

NORMALITY CRITERIA FOR FAMILIES OF MEROMORPHIC FUNCTIONS WITH SHARED VALUES

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ABSTRACT. In this paper we have discussed normality criteria of a family of meromorphic functions. We have studied whether a family of meromorphic functions $\mathcal{F}$ is normal in D if for a normal family G and for each function $f \in \mathcal{F}$ there exists $g \in G$ such that $(f^{(k)})^n = a_i$ implies $(g^{(k)})^n = a_i$, ($i = 1, \dots, 2$) for two distinct non zero constants a_i and $n (\geq 2)$, k being positive integers. In this approach we have considered the functions with multiple zeros and multiple poles. We also have proved another result which improves the result of Yuan et al. [1].

1. INTRODUCTION

Let D be a domain in the open complex plane $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions defined in D . The family is said to be normal in D , in the sense of Montel [2], if for any sequence $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$ converging spherically locally uniformly to a meromorphic function or ∞ .

Let f and g be two meromorphic functions and $a \in \mathbb{C}$ then we say that f and g share a counting (ignoring) multiplicities if $f - a$ and $g - a$ have same zeros counting (ignoring) multiplicities. Again, $g = a$ whenever $f = a$, we denote it by $f = a \implies g = a$.

Schwick [3] first proved an interesting problem concerning sharing values of family of meromorphic functions.

Hayman [4] proposed a conjecture: If $\mathcal{F}$ be a family of meromorphic functions in D and $n \in \mathbb{N}$ and each $f \in \mathcal{F}$ satisfies $f^n(z) f'(z) \neq 1$ then $\mathcal{F}$ is normal in D .

In 1988 Gu [5] proved the conjecture for $n \geq 3$. Pang [6] proved the conjecture for $n = 2$. In 1995 Chen and Fang [7] proved the same for $n = 1$.

In 2004 Fang and Zalcman [8] studied a problem concerning sharing values and they obtained the following result:

Theorem 1.1. [8]. *Let $\mathcal{F}$ be a family of meromorphic functions in D and $n \in \mathbb{N}$. If for each pair of functions f and g in $\mathcal{F}$, f and g share the value*

0, and $f^n f'$, $g^n g'$ share a non-zero value a in D then $\mathcal{F}$ is normal in D .

In 2008 Zhang [9] obtained some normality criteria concerning multiplicities of zeros and poles of the functions of the family of meromorphic functions. He proved the following theorems.

Theorem 1.2. [9]. *Let $\mathcal{F}$ be a family of meromorphic functions in D . If for each $f \in \mathcal{F}$ have poles and zeros of multiplicities at least 3. If for each f and g in $\mathcal{F}$, f' and g' share a nonzero value a in D then $\mathcal{F}$ is normal in D .*

Theorem 1.3. [9]. *Let $\mathcal{F}$ be a family of meromorphic functions on D and $n (\geq 2)$ be an integer. If for each pair of functions f and g in $\mathcal{F}$, $f^n f'$ and $g^n g'$ share a nonzero value a in D , then $\mathcal{F}$ is normal.*

In 2011 Yuan et al. [10] improved Theorem 1.2 by diminishing multiplicity of poles. They proved the following Theorem 1.4.

Theorem 1.4. [10]. *Let $\mathcal{F}$ be a family of meromorphic functions in D . If for each $f \in \mathcal{F}$ all poles have multiplicities at least 2 and zeros of multiplicities at least 4 and also for each f and g in $\mathcal{F}$, f' and g' share a nonzero value a in D then $\mathcal{F}$ is normal in D .*

Yuan et al [10] also proved Theorem 1.3 for $n = 1$, by proving

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Theorem 1.5. [10]. Let $\mathcal{F}$ be a family of meromorphic functions on D and each function $f \in \mathcal{F}$ have multiple zeros and for each pair of functions f and g in $\mathcal{F}$, ff' and gg' share a nonzero value a in D , then $\mathcal{F}$ is normal.

