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UNIVALENCE CONDITIONS FOR A NEW INTEGRAL OPERATOR

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ABSTRACT. In the present paper, we will obtain norm estimates of the pre-Schwarzian derivatives for $F_{\lambda,\mu}(z)$, such that

$$F_{\lambda,\mu}(z)=\int_0^z \prod_{i=1}^{n}(f'_i(t))^{\lambda_i}\left(\frac{f_i(t)}{t}\right)^{\mu_i} dt \quad ;(z\in D),$$

where $\lambda_i,\mu_i\in\mathbb{R}$, $\lambda_i=(\lambda_1,\lambda_2,\cdots\lambda_n),\mu=(\mu_1,\mu_2,\cdots\mu_n)$ and f_i belongs to the class of convex univalent functions $C\subset S$.

1. INTRODUCTION

Let $D=\{z\in\mathbb{C}:|z|<1\}$ be the open unit disk in the complex plane and H denotes the space of holomorphic functions on D. Further, let $\mathcal{A}$ denote the class of functions $f(z)$ of the form :

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

which are analytic in the open unit disk D and satisfy the condition $f(0)=f'(z)-1=0$. The subclass of $\mathcal{A}$, consisting of all univalent functions $f(z)$ in D is denoted by S.

A function $f\in\mathcal{A}$ is a convex function of order $\alpha, 0\le\alpha<1$, if f satisfies the following inequality

$$Re\left\{\frac{zf''(z)}{f'(z)}+1\right\}>\alpha,\quad z\in D,$$

and we denote this class by $C(\alpha)$.

Similarly, if $f\in\mathcal{A}$ satisfies the following inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad z\in D,$$

for some $\alpha, 0\le\alpha<1$, then f is said to be starlike of of order α and we denote this class by $S^*(\alpha)$, see [6]. We note that

$$f\in C\Leftrightarrow zf'(z)\in S^*,\quad z\in D.$$

In particular case, the classes $C(0)=C, S^*(0)=S^*$ are familiar classes of starlike and convex functions in D.

For a locally univalent holomophic function f, we define

$$T_f(z)=\frac{f''(z)}{f'(z)},$$

which is said to be pre-Schwarzian derivative. For a locally univalent function of f in D, we define the norm of T_f by

$$\|T_f\|=\sup_{|z|<1}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|.$$

It is well-known that from Becker's univalence criterion, see [4], every analytic function f in D with $\|T_f\|\le 1$ is univalent in D. Conversely $\|T_f\|\le 6$ holds if f be univalent.

Theorem 1.1. *Let f be analytic and locally univalent in D. Then*
i) if $\|T_f\|\le 1$ then f is univalent, and
ii) if $\|T_f\|\le 2$, then f is bounded.

Proof. See [4, 5]. □

Theorem 1.2. *Let $0\le\alpha<1$ and $f\in S$.*
i) If f is starlike of order α, i.e $Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha$ then $\|T_f\|\le 6-4\alpha$.
ii) If f is convex of order α, i.e $Re\left\{\frac{zf''(z)}{f'(z)}+1\right\}>\alpha$, then $\|T_f\|\le 4(1-\alpha)$.

Proof. See [8]. □

The study of the integral operators has been rapidly investigated by many authors in the field of

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$$\Lambda[f](z)=\int_0^z \frac{f(\zeta)}{\zeta}d\zeta,$$

was introduced by Alexander, see [1]. Note that $f\in S^*(\alpha)\Leftrightarrow \Lambda(f)\in C(\alpha)$.

