a solution of any solvable quasi-basic problem.

Hence, assume that $h_0 \le k-1$. Note that, in view of (4.2), we have that

$$\dim \mathcal{P}_{n,\mathcal{Y}_0} = h_0. \tag{4.5}$$

By using Lemma 4.1, we may assume that we have solutions of subproblems of $\{\mathcal{X},\mathcal{S}\}_{k+1}^n$ of cardinality k *within* $\mathcal{P}_{n,\mathcal{Y}_0}$.

Then, let us denote by $M := \{1,\dots,k+1\}$. Set, $M_i := M \setminus \{i\}$, $i = 1,\dots,h_0+2$. (Recall that $h_0 \le k-1$, and hence $h_0+2 \le k+1$.) We have that each subproblem $\{\mathcal{X},\mathcal{S}\}_{M_i}^n$ is of cardinality k. Assume that the following polynomial is a solution of it within $\mathcal{P}_{n,\mathcal{Y}_0}$:

$$q_i \in \mathcal{P}_{n,\mathcal{Y}_0}, \text{ where } i = 1,\dots,h_0+2. \tag{4.6}$$

Next, let us verify that one can find multipliers $\omega_1,\dots,\omega_{h_0+2} \in \mathbb{R}$, not all zero, such that

$$\sum_{i=1}^{h_0+2} \omega_i p_i = 0, \quad \sum_{i=1}^{h_0+2} \omega_i = 0. \tag{4.7}$$

Indeed, first relation of (4.7), in view of (4.5) and (4.6), can be reduced to h_0 scalar linear homogeneous equations. Thus, (4.7) is equivalent to a system of h_0+1 homogeneous equations in h_0+2 unknowns, and hence has a nontrivial solution.

Denote by E_+ the set of subscripts of positive multipliers ω_i in (4.7), and by E_- the set of subscripts of negative or zero multipliers. Then we have from (4.7):

$$q := \sum_{i\in E_+} \omega_i p_i = -\sum_{i\in E_-} \omega_i p_i.$$

Now, one can verify readily that the polynomial $Q := (1/\omega)q$, where $\omega = \sum_{i\in E_+} \omega_i = -\sum_{i\in E_-} \omega_i$, is a desired solution of $\{\mathcal{X},\mathcal{S}\}_{k+1}^n$. Indeed, Q is a convex combination of $\{p_i : i \in E_+\}$ and p_i satisfies all relations of (4.4) except possibly the ith one. Hence, Q satisfies all relations of (4.4) except possibly the $i$ths with $i \in E_+$. At the same time Q is a convex combination of $\{p_i : i \in E_-\}$ and hence Q satisfies all relations of (4.4) except possibly the $i$ths with $i \in E_-$. On the other hand we have $E_- \cap E_+ = \emptyset$, therefore Q satisfies all the relations of (4.4).

Remark 4.2. *Note that the above described method of solving at the same time presents another proof of Theorem 4.2 in the case of finite segmental problems.*

5. The univariate segmental problem

Denote the space of univariate polynomials of total degree at most n by

$$\pi_n := \{p = a_0 + a_1 x + \cdots + a_n x^n\}, \quad \dim \pi_n = n+1.$$

Let $\mathcal{X}_I = \{x_\nu : \nu \in I\} \subset \mathbb{R}$ be any set of points. Let $\mathcal{S}_I = \{[\alpha_\nu,\beta_\nu] : \nu \in I\}$ be a respective set of any segments. The univariate segmental interpolation problem $\{\mathcal{X},\mathcal{S}\}_I^n$ is to find out whether there is a polynomial $p \in \pi_n$, satisfying the conditions

$$\alpha_\nu \le p(x_\nu) \le \beta_\nu, \quad \forall \nu \in I.$$

In the case when I is finite we use the notation $\mathcal{X}_m = \{x_1, x_2, \dots, x_m\} \subset \mathbb{R}$ for the set of knots and $\mathcal{S}_m = \{[\alpha_i,\beta_i] : i = 1,\dots,m\}$ for the set of segments. The corresponding finite segmental problem is denoted by $\{\mathcal{X},\mathcal{S}\}_m^n$. Denote the set of all its solutions, as in the multivariate case, by $Sol\{\mathcal{X},\mathcal{S}\}_m^n$. In the univariate case any set of knots of cardinality at most $n+1$ is n-independent and any set of cardinality $n+2$ is essentially n-dependent. Therefore, Definition 3.1 in the univariate case simply reduces to:

Definition 5.1. *We call a subproblem* $\{\mathcal{X},\mathcal{S}\}_{\mathfrak{b}}^n$, $\mathfrak{b} \subset I$, *of* $\{\mathcal{X},\mathcal{S}\}_I^n$ basic *if* $\#\mathfrak{b} = n+2$.

Now, we get from Theorem 4.2:

Theorem 5.1. *The univariate segmental interpolation problem* $\{\mathcal{X},\mathcal{S}\}_I^n$ *is solvable if and only if all its subproblems of cardinality* $n+2$*, are solvable.*

In the last two subsections we bring the two characterizations of solvability of basic segmental problem in the univariate case, in more details. In particular, we find explicitly the values of signs of fundamental polynomials taking part in relations (3.5) and (3.9).

5.1. **The solvability of univariate basic problem, I.** Here we start with a subproblem of the basic segmental problem $\{\mathcal{X},\mathcal{S}\}_{n+2}^{n}$ where the last knot is absent, i.e., $\{\mathcal{X},\mathcal{S}\}_{n+1}^{n}$.