Recently Yuan et al. [1] also proved a result concerning sharing values between two families of functions. They proved

Theorem 1.6. [1]. Let $\mathcal{F}$ and G be two families of meromorphic functions in $D \subset \mathbb{C}$, k being a positive integer and a_i , ($i = 1, 2$) be two distinct nonzero constants. Suppose for each $f \in \mathcal{F}$, all of its zeros are of multiplicities at least $k + 1$ and all its poles are multiple. If G be normal and for each $f \in \mathcal{F}$ there exists $g \in G$ such that $f^{(k)} = a_i \Rightarrow g^{(k)} = a_i$ then $\mathcal{F}$ is normal in D .

They also proved another result for $k = 1$ in Theorem 1.6 by considering the family of functions containing only multiple zeros.

Theorem 1.7. [1]. Let $\mathcal{F}$ and G be two families of meromorphic functions in $D \subset \mathbb{C}$. If all zeros of each $f \in \mathcal{F}$ have multiplicities at least 3 and a_i ($i = 1, 2$) be two distinct nonzero constants. If G be normal and for each $f \in \mathcal{F}$ there exists $g \in G$ such that $f' = a_i \Rightarrow g' = a_i$ then $\mathcal{F}$ is normal in D .

In this paper we have taken $f^{(k)^n}$ instead of $f^{(k)}$ in Theorem 1.6 where $n \geq 2$. which is presented in Section 3. Also we improve Theorem 1.7 by diminishing multiplicity of poles of the functions.

2. PRELIMINARIES

In order to prove our theorems we require the following results.

Lemma 2.1. [11]. Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc such that all zeros of each $f \in \mathcal{F}$ have multiplicity greater than p and all poles have multiplicity greater than q . Let α be a real number satisfying $-q < \alpha < p$. Then $\mathcal{F}$ is not normal at 0 if and only if there exist

- (1) a number r , $0 < r < 1$;
- (2) points z_n , $|z_n| < r$;
- (3) functions $f_n \in \mathcal{F}$;
- (4) positive numbers $\rho_n \rightarrow 0$;

such that

$$g_n(\tau) = \rho_n^{-\alpha} f_n(z_n + \rho_n \tau)$$

converges spherically uniformly on each compact subset on $\mathbb{C}$ to a nonconstant meromorphic function $g(\tau)$, its all zeros are of multiplicity greater

than p and its all poles are of multiplicity greater than q and its order is at most 2.

Lemma 2.2. [12]. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 + \frac{q(z)}{p(z)}$ where $a_0, \dots, a_n$ are constants with $a_n \neq 0$ and $q(z)$, $p(z)$ are two coprime polynomials, neither of which vanishes identically with $\deg q < \deg p$. Also let k be a positive integer b a nonzero complex number such that $f^{(k)} \neq b$ and zeros of f have multiplicities at least $k + 1$ then $f(z) = \frac{b(z-d)^{k+1}}{k!(z-c)}$, where c, d are distinct complex numbers.

Lemma 2.3. [13] Let f be a meromorphic function, $k \geq 1$ and $\epsilon > 0$ then we have:

$$\begin{aligned} (k-2)\overline{N}(r, f) + N(r, \frac{1}{f}) &\leq \\ &\leq 2\overline{N}(r, \frac{1}{f}) + N\left(r, \frac{1}{f^{(k)}}\right) + \epsilon T(r, f) + S(r, f). \end{aligned}$$

Lemma 2.4. [12] Let f be a meromorphic function of finite order n (≥ 2), an integer, and f has only zeros of order at least n and poles of order at least 2 then for each k , ($1 \leq k \leq n-1$), $f^{(k)}$ assumes every nonzero finite value infinitely often.

3. MAIN RESULTS

Lemma 3.1. Let f be a meromorphic function with finite order and k, n (≥ 2) be two positive integers. If zeros of f are of multiplicity at least $k + 1$ and no poles are simple and if $f^{(k)^n}(z) \neq a$ then $f(z)$ is constant.