For the complex number γ, Kim and Merkes, see [9] considered the nonlinear integral transform

$$\Lambda_\gamma(f)[z]=\int_0^z (\frac{f(\zeta)}{\zeta})^\gamma d\zeta\,.$$

For $f_i(z)\in A$ and $\gamma_i>0$, for all $i\in\{1,2,\dots n\}$. Breaz and Breaz in [2] introduced the following integral operator:

$$F_n[f](z)=\int_0^z\prod_{i=1}^n\left(\frac{f_i(t)}{t}\right)^{\gamma_i}dt\,.$$

In [10] Kim, Ponnusamy and Sugawa defined the following integral operator

$$I_\gamma[f](z)=\int_0^z (f'(t))^\gamma dt\,,$$

for $\gamma\in\mathbb{C}$, $f\in A$. Breaz et.al, [3] introduced the following integral operator

$$F_{\gamma_1,\gamma_2,\cdots,\gamma_n}[f](z)=\int_0^z\prod_{i=1}^n (f'_i(t))^{\gamma_i}dt.$$

Recently Frasin introduced the following integral operator in [7],

(1.1)

$$F_{\lambda,\mu}(z)=\int_0^z\prod_{i=1}^n (f'_i(t))^{\lambda_i}\left(\frac{f_i(t)}{t}\right)^{\mu_i}dt\quad;(z\in D),$$

for $\lambda_i,\mu_i\in\mathbb{R}$, $f_i(t)\in C$.

In this paper, we shall give the best estimate for the norm of pre-Schwarzian derivatives of integral $F_{\lambda,\mu}(z)$.

2. Main Results

Theorem 2.1. *Let $\lambda_i\in\mathbb{R}$, $i\in\{1,2,\cdots n\}$, $f_i\in C$ and $g_i\in S^*$. Suppose that $F_{\lambda,\mu}(z)$ is locally univalent in D.*

1) If

(2.1)

$$\|T_{f_i}\|\le\frac{\frac{1}{2}}{\sum_{i=1}^n\lambda_i},\qquad \|T_{g_i}\|\le\frac{\frac{1}{2}}{\sum_{i=1}^n\mu_i},$$

then $F_{\lambda,\mu}$ is univalent.

2) If

(2.2)

$$\|T_{f_i}\|\le\frac{1}{\sum_{i=1}^n\lambda_i},\qquad \|T_{g_i}\|\le\frac{1}{\sum_{i=1}^n\mu_i},$$

then $F_{\lambda,\mu}$ is bounded, where $F_{\lambda,\mu}$ is the integral operator defined as in (1.1).

Proof. Since

$$\|T_{F_{\lambda,\mu}}\|=\sup_{|z|<1}(1-|z|^2)\left|\frac{F_{\lambda,\mu}{}''(z)}{F_{\lambda,\mu}{}'(z)}\right|,$$

then,

$$\|T_{F_{\lambda,\mu}}\|=\sup_{|z|<1}(1-|z|^2)\times\left|\frac{(\int_0^z\prod_{i=1}^n(f'_i(t))^{\lambda_i}\left(\frac{f_i(t)}{t}\right)^{\mu_i}dt)''}{(\int_0^z\prod_{i=1}^n(f'_i(t))^{\lambda_i}\left(\frac{f_i(t)}{t}\right)^{\mu_i}dt)'}\right|$$

$$=\sup_{|z|<1}(1-|z|^2)\times\left|\frac{\lambda_1[f''_1(z)][f'_1(z)]^{\lambda_1-1}\left(\frac{f_1(z)}{z}\right)^{\mu_1}}{[f'_1(z)]^{\lambda_1}\left(\frac{f_1(z)}{z}\right)^{\mu_1}}\times\frac{[f'_2(z)]^{\lambda_2}\left(\frac{f_2(z)}{z}\right)^{\mu_2}}{[f'_2(z)]^{\lambda_2}\left(\frac{f_2(z)}{z}\right)^{\mu_2}}\times\frac{\cdots[f'_n(z)]^{\lambda_n}\left(\frac{f_n(z)}{z}\right)^{\mu_n}}{\cdots[f'_n(z)]^{\lambda_n}\left(\frac{f_n(z)}{z}\right)^{\mu_n}}\right.$$