Let us determine the set of values of solutions of the above subproblem at *any* fixed point $x\in\mathbb{R}$, i.e., we determine the set

$$\mathcal{A}_x := \{p(x),\ p\in Sol\{\mathcal{X},\mathcal{S}\}_{n+1}^{n}\}.$$

We know that this set of values forms an interval $[\alpha(x),\beta(x)]$. Therefore, a necessary and sufficient condition for the solvability of the univariate basic segmental problem $\{\mathcal{X},\mathcal{S}\}_{n+2}^{n}$ becomes the condition

$$[\alpha(x_{n+2}),\beta(x_{n+2})]\cap[\alpha_{n+2},\beta_{n+2}]\neq\emptyset.$$

To find $\alpha(x)$ and $\beta(x)$ we use the Lagrange interpolation formula, according to which, we have for any polynomial $p\in\pi_n$

$$p(x)=\sum_{k=1}^{n+1}p(x_k)p_k^\star(x), \tag{5.1}$$

where

$$p_k^\star(x)=\prod_{i=1,\ i\neq k}^{n+1}\frac{x-x_i}{x_k-x_i}.$$

Next, since we are going to determine the sign of $p_k^\star(x)$, assume, without loss of generality, that the sequence $\{x_i\}_{i=1}^{n+1}$ is increasing. Consider the intervals $\mathcal{I}_\nu,\ \nu=0,\ldots,n+1$, where

$$\mathcal{I}_\nu=\begin{cases}(-\infty,x_1) & \text{if } \nu=0,\\ (x_\nu,x_{\nu+1}) & \text{if } \nu=1,\ldots,n,\\ (x_{n+1},+\infty) & \text{if } \nu=n+1.\end{cases}$$

Now, assume that $x\in\mathcal{I}_\nu$, and let us determine the interval $\mathcal{A}_x=[\alpha(x),\beta(x)]=:[\alpha_\nu(x),\beta_\nu(x)]$, where $0\le\nu\le n+1$ is fixed. Indeed, we get the possible minimal (maximal) value $\alpha_\nu(x)$ ($\beta_\nu(x)$) of $p\in Sol\{\mathcal{X},\mathcal{S}\}_{n+1}^{n}$ by replacing the value $p(x_k)$ in the right hand side of (5.1) with α_k (β_k), if $p_k^\star(x)$ is positive, and with β_k (α_k), otherwise.

Now, notice that the fundamental polynomials $p_\nu^\star(x)$ and $p_{\nu+1}^\star(x)$ both are positive on $\mathcal{I}_\nu$. While the polynomials $p_k^\star(x),\ k=\nu,\nu-1,\ldots,$ as well as the polynomials $p_k^\star(x),\ k=\nu+1,\nu+2,\ldots,$ alternate their signs on $\mathcal{I}_\nu$. Thus, we get

$$\alpha_\nu(x)=\sum_{k=1}^{n}\gamma_k p_k^\star(x),\quad \beta_\nu(x)=\sum_{k=1}^{n}\gamma_k' p_k^\star(x),\ \ x\in\mathcal{I}_\nu,$$

where $\gamma_k=\alpha_k,\gamma_k'=\beta_k$ for $k=\nu,\nu-2,\nu-4,\ldots,$ and $k=\nu+1,\nu+3,\nu+5,\ldots;$ $\gamma_k=\beta_k,\gamma_k'=\alpha_k$ otherwise.

5.2. **The solvability of univariate basic problem, II.** In the univariate case the approach presented in Section 3.2 becomes much more simple. Namely, we start with the set of polynomials of degree not exceeding $n+1$ satisfying the conditions of basic segmental problem $\{\mathcal{X},\mathcal{S}\}_{n+2}^{n}$. Then we choose from this set of polynomials a polynomial whose leading coefficient, i.e., the coefficient of the monomial x^{n+1}, vanishes.

Notice that the leading coefficient of any polynomial $p\in\pi_{n+1}$ coincides with the divided difference $[x_1,\ldots,x_{n+2}]p$ (for any $n+2$ knots). To start let us consider

$$\mathcal{B}:=\left\{[x_1,\ldots,x_{n+2}]p,\ p\in Sol\{\mathcal{X},\mathcal{S}\}_{n+2}^{n+1}\right\}.$$

We know that the set $\mathcal{B}$ forms an interval $[a,b]$. Thus, a necessary and sufficient condition for the solvability of the univariate basic segmental problem $\{\mathcal{X},\mathcal{S}\}_{n+2}^{n}$ becomes the condition

$$0\in[a,b],\ \text{ or in other words, }\ a\le 0, b\ge 0.$$

To determine the minimal and maximal values: a and b, we use the Lagrange formula for the divided difference:

$$[x_1,\ldots,x_{n+2}]p=\sum_{k=1}^{n+2}\lambda_k p(x_k),$$

where

$$\lambda_k=\prod_{i=1,\ i\neq k}^{n+2}\frac{1}{x_k-x_i}.$$

Assume, without loss of generality, that the sequence of knots $\{x_i\}_{i=1}^{n+2}$ is increasing. Then we have that

$$\mathrm{sgn}\lambda_k=(-1)^{n-k},\quad k=1,\ldots,n+2.$$

Therefore, we obtain that

$$a=\sum_{k=1}^{n+2}\gamma_k\lambda_k\le[x_0,\ldots,x_{n+1}]p\le\sum_{k=1}^{n+2}\gamma_k'\lambda_k=b,$$

where $\gamma_k=\alpha_k,\gamma_k'=\beta_k$, if $k=n+2,n,n-2,\ldots,$ and $\gamma_k=\beta_k,\gamma_k'=\alpha_k$, otherwise.

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