Proof. Obviously we have $f^{(k)}(z) \neq a$. If we have $f^{(k)}(z) = a$ then f is a polynomial of degree k but zeros of $f(z)$ has degree at least $k + 1$ which is a contradiction.

We consider three cases:

- Case 1. Suppose f be a polynomial. As above we conclude that f can not be a polynomial.
- Case 2. Let f be a transcendental meromorphic function of finite order. If $k = 1$ then by Lemma 2.4 we get $f'(z)$ assumes every value infinitely often which contradicts $f^{(k)}(z) \neq a$. If $k \geq 2$ then by Lemma 2.3 we have:

$$\begin{aligned} \overline{N}(r, f) + N(r, \frac{1}{f}) &\leq 2 \overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}}) + \\ &+ \epsilon T(r, f) + S(r, f). \end{aligned}$$

From Milloux inequality for $a \neq 0, \infty$ we have

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \\ &+ N\left(r, \frac{1}{f^{(k)} - a}\right) - \\ &- N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f). \end{aligned}$$

From the above two inequalities we have

$$\begin{aligned} T(r, f) &\leq 2 \overline{N}(r, \frac{1}{f}) + N\left(r, \frac{1}{f^{(k)} - a}\right) + \\ &+ \epsilon T(r, f) + S(r, f). \end{aligned}$$

Considering zeros of order $\geq k+1 > 2$ and $\epsilon = \frac{1}{6}$ we have

$$T(r, f) \leq 6N\left(r, \frac{1}{f^{(k)} - a}\right) + S(r, f).$$

which contradicts $f^{(k)}(z) \neq a$.

Case 3. Let f be rational. Then by Lemma 2.2 we have $f(z) = \frac{b(z-d)^{k+1}}{k!(z-c)}$, which contradicts that poles of f are multiple.

Hence $f(z)$ is constant. $\square$

Theorem 3.1. *Let $\mathcal{F}$ and G be two families of meromorphic functions in $D \subset \mathbb{C}$, k being a positive integer and a_i , ($i = 1, 2$) be two distinct nonzero constants. Suppose for each $f \in \mathcal{F}$, all of its zeros are of multiplicities at least $k+1$ and no poles are simple. If G is normal and for each $f \in \mathcal{F}$ there exists $g \in G$ such that $f^{(k)n} = a_i \Rightarrow g^{(k)n} = a_i$ and $n \geq 2$ then $\mathcal{F}$ is normal in D .*

Proof. Suppose $\mathcal{F}$ is not normal in D there exists at least one $z_0 \in D$ where $\mathcal{F}$ is not normal. Without loss of generality let $z_0 = 0$. Then by Lemma 2.1 there exists sequence $\{z_j\}$ of complex numbers with $z_j \rightarrow 0$ and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \rightarrow 0$ such that:

$$F_j(\xi) = \rho_j^{-\frac{k}{n}} f_j(z_j + \rho_j \xi) \rightarrow F(\xi),$$

locally uniformly with respect to the spherical metric, where F is non constant meromorphic in $\mathbb{C}$ all of whose zeros have multiplicity at least $k+1$. Moreover order of F is less than two.

Now let

$$\left(F_j^{(k)n}\right)_j(\xi) = \left(f_j^{(k)n}\right)(z_j + \rho_j \xi) \rightarrow \left(F^{(k)n}\right)(\xi)$$

also converges uniformly with respect to the spherical metric. Then using Lemma 3.1 we know that $\left(F^{(k)n}\right)(\xi)$ takes two distinct non-zero finite values $\{a_1, a_2\}$. We set ξ_0 , and ξ^*_0 to be two zeros of