$$\left.+\frac{\mu_1\left(\frac{f_1(z)}{z}\right)'\left(\frac{f_1(z)}{z}\right)^{\mu_1-1}[f'_1(z)]^{\lambda_1}[f'_2(z)]^{\lambda_2}\left(\frac{f_2(z)}{z}\right)^{\mu_2}}{[f'_1(z)]^{\lambda_1}\left(\frac{f_1(z)}{z}\right)^{\mu_1}[f'_2(z)]^{\lambda_2}\left(\frac{f_2(z)}{z}\right)^{\mu_2}}\times\frac{\cdots[f'_n(z)]^{\lambda_n}\left(\frac{f_n(z)}{z}\right)^{\mu_n}}{\cdots[f'_n(z)]^{\lambda_n}\left(\frac{f_n(z)}{z}\right)^{\mu_n}}+\cdots\right|$$

$$=\sup_{|z|<1}(1-|z|^2)\times\left|\lambda_1\frac{f_1{}''(z)}{f_1{}'(z)}+\mu_1\frac{\left(\frac{f_1(z)}{z}\right)'}{\left(\frac{f_1(z)}{z}\right)}+\cdots\lambda_n\frac{f_n{}''(z)}{f_n{}'(z)}+\mu_n\frac{\left(\frac{f_n(z)}{z}\right)'}{\left(\frac{f_n(z)}{z}\right)}\right|$$

$$=\sup_{|z|<1}(1-|z|^2)\times\left|\sum_{i=1}^n\lambda_i\frac{f''_i(z)}{f'_i(z)}+\sum_{i=1}^n\mu_i\frac{\left(\frac{f_i(z)}{z}\right)'}{\left(\frac{f_i(z)}{z}\right)}\right|.$$

By Alexander theorem we have $f_i(z)=zg'_i(z)$ where $g_i(z)\in S^*$.

Hence we have

$$\|T_{F_{\lambda,\mu}}\| =$$

$$\sup_{|z|<1}(1-|z|^2)\left|\sum_{i=1}^{n}\lambda_i\frac{f_i''(z)}{f_i'(z)}+\sum_{i=1}^{n}\mu_i\frac{g_i''(z)}{g_i'(z)}\right|$$

$$\le \sup_{|z|<1}(1-|z|^2)\sum_{i=1}^{n}\lambda_i\left|\frac{f_i''(z)}{f_i'(z)}\right|$$

$$+\sup_{|z|<1}(1-|z|^2)\sum_{i=1}^{n}\mu_i\left|\frac{g_i''(z)}{g_i'(z)}\right|.$$

Then,

$$\|T_{F_{\lambda,\mu}}\| \le \sum_{i=1}^{n}\lambda_i \sup_{|z|<1}(1-|z|^2)\left|\frac{f_i''(z)}{f_i'(z)}\right|$$

$$+\sum_{i=1}^{n}\mu_i \sup_{|z|<1}(1-|z|^2)\left|\frac{g_i''(z)}{g_i'(z)}\right|.$$

So,

$$\|T_{F_{\lambda,\mu}}\| \le \sum_{i=1}^{n}\lambda_i\|T_{f_i}\|+\sum_{i=1}^{n}\mu_i\|T_{g_i}\| \tag{2.3}$$

From (2.1), (2.2) and (2.3) and applying theorem 1.1, we obtain the assertions. □

Theorem 2.2. *Let f_i, g_i and $i \in \{1,2,3,\cdots n\}$ be a family of analytic functions and f_i are convex of order $\beta_i, i \in \{1,2,3,\cdots n\}$ and g_i are starlike of order $\beta_i, i \in \{1,2,3,\cdots n\}$ then,*

$$\|T_{F_{\lambda,\mu}}\| \le \sum_{i=1}^{n}\lambda_i(4-4\beta_i)+\sum_{i=1}^{n}\mu_i(6-4\beta_i)$$

Proof. From (2.3) and by using theorem 1.2 we conclude the proof. □

Corollary 2.1. *Let f_i, g_i and $i \in \{1,2,3,\cdots n\}$ be a family of analytic functions and f_i are convex of order β and g_i are starlike of order β then,*

$$\|T_{F_{\lambda,\mu}}\| \le (4-4\beta)\sum_{i=1}^{n}\lambda_i+(6-4\beta)\sum_{i=1}^{n}\mu_i\,.$$

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