$\left(F^{(k)n}\right) - a_1$ and $\left(F^{(k)n}\right) - a_2$ respectively. Obviously $\xi_0 \neq \xi^*_0$ and then choose $\delta > 0$ small enough such that $D(\xi_0, \delta) \cap D(\xi^*_0, \delta) = \emptyset$ where:

$$D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\} \text{ and}$$

$$D(\xi^*_0, \delta) = \{\xi : |\xi - \xi^*_0| < \delta\}.$$

By Hurwitz's theorem there exist points $\xi_j \in D(\xi_0, \delta)$ and $\xi^*_j \in D(\xi^*_0, \delta)$ such that for sufficiently large j ,

$$\left(f_j^{(k)n}\right)(z_j + \rho_j \xi) - a_1 = 0, \text{ and}$$

$$\left(f_j^{(k)n}\right)(z_j + \rho_j \xi) - a_2 = 0.$$

From the hypothesis: $\left(g_j^{(k)n}\right)(z_j + \rho_j \xi) = a_1$ and $\left(g_j^{(k)n}\right)(z_j + \rho_j \xi) = a_2$.

Since G is normal without loss of generality we assume that:

$$g_j(z) \rightarrow g(z)$$

converges locally spherically uniformly. As $j \rightarrow \infty$ we have:

$$0 < |a_1 - a_2|$$

$$= \left| \left(g_j^{(k)n}\right)(z_j + \rho_j \xi) - \left(g_j^{(k)n}\right)(z_j + \rho_j \xi) \right|,$$

i.e., $0 < |a_1 - a_2| = \left| \left(g_j^{(k)n}\right)(0) - \left(g_j^{(k)n}\right)(0) \right| = 0$, which is a contradiction. This completes the proof of the theorem. $\square$

Theorem 3.2. *Let $\mathcal{F}$ and G be two families of meromorphic functions in $D \subset \mathbb{C}$. If each $f \in \mathcal{F}$, all zeros and poles have multiplicities at least 2 and a_i ($i = 1, 2$) be two distinct nonzero constants. If G be normal and for each $f \in \mathcal{F}$ there exists $g \in G$ such that $f' = a_i \Rightarrow g' = a_i$ then $\mathcal{F}$ is normal in D .*

Proof. Suppose $\mathcal{F}$ is not normal in D there exists at least one $z_0 \in D$ where $\mathcal{F}$ is not normal. Without loss of generality let $z_0 = 0$. Then by Lemma 2.4 there exists sequence $\{z_j\}$ of complex numbers with $z_j \rightarrow 0$ and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \rightarrow 0$ such that:

$$F_j(\xi) = \rho_j^{-1} f_j(z_j + \rho_j \xi) \rightarrow F(\xi)$$

and poles are multiple. Again we have

$$F_j'(\xi) = f_j'(z_j + \rho_j \xi) \rightarrow F'(\xi)$$

converges uniformly with respect to the spherical metric.

Combining with Lemma 2.4 we know that $(F')(\xi)$ takes two distinct non-zero finite values $\{a_1, a_2\}$. Set ξ_0, ξ^*_0 two zeros of $(F') - a_1$ and $(F') - a_2$ respectively. Obviously $\xi_0 \neq \xi^*_0$ and then choose $\delta > 0$ small enough such that $D(\xi_0, \delta) \cap D(\xi^*_0, \delta) = \emptyset$ where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi^*_0, \delta) = \{\xi : |\xi - \xi^*_0| < \delta\}$.

By Hurwitz's theorem there exist $\xi_j \in D(\xi_0, \delta)$ and $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j it holds:

$$\begin{aligned} (f_j') (z_j + \rho_j \xi) - a_1 &= 0, \\ (f_j') (z_j + \rho_j \xi) - a_2 &= 0. \end{aligned}$$

From the hypothesis

$$(g_j') (z_j + \rho_j \xi) = a_1, \text{ and } (g_j') (z_j + \rho_j \xi) = a_2.$$

Since G is normal without loss of generality we assume that:

$$g_j(z) \rightarrow g(z),$$

converges locally spherically uniformly. As $j \rightarrow \infty$ we have

$$0 < |a_1 - a_2| = |(g_j') (z_j + \rho_j \xi) - (g_j') (z_j + \rho_j \xi)|,$$

i.e $0 < |a_1 - a_2| = |(g') (0) - (g') (0)| = 0$, which is a contradiction. $\square$

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